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PRODUCT DIMENSION OF UNIFORM SPIDER

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Abstract In this paper, we define a category of a spider graph called uniform spider. We study product dimension of the uniform spider $S(n,t)$ and obtain the exact dimension for the case $n = 2$. We obtain a lower bound of product dimension of $S(n,t)$ and give a different technique to extend the labeling corresponding to the product dimension of a tree T to $T^{(2)}$ which is a one-point union of two copies of T. This result leads us to an upper bound of product dimension of $S(n,t)$, $n > 2$. For certain values of n and t, we have obtained the exact product dimension of $S(n,t)$.

1 Introduction

The beginnings of this topic lie in the classical problem where we are given a class $\mathscr{C}$ of objects of certain type such as posets, graphs and we have a subclass $\mathscr{B}$ of $\mathscr{C}$ such that every $C \in \mathscr{C}$ can be embedded into a $\prod_{i=1}^{n} B_i$ with $B_i \in \mathscr{B}$. It is natural to regard the necessary number of B_i as the dimension of $\mathscr{C}$(with respect to $\mathscr{B}$ and $\mathscr{C}$). One such class of objects was studied by Dushnik-Miller in [1]. Taking it further, the dimension (known as product dimension in case of graphs) was initially studied by Lovász et al.[6]. It is well known that every graph can be embedded into a tensor product of suitably many complete graphs. (See Lovász, Nešetřill and Pultr [6]). The dimension (or product dimension) of a graph G denoted by $\text{pdim}(G)$ is defined as the minimal number of complete graphs whose tensor product contains G as an induced subgraph. The study of product dimension has diverse applications. (See D. Greenwell and L. Lovász [3], Koubek et al.[5], Nešetřill and V. Rödl [7]).

The product dimension of complete graphs, discrete graphs, paths and cycles have been studied by Lovász et al. [6]. The product dimension of Caterpillar and Lobster have been studied by Yahyaei and Katre [4, 10]. The product dimension of the set graph has been studied by Shinde et al.[8].

In this paper, we define the notion of uniform spider and study its product dimension. Section 2 includes basic definitions and preliminaries. In Section 3, we obtain the exact product dimension of $S(2,t)$. We also find a lower bound of product dimension of $S(n,t)$ and give an estimate for the difference between product dimension of a tree T to $T^{(2)}$ which is a one-point union of two copies of T. This result leads us to an upper bound of product dimension of $S(n,t), n > 2$. When t is a power of 2, we have obtained the exact product dimension of $S(n,t)$ for all values of n. Also when n is a power of 2 and t satisfies certain conditions, we have obtained the exact product dimension of $S(n,t)$. Finally, we have obtained the exact product dimension of $S(n,t)$ when n and t are not powers of t but satisfy certain conditions.

2 Preliminaries

Definition 2.1 (Shee and Yong [9]). A vertex v in a graph G is called a *root* of G if it is distinguished from all other vertices of G. A graph G is called a *rooted graph* if it has a root. We denote by $G^{(n)}$ the one-point union of n copies of a rooted graph G by identifying their root.

Definition 2.2 (Gallian [2]). A *spider* is a tree that has at most one vertex (called the center) of degree greater than 2. Every path from the center to a pendant vertex is called a *leg* of the spider.

Definition 2.3. A *uniform spider* is defined as a spider with all legs of the same length. We denote by $S(n,t)$, the uniform spider with t legs of length n each, where $n, t \in \mathbb{N}, n \geq 1, t \geq 2$.

Let v denote the center of $S(n,t)$. The vertex set and edge set of $S(n,t)$ are $V(S(n,t)) = \{v\} \cup \{v_i^j : 1 \leq i \leq n, 1 \leq j \leq t\}$ and $E(S(n,t)) = \{vv_1^j : 1 \leq j \leq t\} \cup \{v_i^j v_{i+1}^j : 1 \leq i \leq (n-1), 1 \leq j \leq t\}$ respectively. Here i is the distance of the vertex v_i^j from the center 'v'. Also $|V(S(n,t))| = nt + 1$ and $|E(S(n,t))| = nt$.

