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ON WEAKLY RIGHT PRIMARY IDEALS

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Abstract Weakly primary ideals in a commutative ring have been introduced and studied by Atani S. Ebrahimi and Farzalipour F. . Here we study weakly (principally) right primary ideals in a non-commutative ring. We define a proper ideal P of the ring R to be weakly (principally) right primary if whenever A and B are (principal) ideals of R such that $\{0\} \neq AB \subseteq P$, then either $A \subseteq P$ or $B^n \subseteq P$ for some positive integer n depending on A and B. If R is a commutative ring then I is a weakly primary ideal if and only if it is a weakly (principally) right primary ideal. Hence coresponding results about weakly primary ideals will follow as special cases from results proved in this note. We show that if P is an ideal of R such $P^2 \neq \{0\}$ then P is principally right primary if and only if P is weakly principally right primary. If P is a weakly principally right primary ideal which is not a principally right primary ideal of R, then R is 2-primal if and only if P is a 2-primal ideal. We also prove a version of Nakayama's Lemma.

1 Introduction

By a proper ideal I of R, we mean an ideal I of R with $I \neq R$. Let R be a commutative ring and I a proper ideal of R. By $\sqrt{I}$ we mean the radical of R, that is, $\{a \in R | a^n \in I$ for some positive integer $n\}$. In particular, $\sqrt{\{0\}}$ denotes the set of all nilpotent elements of R. Recall that for a commutative ring R an ideal I is called primary if for elements $a, b \in R$ such that $ab \in I$ with $a \notin I$, then $b \in \sqrt{I}$. In [2] Atani and Farzalipour introduced the concept of weakly primary ideals. A proper ideal I of R is called a weakly primary ideal of R if whenever $a, b \in R$ and $0 \neq ab \in I$, then $a \in I$ or $b \in \sqrt{I}$. So a primary ideal is weakly primary. However, since $\{0\}$ is always weakly primary (by definition), a weakly primary ideal does not need to be primary. The concept of primary ideals in commutative rings has been generalized to a non-commutative setting by several authors, e.g., Barnes [3], Chatters and Hajarnavis [6], and Fuchs [14]. This was done with a vision of extending the Noether theory of primary ideal decompositions, [14]. In [9] and [10] and [13] the authors examine several such generalizations and investigate their interrelations and their relations to structural properties. In this paper we introduce the weakly version of some of these generalizations. Among many results in this paper, it is shown that if I is a weakly principally right primary ideal of R that is not principally right primary, then $I^2 = \{0\}$ and hence $I \subseteq \mathcal{P}(R)$ where $\mathcal{P}(R)$ is the prime radical of the ring R (Theorem 2.20). If I is a proper ideal of R and $I^2 = \{0\}$, then I need not be a weakly principally right primary ideal of R (Example 2.21). We have that if R satisfies either the a.c.c. on ideals or is either left or right Artinian then every weakly principally right primary ideal of R is weakly right primary (Proposition 2.12).

2 Preliminaries

Throughout this paper the rings are associative but not necessarily assumed to have a unity unless indicated otherwise. Also an ideal means a two-sided ideal.

We adopt the following notation:

(i) $A \lhd R$, $A \lhd_r R$, $A \lhd_l R$ mean that A is a two-sided, right, left ideal of R, respectively;

(ii) for a nonempty subset X of R, we use $\langle X \rangle$, $\langle X \rangle_r$, and $\langle X \rangle_l$ for the two- sided, right, left ideal, respectively, of R generated by X;

(iii) $\mathbb{N}$ and $\mathbb{Z}$ are the set of natural numbers and the set of rational integers, respectively.

Definition 2.1. Let R be a ring and I an ideal of R.

(i) The ideal I is called a (principally) right primary ideal if whenever A and B are (principal) ideals of R with $AB \subseteq I$, then either $A \subseteq I$ or $B^n \subseteq I$ for some positive integer n depending on A and B.

(ii) R is said to be a (principally) right primary ring if the zero ideal is a (principal) right primary ideal of R.

(iii) The ideal I is called a **weakly (principally) right primary ideal** if whenever A and B are (principal) ideals of R with $\{0\} \neq AB \subseteq I$, then either $A \subseteq I$ or $B^n \subseteq I$ for some positive integer n depending on A and B.

