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A NOTE ON CLOPEN TOPOLOGIES

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Abstract Let X be a set, $\mathbf{Clop}(X)$ the set of all clopen topologies on X and $\prod(X)$ the set of all partitions of X. Our aim here is to provide a direct simple proof of the fact that the sets $\mathbf{Clop}(X)$ and $\prod(X)$ are equipotent.

1 Introduction

A subset of a topological space which is both closed and open is referred to as a clopen set. A topology in which every open set is closed will be called a *clopen topology* (clopen topologies are also called partition topologies). Let X be a set, $\mathbf{Clop}(X)$ the set of all clopen topologies on X and $\prod(X)$ the set of all partitions of X. In [2, Theorem 2.3], the authors have proved that these two sets are equipotent in case X is finite. On the other hand, Ayache et al. have shown in [1, Theorem 1.4] that this result holds true even if X is infinite. Our aim here, is to provide a direct and simple proof of this fact.

Any undefined terminology is standard as in [3].

2 The proof

Let $\theta : \mathbf{Clop}(X) \to \prod(X),\ \mathcal{T} \mapsto \{\overline{\{x\}}^{\mathcal{T}} \mid x \in X\}$.

Theorem 2.1. *θ is a bijective mapping from* $\mathbf{Clop}(X)$ *onto* $\prod(X)$.

Proof. θ is well-defined. Indeed, let $x, y \in X$ such that $\overline{\{x\}}^{\mathcal{T}} \cap \overline{\{y\}}^{\mathcal{T}} \neq \emptyset$. Pick an element $z \in \overline{\{x\}}^{\mathcal{T}} \cap \overline{\{y\}}^{\mathcal{T}}$. We claim that $\overline{\{x\}}^{\mathcal{T}} = \overline{\{z\}}^{\mathcal{T}}$. As $z \in \overline{\{x\}}^{\mathcal{T}}$, then $\overline{\{z\}}^{\mathcal{T}} \subseteq \overline{\{x\}}^{\mathcal{T}}$. Conversely, let $x \in O \in \mathcal{T}$. As $O = \overline{O}^{\mathcal{T}}$, we get $\overline{\{x\}}^{\mathcal{T}} \subseteq O$. Thus, $z \in O$. It follows that $x \in \overline{\{z\}}^{\mathcal{T}}$ and so $\overline{\{x\}}^{\mathcal{T}} \subseteq \overline{\{z\}}^{\mathcal{T}}$. Therefore, $\overline{\{x\}}^{\mathcal{T}} = \overline{\{z\}}^{\mathcal{T}}$, as claimed. Similarly, we show that $\overline{\{y\}}^{\mathcal{T}} = \overline{\{z\}}^{\mathcal{T}}$. Hence, $\overline{\{x\}}^{\mathcal{T}} = \overline{\{y\}}^{\mathcal{T}}$. On the other hand, as $X = \bigcup_{x \in X} \overline{\{x\}}^{\mathcal{T}}$, we infer that $\{\overline{\{x\}}^{\mathcal{T}} \mid x \in X\} \in \prod(X)$.

Now, we show that θ is injective. For, let $\mathcal{T}, \mathcal{T}' \in \mathbf{Clop}(X)$ such that $\theta(\mathcal{T}) = \theta(\mathcal{T}')$. Let $O \in \mathcal{T}$. One can easily check that $O = \bigcup_{x \in O} \overline{\{x\}}^{\mathcal{T}}$. For any $x \in O$, there exists $x^* \in X$ such that $\overline{\{x\}}^{\mathcal{T}} = \overline{\{x^*\}}^{\mathcal{T}'}$. Thus, $O = \bigcup_{x \in O} \overline{\{x^*\}}^{\mathcal{T}'} \in \mathcal{T}'$. Hence, $\mathcal{T} \subseteq \mathcal{T}'$. A similar argument shows that $\mathcal{T}' \subseteq \mathcal{T}$ and hence, $\mathcal{T} = \mathcal{T}'$.

It remains to show that θ is surjective. To this end, let $\Pi := \{A_i \mid i \in I\} \in \prod(X)$ and set $\mathcal{T} := \{\emptyset, \bigcup_{i \in J} A_i, J \subseteq I\}$. Clearly, $\mathcal{T} \in \mathbf{Clop}(X)$. For any $i \in I$, pick $x_i \in A_i$. We claim that $\overline{\{x_i\}}^{\mathcal{T}} = A_i$. Indeed, as A_i is closed, we get $\overline{\{x_i\}}^{\mathcal{T}} \subseteq A_i$. Now, let $y \in A_i$ and let $O \in \mathcal{T}$ such that $y \in O$. There exists $J \subseteq I$ such that $O = A_i \cup (\bigcup_{i \neq j \in J} A_j)$. Hence, $x_i \in O$. It follows that $y \in \overline{\{x_i\}}^{\mathcal{T}}$. Therefore $A_i \subseteq \overline{\{x_i\}}^{\mathcal{T}}$ and so $A_i = \overline{\{x_i\}}^{\mathcal{T}}$. It follows that $\Pi \subseteq \theta(\mathcal{T})$. For the reverse inclusion, let $x \in X$. As $X = \bigcup_{i \in I} A_i$, then there exists $i \in I$ such that $x \in A_i$. Proceed along the same lines as above, we infer that $\overline{\{x\}}^{\mathcal{T}} = A_i$. Therefore, $\theta(\mathcal{T}) \subseteq \Pi$. Hence, $\theta(\mathcal{T}) = \Pi$. The proof is complete. □

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