Definition 2.4 (Lovász, Nešetřill and Pultr [6]). The *product dimension* of a graph G is defined as the minimal number of complete graphs whose tensor product contains G as an induced subgraph. It is denoted by $\text{pdim}(G)$.

An equivalent definition of pdim is as follows:

Definition 2.5 (Lovász, Nešetřill and Pultr [6]). Let G be a graph. For any positive integer n, we define an encoding of G in $\mathbb{N}^n$ as an injection $f : V(G) \to \mathbb{N}^n$, where $uv \in E(G)$ if and only if $f(u)$ and $f(v)$ differ in all coordinates. We let $f_k(v)$ denote the k^{th} coordinate of the n-dimensional vector assigned to the vertex v. Thus $f(v) = (f_1(v), f_2(v), \ldots, f_n(v))$. Here, $\mathbb{N}^n$ denotes the cartesian product of the set of non-negative integers with itself n times. The *product dimension* of G is the least positive integer n such that an encoding of G exists.

It has been proved in [7] that the product dimension of every graph is a well defined positive integer.

Theorem 2.6 (Lovász, Nešetřill and Pultr [6]). *Let P_n be a path of length n. Then* $\text{pdim}(P_0) = \text{pdim}(P_1) = 1$, $\text{pdim}(P_2) = 2$ *and for* $n > 2$, $\text{pdim}(P_n) = \lceil \log_2 n \rceil$.

The following Proposition is used in Lemma 3.1 and Lemma 3.2.

Proposition 2.7 (Lovász, Nešetřill and Pultr [6]). *Let $x^1, x^2, \ldots, x^\ell$ be distinct elements of $V(G)$ such that for some $y^1, y^2, \ldots, y^\ell \in V(G)$, $\{x^i, y^i\} \in E(G)$ and $\{x^i, y^j\} \notin E(G)$ for $i < j$, then* $\text{pdim}(G) \geq \log_2 \ell$.

3 Product Dimension of $S(n,t)$

For $n = 1$, $S(n,t)$ is the star graph $K_{1,t}$. It can be easily proved that $\text{pdim}(K_{1,t}) = 2$. Our study covers the cases for $S(n,t)$, where $n \geq 2$.
One of the ways to get the product dimension of a graph G is to obtain a lower bound of the product dimension and then show that the lower bound is achieved by a certain labelling. We first obtain a lower bound of $\text{pdim}(S(2,t))$ using Proposition 2.7.

Lemma 3.1. $\text{pdim}(S(2,t)) \geq \lceil \log_2 2t \rceil$*, where $S(2,t)$ denotes a uniform spider with t legs each of length 2.*

Proof. We first prove that $\log_2 2t$ is a lower bound of $\text{pdim}(S(2,t))$. For $1 \leq i \leq 2t$, let

$$x^i = \begin{cases} v_1^{\frac{i+1}{2}}, & \text{if } i \text{ is odd;} \\ v_2^{\frac{i}{2}}, & \text{if } i \text{ is even} \end{cases}$$

and

$$y^i = \begin{cases} v_2^{\frac{i+1}{2}}, & \text{if } i \text{ is odd;} \\ v_1^{\frac{i}{2}}, & \text{if } i \text{ is even.} \end{cases}$$