(iv) Let $\sqrt{I} = \sum\{V \lhd R | V^n \subseteq I$ for some positive integer $n\}$ and as in [5] we call $\sqrt{I}$ the pseudo radical of I. Also $\sqrt{\{0\}} = \sum\{V \lhd R | V^n = \{0\}$ for some positive integer $n\}$.

(v) The **prime radical**, $\rho(I)$, of I is the intersection of all prime ideals of R containing I. Thus $\mathcal{P}(R)$ is $\rho(\{0\})$ and $\sqrt{I} \subseteq \rho(I)$.

Right primary ideals and principally right primary ideals were defined in [10] where they are called “generalized right primary” and "principally generalized right primary ideals". Similarly, one defines left primary and principal left primary rings and ideals. Some of the results will be stated for right-sided conditions, with the left-handed analogs being obvious to the reader.

Remark 2.2. From Birkenmeier [5, Proposition 1.3 (iv)] we have that for a commutative ring R an ideal P of R is a primary ideal if and only if it is a principal right primary ideal of R.

Lemma 2.3. *If R is a commutative ring then an ideal P of R is a weakly primary ideal if and only if it is a weakly principal right primary ideal of R.*

Proof. Let P be a weakly principal right primary ideal of R and let $a, b \in R$ such that $0 \neq ab \in P$. Since R is a commutative ring, we have $\{0\} \neq \langle a \rangle \langle b \rangle \subseteq P$. Hence $\langle a \rangle \subseteq P$ or $\langle b \rangle^n \subseteq P$ for some positive integer n i.e. $a \in P$ or $b^n \in P$ for some positive integer n. The other direction is clear. □

Remark 2.4. Thus for all the results proved in this note the corresponding results for commutative rings will follow as special cases.

Lemma 2.5. *[5, Lemma 1.2.] Let A, B and I be ideals of a ring R. Then we have the following.*

(i) $A \subseteq B$ implies $\sqrt{A} \subseteq \sqrt{B}$.

(ii) Assume that $A \subseteq \sqrt{I}$. If A is finitely generated or $\left(\sqrt{I}\right)^m \subseteq I$ for some positive integer m, then $A^n \subseteq I$ for some positive integer n. In particular, if $\sqrt{I}$ is finitely generated, then $\left(\sqrt{I}\right)^n \subseteq I$ for some positive integer n.

(iii) If $\left(\sqrt{I}\right)^m \subseteq I$ for some positive integer m, then $\sqrt{I} = \rho(I) = \sqrt{\sqrt{I}}$.

Definition 2.6. If P is a weakly principally right primary ideal then $a, b \in R$ is said to be a **twin zero** of P provided that $\langle a \rangle \langle b \rangle = \{0\}$, $\langle a \rangle \nsubseteq P$ and $(\langle b \rangle)^n \nsubseteq P$ for every $n \in \mathbb{N}$.

Note that if I is a weakly principally right primary ideal of R that is not a principally right primary ideal then I has a twin-zero (a, b) for some $a, b \in R$.

Lemma 2.7. *[9, Lemma 3.1] Let $I \lhd R$, if $b \in \sqrt{I}$ then there exists a positive integer m such that $(\langle b \rangle)^m \subseteq I$.*

Hence if A and I are ideals of R and $A \nsubseteq \sqrt{I}$ then $(A)^n \nsubseteq I$ for every $n \in \mathbb{N}$.

Remark 2.8. It follows from Lemma 2.7 that if P is a weakly principally right primary ideal then $a, b \in R$ is a **twin zero** of P if $\langle a \rangle \langle b \rangle = \{0\}$, $a \notin P$ and $b \notin \sqrt{P}$.

Proposition 2.9. *Let $I \lhd R$. The following are equivalent:*

(i) I is a weakly right primary ideal;

(ii) if $A, B \lhd_r R$ such that $\{0\} \neq AB \subseteq I$, then either $A \subseteq I$ or $B^n \subseteq I$, for some $n \in \mathbb{N}$;

(iii) if $A, B \lhd_l R$ such that $\{0\} \neq AB \subseteq I$, then either $A \subseteq I$ or $B^n \subseteq I$, for some $n \in \mathbb{N}$;

(iv) if $A_1, ..., A_n$ are ideals of R with $\{0\} \neq A_1 \cdots A_n \subseteq I$ and $A_j \nsubseteq I$ for $j = 1, ..., n$, then there exists $m \in \mathbb{N}$ such that $A_k^m \subseteq I$ for at least one $k > 1$.