Clearly for each $i, \{x^i, y^i\} \in E(S(2,t))$. Now, let $1 \leq i < j \leq 2t$. If i is even then x^i is a pendant vertex and as such is adjacent to only y^i, hence x^i is not adjacent to y^j, if $i < j$.
If i is odd then x^i is a vertex of degree 2, hence x^i is adjacent to the central vertex v (which does not feature in the selection of x^i's and y^i's) and to y^i.
Now, since $x^1, x^2, \ldots, x^{2t}$ are distinct elements of $V(S(2,t))$ such that for
$y^1, y^2, \ldots, y^{2t} \in V(S(2,t))$ (as defined above), $\{x^i, y^i\} \in E(S(2,t))$ and $\{x^i, y^j\} \notin E((S(2,t))$ for $i < j$, so $\text{pdim}\, S(2,t) \geq \log_2 2t$. Thus $\lceil \log_2 2t \rceil$ is a lower bound of $\text{pdim}(S(2,t))$. □

We will now get a labelling of $S(2,t)$ requiring $\lceil \log_2 2t \rceil$ coordinates so that $\lceil \log_2 2t \rceil$ becomes an upper bound of $\text{pdim}(S(2,t))$.

Lemma 3.2. $\text{pdim}(S(2,t)) \leq \lceil \log_2 2t \rceil$, *where* $t \geq 2$.

Proof. We first prove the result when t is of the form 2^β, where $\beta \in \mathbb{N}, \beta \geq 1$ that is $\text{pdim}(S(2,2^\beta)) \leq \beta + 1$. We prove this result by induction on β.
Step 1: For $\beta = 1$, we get a path for which the result is already proved in [6].
Step 2: Assume that the result is true for some β; that is $\text{pdim}(S(2,2^\beta)) \leq \beta + 1$.
Hence we have a labelling $f : V(S(2,2^\beta)) \to \mathbb{N}^{\beta+1}$ satisfying conditions of pdim.
We now consider $S(2,2^{\beta+1})$. Observe that $S(2,2^{\beta+1})$ is a one-point union of two copies of $S(2,2^\beta)$ by identifying the center as the root. For ease of reference, we use the following notation for the vertices of $S(2,2^{\beta+1})$. In the first copy of $S(2,2^\beta)$ we retain the earlier notation, that is 'v' for the central vertex and v_i^j for the i^{th} vertex on the j^{th} leg. In the second copy of $S(2,2^\beta)$, we replace each v_i^j by w_i^j.
We define a new labelling $g : V(S(2,2^{\beta+1})) \to \mathbb{N}^{\beta+2}$ as follows:
For $1 \leq k \leq \beta+1$, let $g_k(v) = f_k(v)$, $g_k(v_i^j) = g_k(w_i^j) = f_k(v_i^j)$ and $g_{\beta+2}(v) = 0$, $g_{\beta+2}(v_1^j) = 1$, $g_{\beta+2}(v_2^j) = 2$, $g_{\beta+2}(w_1^j) = 2$, $g_{\beta+2}(w_2^j) = 1$.
In other words, we have retained the first $\beta+1$ coordinates as it is for all vertices in the first copy of $S(2,\beta)$; also each vertex w_i^j in the second copy of $S(2,\beta)$ matches in the first $\beta+1$ coordinates with the corresponding vertex v_i^j in the first copy.
By choice of the last coordinates, it is evident that no two vertices have identical labels.
Let us now check the condition for adjacent vertices. Adjacent vertices fall in one of the following categories:
(i) $\{v, v_1^j\}$, $\{v, w_1^j\}$
(ii) $\{v_1^j, v_2^j\}$, $\{w_1^j, w_2^j\}$.
For (i): Since v and v_1^j are different in the first '$\beta+1$' coordinates and also the last, they differ in all coordinates. Likewise, for the pair $\{v, w_1^j\}$.
For (ii): v_1^j and v_2^j clearly differ in the first '$\beta+1$' coordinates and by choice we have taken the last coordinate on v_1^j and v_2^j as $1, 2$ respectively. Hence they also differ in the last coordinate. The argument is similar for the pair w_1^j, w_2^j.
Finally, we check the condition for every pair of non adjacent vertices in $S(2,2^{\beta+1})$. If both vertices lie in the first copy they already have at least one coordinate common from amongst the first '$\beta+1$' coordinates. Likewise, if both vertices lie in the second copy.
We now look at non adjacent pairs of vertices where one vertex lies in the first copy and one in the second copy of $S(2,2^\beta)$.
We consider the pair $\{v_i^j, w_{i'}^{j'}\}$. Observe that for $1 \leq k \leq \beta+1$, $g_k(w_{i'}^{j'}) = g_k(v_{i'}^{j'})$ and $g_{\beta+2}(w_{i'}^{j'}) \neq g_{\beta+2}(v_{i'}^{j'})$.
If v_i^j and $v_{i'}^{j'}$ are adjacent, then $g_k(v_i^j) \neq g_k(v_{i'}^{j'})$ for $1 \leq k \leq \beta+2$; thus $g_{\beta+2}(v_i^j) = g_{\beta+2}(w_{i'}^{j'})$; hence v_i^j and $w_{i'}^{j'}$ match in the last coordinate.
However, suppose v_i^j and $v_{i'}^{j'}$ are non adjacent, then $g_k(v_i^j) = g_k(v_{i'}^{j'})$ for at least one k, $1 \leq k \leq \beta+1$; hence v_i^j and $w_{i'}^{j'}$ match in at least one coordinate amongst the first $\beta+1$.
Thus we have obtained a labelling for $S(2,2^{\beta+1})$ having $\beta+2$ coordinates and satisfying conditions of pdim. Therefore $\text{pdim}(S(2,2^{\beta+1})) \leq \beta+2$. Thus the result is proved for $\beta+1$.
Lastly, for any other t, there exists δ such that $2^\delta < t < 2^{\delta+1}$. Since $S(2,t)$ is an induced subgraph of $S(2,2^{\delta+1})$; hence $\text{pdim}(S(2,t)) \leq \text{pdim}(S(2,2^{\delta+1})) \leq \delta+2$.
Further, $2^\delta < t < 2^{\delta+1}$ implies that $2^{\delta+1} < t < 2^{\delta+2}$. Hence $\delta+1 < \log_2 2t < \delta+2$ and therefore $\lceil \log_2 2t \rceil = \delta+2$. □

From Lemma 3.1 and Lemma 3.2, it is easy to prove the following theorem.

Theorem 3.3. $\text{pdim}(S(2,t)) = \lceil \log_2 2t \rceil$, *where* $t \geq 2$.

We next obtain a lower bound of $\text{pdim}(S(n,t))$, where $n > 2$.

Theorem 3.4. *For* $n > 2$ *and* $t \geq 2$, $\text{pdim}(S(n,t)) \geq \lceil \log_2\big((n-1)t+1\big) \rceil$.

Proof. For each r; $1 \leq r \leq (n-1), 0 \leq i < (t-1)$, we define $x^{r+i(n-1)} = {v_r}^{i+1}$ and for $i = t-1$,

$$x^{r+(t-1)(n-1)} = \begin{cases} v, & \text{for } r = 1; \\ v^t_{r-1}, & \text{for } 2 \leq r \leq n \end{cases}$$

Also, $y^{r+i(n-1)} = {v_{r+1}}^{i+1}$ for $1 \leq r \leq (n-1), 0 \leq i < (t-1)$ and $y^{r+(t-1)(n-1)} = v^t_r$ for $1 \leq r \leq n$. We can represent this as follows:

$v^1_1\ v^1_2\ \ldots\ v^1_{n-1}$	$v^2_1\ v^2_2\ \ldots\ v^2_{n-1}$	$\ldots$	$v^i_1\ v^i_2\ \ldots\ v^i_{n-1}$	$\ldots$	$v\ v^t_1\ v^t_2\ \ldots\ v^t_{n-1}$
$v^1_2\ v^1_3\ \ldots\ v^1_n$	$v^2_2\ v^2_3\ \ldots\ v^2_n$	$\ldots$	$v^i_2\ v^i_3\ \ldots\ v^i_n$	$\ldots$	$v^t_1\ v^t_2\ v^t_3\ \ldots\ v^t_n$

where the x^i's are in the first row and y^i's in the second row.
From the above table, it is immediately clear that x^i and y^i are adjacent for all i and x^i is not adjacent to y^j for $i < j$. Hence by Proposition 2.7, pdim$(S(n,t)) \geq \lceil \log_2((n-1)t+1) \rceil$. □