Proof. 1. $\Longleftrightarrow$ 2. Assume 1. holds. Suppose $A, B \lhd_r R$ such that $\{0\} \neq AB \subseteq I$. Let $\langle A \rangle$, $\langle B \rangle$ be the ideals generated by A, B respectively. Then $\{0\} \neq \langle A \rangle \langle B \rangle \subseteq I$, whence $A \subseteq \langle A \rangle \subseteq I$ or $B^n \subseteq \langle B \rangle^n \subseteq I$, for some $n \in \mathbb{N}$. Thus 1. implies 2., and the converse is immediate.

1. $\Longleftrightarrow$ 3. Proceed similarly to establish 1. is equivalent to 3.

1. $\Longleftrightarrow$ 4. To see that 1. implies 4, suppose $A_1, ..., A_n$ are ideals of R with $\{0\} \neq A_1 \cdots A_n \subseteq I$. Since I is a weakly right primary ideal, then either $A_1 \cdots A_{n-1} \subseteq I$ or there exists $m \in \mathbb{N}$ such that $(A_n)^m \subseteq I$. If $A_1 \cdots A_{n-1} \subseteq I$ then either $A_1 \cdots A_{n-2} \subseteq I$ or there exists $k \in \mathbb{N}$ such that $A_{n-1}^k \subseteq I$. Repeating this process yields that $(A_j)^t \subseteq I$ for some $t \in \mathbb{N}$ and some j, $2 \leq j \leq n$. Thus 1. implies 4., and the converse is trivial. □

Proposition 2.10. *Let $I \lhd R$. The following are equivalent:*

(i) I is a weakly principally right primary ideal;

(ii) let A and B be ideals of R such that $\{0\} \neq AB \subseteq I$ then $A \subseteq I$ or $B \subseteq \sqrt{I}$;

(iii) let A and B be ideals of R. If B is finitely generated and $\{0\} \neq AB \subseteq I$ then $A \subseteq I$ or $B^n \subseteq I$ for some $n \in \mathbb{N}$;

(iv) if $A \lhd R$ and $b \in R$ such that $\{0\} \neq A \langle b \rangle \subseteq I$, then either $A \subseteq I$ or $(\langle b \rangle)^n \subseteq I$, for some $n \in \mathbb{N}$.

Proof. 1. $\Longrightarrow$ 2. Suppose $AB \subseteq I$ and $A \nsubseteq I$ and $B \nsubseteq \sqrt{I}$. Let $a \in A \setminus I$, $b \in B \setminus \sqrt{I}$, $a^{'} \in A \cap I$ and $b^{'} \in B \cap \sqrt{I}$ be arbitrary. Now $\left(a + a^{'}\right) \notin I$ and $\left(b + b^{'}\right) \notin \sqrt{I}$. Hence $\left\langle \left(a + a^{'}\right) \right\rangle \nsubseteq I$ and $\left(\left\langle \left(b + b^{'}\right) \right\rangle\right)^n \nsubseteq I$ for every $n \in \mathbb{N}$. Since I is a weakly principally right primary ideal and $\left\langle \left(a + a^{'}\right) \right\rangle \left\langle \left(b + b^{'}\right) \right\rangle \subseteq I$ we have $\left\langle \left(a + a^{'}\right) \right\rangle \left\langle \left(b + b^{'}\right) \right\rangle = \{0\}$ hence $\left(a + a^{'}\right) \left(b + b^{'}\right) = ab + ab^{'} + a^{'}b + a^{'}b^{'} = 0$. Considering all combinations where $a^{'}$ and or $b^{'}$ equal zero shows that $ab = ab^{'} = a^{'}b = a^{'}b^{'} = 0$. Hence $AB = \{0\}$ and we are done.

2. $\Longrightarrow$ 3. Suppose $\{0\} \neq AB \subseteq I$ with B finitely generated. From 2. it follows that $A \subseteq I$ or $B \subseteq \sqrt{I}$. It now follows from Lemma 2.5 that $A \subseteq I$ or $B^n \subseteq I$ for some $n \in \mathbb{N}$ and we are done.