Theorem 3.5. *Let T be a tree and let v be any vertex of T. Then*

$$\text{pdim}(T^{(2)}) - \text{pdim}(T) \leq 1,$$

where $T^{(2)}$ is a one-point union of two copies of T by identifying the root as v.

Proof. Let $V(T) = \{v, v_1, v_2, \cdots, v_n\}$ and $T^{(2)}$ be a one-point union of two copies of T. Let the vertices be denoted as before in one copy of T and except for v, let the vertices in the second copy be denoted by $w_1, w_2, \cdots, w_n$; where the vertex w_i in the second copy corresponds to the vertex v_i in the first copy of T. We note here that the two copies have only the vertex v in common and are edge disjoint. Thus $V(T^{(2)}) = \{v, v_1, v_2, \cdots, v_n, w_1, w_2, \cdots, w_n\}$. Let pdim$(T) = a$. Thus there is a labeling $f : V(T) \to \mathbb{N}^a$ of vertices of T by vectors of length a satisfying the conditions of pdim.
We define a labeling $g : V(T^{(2)}) \to \mathbb{N}^{a+1}$ of vertices of $T^{(2)}$ by vectors of length $a+1$ as follows: For $1 \leq k \leq a$, let

$$g_k(v) = f_k(v),$$
$$g_k(v_i) = g_k(w_i) = f_k(v_i)$$

and

$$\begin{aligned} g_{a+1}(v) &= 0, \\ g_{a+1}(v_i) &= 1, \quad \text{if } d(v, v_i) \text{ is odd} \\ &= 2, \quad \text{if } d(v, v_i) \text{ is even} \\ g_{a+1}(w_i) &= 1, \quad \text{if } d(v, w_i) \text{ is even} \\ &= 2, \quad \text{if } d(v, w_i) \text{ is odd;} \end{aligned}$$

here $d(v, v_i)$ denotes the distance between v and v_i. By the choice of the last coordinates, it is evident that no two vertices have identical labels.
We now ensure that for all adjacent pairs of vertices in $T^{(2)}$, all corresponding coordinates differ. Adjacent vertices fall in one of the following categories:
(i) $\{v, v_i\}, \{v, w_i\}$
(ii) $\{v_i, v_j\}, \{w_i, w_j\}$.
For (i): Consider the adjacent pair $\{v, v_i\}$; then $f_k(v) \neq f_k(v_i)$, for any k, $1 \leq k \leq a$. Also $f_{a+1}(v) = 0$, whereas $f_{a+1}(v_1)$ is either 1 or 2. Therefore $g_k(v) \neq g_k(v_i)$ for each k, $1 \leq k \leq a+1$. Hence $g(v)$ and $g(v_i)$ differ in all $a+1$ coordinates. Likewise for the pair $\{v, w_i\}$.
Next, consider the adjacent pair of vertices $\{v_i, v_j\}$. Clearly, $f_k(v_i) \neq f_k(v_j)$ for each k, $1 \leq k \leq a$.
Next suppose $d(v, v_i) = b$, then $d(v, v_j)$ is either $b-1$ or $b+1$ as there is precisely one path between each pair of vertices in a tree. Hence if $g_{a+1}(v_i)$ is 1(or 2) then $g_{a+1}(v_j)$ is 2 (or 1). Thus $g(v_i)$ and $g(v_j)$ differ in all coordinates. Similar argument works for the pair $\{w_i, w_j\}$.
Finally, we check the conditions for non adjacent pairs of vertices in $T^{(2)}$. If both vertices lie in the first copy or both vertices lie in the second copy and are non-adjacent, then they have at least

one coordinate common from amongst the first 'a' coordinates.
Hence we only need to check the condition for non-adjacent pairs of the type $\{v_i, w_j\}$.
We begin by noting that $g_k(w_j) = g_k(v_j)$, for $1 \leq k \leq a$ and $g_{a+1}(w_j) \neq g_{a+1}(v_j)$. Firstly, suppose v_i and v_j are non-adjacent, then $f_k(v_i) = f_k(v_j)$ for at least one k, $1 \leq k \leq a$. Hence $f_k(v_i) = f_k(w_j)$ for at least one k. Next suppose v_i and v_j are adjacent, then $f_k(v_i) \neq f_k(v_j)$, for each k, $1 \leq k \leq a$. Also by choice, as v_i and v_j are adjacent vertices, $g_{a+1}(v_i) \neq g_{a+1}(v_j)$. Thus $g_{a+1}(v_i) = g_{a+1}(w_j)$. Hence v_i and w_j agree in the $(a+1)^{\text{th}}$ coordinate. Thus we have obtained a labeling of $T^{(2)}$ using $a+1$ coordinates and satisfying conditions of pdim. Therefore pdim$(T^{(2)})$ is either a or $a+1$ that is pdim(T) or pdim$(T)+1$. □

Corollary 3.6. pdim$(S(n,t'))$ – pdim$(S(n,t)) \leq 1$ *for all* $n \in \mathbb{N}$, $n \geq 1$ *and* $t \leq t' \leq 2t$.

Proof. Since $S(n, 2t)$ is a one-point union of two copies of $S(n,t)$, the results follows for $t' = 2t$. For all other values of t', $S(n,t')$ is an induced subgraph of $S(n, 2t)$, hence the result will follow. □

Corollary 3.7. pdim$(S(n,t)) \leq \lceil \log_2 n \rceil + \beta$ *for all* $\beta \geq 1$ *and* $n \geq 2$, $2^{\beta-1} < t \leq 2^{\beta}$.

Proof. We first prove the result when t is of the form 2^β, $\beta \geq 1$; that is, pdim$(S(n, 2^\beta)) \leq \lceil \log_2(2^\beta n) \rceil$.
We prove this result by induction on β.
For $\beta = 1$, $S(n,2) = P_{2n}$ which is a path of length $2n$. Hence by [6],

$$\text{pdim}(S(n,2)) = \text{pdim}(P_{2n}) = \lceil \log_2(2n) \rceil.$$

Let us now assume that the result is true for some $\beta = k$, that is

$$\text{pdim}(S(n,2^k)) \leq \lceil \log_2(2^k n) \rceil.$$

By Corollary 3.6, it follows that

$$\begin{aligned}\text{pdim}(S(n,2^{k+1})) &\leq \text{pdim}(S(n,2^k)) + 1\\ &\leq \lceil \log_2(2^k n) \rceil + 1\\ &= \lceil \log_2(2^{k+1} n) \rceil.\end{aligned}$$

Hence the result is true for $\beta = k+1$.
For other values of t, that is, $2^{\beta-1} < t < 2^\beta$, the result follows as $S(n,t)$ is an induced subgraph of $S(n, 2^\beta)$. □

Theorem 3.8. pdim$(S(n, 2^\beta)) = \alpha + \beta + 1$, *where* $2^\alpha < n \leq 2^{\alpha+1}$, $\alpha \geq 1$, $\beta \geq 1$.