3. $\Longrightarrow$ 4. Let $A \lhd R$ and $b \in R$ such that $\{0\} \neq A \langle b \rangle \subseteq I$. From 3. we have $A \subseteq I$ or $(\langle b \rangle)^n \subseteq I$.

4. $\Longrightarrow$ 1. This is clear with $A = \langle a \rangle$. □

Theorem 2.11. *Let R be a ring and I be a weakly principally right primary ideal of R. Suppose that $AB \subseteq I$ for some ideals A, B of R, and that for some $a \in A$ and $b \in B$, $\langle a \rangle \langle b \rangle = \{0\}$ but $a \notin I$ and $b \notin \sqrt{I}$ then $AB = \{0\}$.*

Proof. Suppose $AB \neq \{0\}$. Hence we have $\{0\} \neq AB \subseteq I$. From Proposition 2.10 we have $A \subseteq I$ or $B \subseteq \sqrt{I}$. Let $a \in A$ and $b \in B$, $\langle a \rangle \langle b \rangle = \{0\}$ but $a \notin I$ and $b \notin \sqrt{I}$. Since $a \notin I$, we have $A \nsubseteq I$. Hence from the above we have $B \subseteq \sqrt{I}$ contradicting the fact that $b \notin \sqrt{I}$. Hence we must have $AB = \{0\}$. □

Theorem 2.12. *If R satisfies either the a.c.c. on ideals or is either right or left Artinian, then every weakly principally right primary ideal of R is weakly right primary.*

Proof. Suppose $\{0\} \neq AB \subseteq I$ for ideals A, B of R and $A \nsubseteq I$. Then there exists $a \in A$ such that $\langle a\rangle \nsubseteq I$ and $\{0\} \neq \langle a\rangle B \subseteq I$. Let $b \in B$ be arbitrary. If $\{0\} \neq \langle a\rangle \langle b\rangle \subseteq I$, then $\langle b\rangle^n \subseteq I$ for some $n \in \mathbb{N}$ since I is weakly principally right primary. If $\{0\} = \langle a\rangle \langle b\rangle \subseteq I$, then also $\langle b\rangle^n \subseteq I$ for if $\langle b\rangle^n \nsubseteq I$ for every n, then $\langle b\rangle \nsubseteq \sqrt{I}$. Hence we have $\langle a\rangle \langle b\rangle = \{0\}$, $\langle a\rangle \nsubseteq I$ and $\langle b\rangle \nsubseteq \sqrt{I}$. It now follows from Theorem 2.11 that $AB = \{0\}$ a contradiction. Since $B = \sum\limits_{b\in B} \langle b\rangle$, we have $B \subseteq \sqrt{I}$. Now, because B is finitely generated it follows from [5, Lemma 1.2 (ii)] that $B^m \subseteq I$ for some $m \in \mathbb{N}$. Hence I is weakly right primary. □

In [12] Hirano et al. extended the notion of a weakly prime ideal to rings, not necessarily commutative nor with identity. They defined a proper ideal P of the ring to be weakly prime if for ideals A, B of the ring R, $\{0\} \neq AB \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$. From [11, Proposition 1.2] it follows that amongst others an ideal I is weakly prime if for $a, b \in R$ with $\{0\} \neq \langle a\rangle \langle b\rangle \subseteq I$, $a \in I$ or $b \in I$.

Clearly every (principally) right primary ideal of a ring R is weakly (principally) right primary. However, since $\{0\}$ is always weakly (principally) right primary (by definition), a weakly (principally) right primary ideal need not be (principally) right primary. Thus the weakly (principally) right primary concept is a generalization of the concept of (principally) right primary. It's easy to see that every weakly prime ideal is weakly right primary, however the converse is not in general true.

Example 2.13. [13, Example 3] Let $R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in F, \text{ where } F \text{ is a field} \right\}$. Then R has three nonzero proper ideals

$$I_1 = \left\{ \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} : a \in F \right\}$$
$$I_2 = \left\{ \begin{bmatrix} b & a \\ 0 & 0 \end{bmatrix} : a, b \in F \right\}$$
$$I_3 = \left\{ \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} : a, b \in F \right\}$$

Since I_2 and I_3 are idempotent and $I_3 I_2 = \{0\}$, $\{0\}$ is weakly right primary but not right primary.