Proof. Let $n = 2^\alpha + r$; then $0 < r \leq 2^\alpha$. By Corollary 3.7, pdim$(S(n, 2^\beta)) \leq \lceil \log_2(2^\beta n) \rceil = \lceil \log_2(2^\beta(2^\alpha + r)) \rceil$. Since $1 \leq r \leq 2^\alpha$, $\alpha + \beta + 1$ is an upper bound of pdim$(S(n, 2^\beta))$.
By Theorem 3.4, $\lceil \log_2((n-1)t+1) \rceil \leq$ pdim$(S(n,t))$.
Now $(n-1)t+1 = (2^\alpha + r - 1)2^\beta + 1 = 2^{\alpha+\beta} + (r-1)2^\beta + 1 \leq 2^{\alpha+\beta} + 2^{\alpha+\beta}$. Also clearly $2^{\alpha+\beta} < 2^{\alpha+\beta} + (r-1)2^\beta + 1 \leq 2^{\alpha+\beta}$. Therefore $\alpha+\beta+1$ is a lower bound of pdim$(S(n, 2^\beta))$.
Hence pdim$(S(n, 2^\beta)) = \alpha + \beta + 1$; for $2^\alpha < n \leq 2^{\alpha+1}$. □

Theorem 3.9. pdim$(S(2^\alpha, t)) = \alpha + \beta + 1$, *where* $2^\beta < t \leq 2^{\beta+1}$, $\alpha, \beta \in \mathbb{N}$, $\alpha > 1$, $\beta \geq 1$ *and* $t - 2^\beta > \dfrac{2^\beta - 1}{2^\alpha - 1}$.

Proof. Let $t = 2^\beta + r$; then $0 < r \leq 2^\beta$. For $2^\beta < t \leq 2^{\beta+1}$, an upper bound of pdim$(S(2^\alpha, t))$ is

$$\lceil \log_2(2(2^\alpha)) \rceil + \beta = \lceil \log_2(2^{\alpha+1}) \rceil + \beta = \alpha + \beta + 1.$$

Note that $\lceil \log_2[(n-1)t+1] \rceil$ is a lower bound of pdim$(S(n,t))$. Thus a lower bound of pdim$(S(2^\alpha, 2^\beta + r))$ is $\lceil \log_2[(2^\alpha - 1)(2^\beta + r) + 1] \rceil$.
Here $2^{\alpha+\beta} < 2^{\alpha+\beta} + r2^\alpha - 2^\beta - r + 1$ if and only if $r > \dfrac{2^\beta - 1}{2^\alpha - 1}$. Thus $\alpha + \beta + 1$ is a lower bound of pdim$(S(2^\alpha, t))$.
Hence pdim$(S(2^\alpha, t)) = \alpha + \beta + 1$, where $2^\beta < t \leq 2^{\beta+1}$. □

The following result gives the exact product dimension of $S(n,t)$ when n and t are not powers of 2 but satisfy certain conditions.

Theorem 3.10. *Let* $1 \leq r_1 < 2^\alpha$, $\alpha \geq 1$, $1 \leq r_2 < 2^\beta$, $\beta \geq 1$. *If* $2^\alpha r_2 + (r_1 - 1)(2^\beta + r_2) + 1 > 2^{\alpha+\beta+1}$ *then* $\mathrm{pdim}(S(2^\alpha + r_1, 2^\beta + r_2)) = \alpha + \beta + 2$.

Proof. Corollary 3.7 gives upper bound of $\mathrm{pdim}(S(2^\alpha + r_1, 2^\beta + r_2))$, that is $\mathrm{pdim}(S(2^\alpha + r_1, 2^\beta + r_2)) \leq \lceil \log_2 (2^\alpha + r_1) \rceil + \beta + 1 = \alpha + \beta + 2$.
From Theorem 3.4, a lower bound of $\mathrm{pdim}(S(2^\alpha + r_1, 2^\beta + r_2))$ is

$$\lceil \log_2 ((2^\alpha + r_1 - 1)(2^\beta + r_2) + 1) \rceil = \alpha + \beta + 2,$$

when $2^\alpha r_2 + (r_1 - 1)(2^\beta + r_2) + 1 > 2^{\alpha+\beta+1}$ is satisfied.

□

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