Example 2.14. [10, Example 2.2] Let A and B be simple nil rings which are not nilpotent. (For examples of such rings see [15]). Then $R = A \oplus B$ is a nil ring, and identifying A and B as ideals in R we have $AB = \{0\} = BA$. Since A and B are simple rings, the ideals A and B are principal. $\{0\}$ is weakly principally right primary (by definition) but not principally right primary.

Example 2.15. (i) Let $R = \mathbb{Z} \times \mathbb{Z}$. The ideal $P = 2\mathbb{Z} \times \{0\}$ is a weakly prime ideal of R. However, $4\mathbb{Z} \times \{0\}$ is a weakly primary ideal of R, that is not a weakly prime ideal of R.

(ii) Let $R = \mathbb{Z}_{12}$ and let $P = \{0, 6\}$. Then P is neither primary nor a weakly primary ideal.

Proposition 2.16. *Let I be a semiprime ideal of R. I is a weakly prime ideal if and only if I is a weakly principally right primary ideal .*

Proof. The one direction is clear. For the converse, let $\{0\} \neq \langle a\rangle \langle b\rangle \subseteq I$. If $\langle a\rangle \subseteq I$, then we are done, so suppose $\langle a\rangle \nsubseteq I$. Since I is a weakly principally right primary ideal we have $\langle b\rangle^n \subseteq I$ for some $n \in \mathbb{N}$ and since I is a semiprime ideal we have $\langle b\rangle \subseteq I$. □

Lemma 2.17. *Let $P \lhd R$ be a weakly principally right primary ideal and (a, b) a twin zero for $a, b \in R$. Then $\langle a\rangle P = P\langle b\rangle = \{0\}$.*

Proof. Suppose $\langle a\rangle P \neq \{0\}$ i.e. there exists $p \in P$ such that $\langle a\rangle \langle p\rangle \neq \{0\}$. Now $\{0\} \neq \langle a\rangle (\langle p\rangle + \langle b\rangle) = \langle a\rangle \langle p\rangle \subseteq P$. Since P is a weakly principally right primary ideal it follows from Proposition 2.10 3. that $\langle a\rangle \subseteq P$ or $(\langle p\rangle + \langle b\rangle)^n \subseteq P$ for some $n \in \mathbb{N}$. Hence $\langle b\rangle^n \subseteq P$ for some $n \in \mathbb{N}$, a contradiction since (a, b) is a twin zero and therefore $\langle a\rangle P = \{0\}$. Now, suppose $P\langle b\rangle \neq \{0\}$ and $p \in P$ such that $\langle p\rangle \langle b\rangle \neq \{0\}$. From this we have $\{0\} \neq (\langle a\rangle + \langle p\rangle)\langle b\rangle = \langle p\rangle \langle b\rangle \subseteq P$. Hence $(\langle a\rangle + \langle p\rangle) \subseteq P$ or $(\langle b\rangle)^n \subseteq P$ for some $n \in \mathbb{N}$. Thus $\langle a\rangle \subseteq P$ or $(\langle b\rangle)^n \subseteq P$ for some $n \in \mathbb{N}$. A contradiction and we have $P\langle b\rangle = \{0\}$. □

Lemma 2.18. *Let $P \lhd R$ be a weakly principally right primary ideal and (a, b) a twin zero for $a, b \in R$. If $r \in R$ such that $r\langle b\rangle \subseteq P$, then $r\langle b\rangle = \{0\}$.*

Proof. Let $r \in R$ such that $r\langle b\rangle \subseteq P$. If $\{0\} \neq r\langle b\rangle \subseteq P$ then $\{0\} \neq \langle r\rangle \langle b\rangle \subseteq P$. Since P is a weakly principally right primary ideal and (a, b) a twin zero of P we have $\langle r\rangle \subseteq P$. Now $\langle r\rangle \langle b\rangle \subseteq P\langle b\rangle = \{0\}$. Thus $r\langle b\rangle = \{0\}$ for all $r \in R$ such that $r\langle b\rangle \subseteq P$. □

Proposition 2.19. *If R is a ring with identity then the following statements are equivalent:*

(i) $P \lhd R$ is a weakly principally right primary ideal;

(ii) if $a, b \in R$ such that $\{0\} \neq aRb \subseteq P$ then it follows that $a \in P$ or $(\langle b\rangle)^n \subseteq P$ for some $n \in \mathbb{N}$;

(iii) if $a, b \in R$ with $\{0\} \neq \langle a\rangle_r \langle b\rangle_r \subseteq P$, then either $\langle a\rangle_r \subseteq P$ or $(\langle b\rangle_r)^n \subseteq P$ for some $n \in \mathbb{N}$;

(iv) if $a, b \in R$ with $\{0\} \neq \langle a\rangle_l \langle b\rangle_l \subseteq P$, then either $\langle a\rangle_l \subseteq P$ or $(\langle b\rangle_l)^n \subseteq P$, for some $n \in \mathbb{N}$.

Proof. 1. $\Longrightarrow$ 2. Consider $a, b \in R$ such that $\{0\} \neq aRb \subseteq P$. Since R has an identity we have $\{0\} \neq aRb \subseteq \langle a\rangle \langle b\rangle \subseteq P$. Since P is a weakly principally right primary ideal, $a \in \langle a\rangle \subseteq P$ or $(\langle b\rangle)^n \subseteq P$ for some $n \in \mathbb{N}$.

2. $\Longrightarrow$ 1. Let $a, b \in R$ such that $\langle a\rangle \langle b\rangle \subseteq P$. If $\langle a\rangle \nsubseteq P$ and $(\langle b\rangle)^n \nsubseteq P$ for every $n \in \mathbb{N}$, then $\langle a\rangle \nsubseteq P$ and $\langle b\rangle \nsubseteq \sqrt{P}$. Let $x \in \langle a\rangle \setminus P$, $y \in \langle b\rangle \setminus \sqrt{P}$, $x' \in \langle a\rangle \cap P$ and $y' \in \langle b\rangle \cap \sqrt{P}$ be arbitrary. Now $\left(x + x'\right) \notin P$ and $\left(y + y'\right) \notin \sqrt{P}$. Hence $\left\langle \left(x + x'\right) \right\rangle \nsubseteq P$ and $\left(\left\langle \left(y + y'\right) \right\rangle\right)^n \nsubseteq P$ for every $n \in \mathbb{N}$. Hence from our assumption $\left(x + x'\right) R \left(y + y'\right) = \{0\}$. Considering all combinations where x' and or y' equal zero shows that $xy = xy' = x'y = x'y' = 0$. Hence $\langle a\rangle \langle b\rangle = \{0\}$ and we are done.

2. $\Longrightarrow$ 3. Let $a, b \in R$ such that $\{0\} \neq \langle a\rangle_r \langle b\rangle_r \subseteq P$. Now, since R has an identity element, $\{0\} \neq aRb \subseteq \langle a\rangle_r \langle b\rangle_r \subseteq P$. From 2. we have $a \in P$ or $(\langle b\rangle_r)^n \subseteq (\langle b\rangle)^n \subseteq P$ and we done.

Similarly obtain 2. implies 4.. Observe that 1. follows immediately from either 3. or 4., which completes the logical circuit.

□

The following theorem is Theorem 2.2 of Atani and Farzalipour [2] in a more general setting.

Theorem 2.20. *If P is a weakly principally right primary ideal which is not a principally right primary ideal, then $\mathcal{P}(R) = \rho(P)$ and $\sqrt{P} = \sqrt{\{0\}} = \sum\{I \lhd R : I^m = \{0\}$ for some $m\}$.*

Proof. We first prove that $P^2 = \{0\}$. Suppose P is a weakly principally right primary ideal which is not a principally right primary ideal. Hence P has a twin zero (a, b) for some $a, b \in R$. Suppose $P^2 \neq \{0\}$ with $p, q \in P$ such that $pq \neq 0$. Now, from Lemma 2.17, $\{0\} \neq (\langle a\rangle + \langle p\rangle)(\langle b\rangle + \langle q\rangle) = \langle p\rangle \langle q\rangle \subseteq P$. From Proposition 2.10 3. we have $(\langle a\rangle + \langle p\rangle) \subseteq P$ or $(\langle b\rangle + \langle q\rangle)^n \subseteq P$ for some $n \in \mathbb{N}$. Hence $\langle a\rangle \subseteq P$ or $(\langle b\rangle)^n \subseteq P$ for some $n \in \mathbb{N}$ contradicting the fact that (a, b) is a twin zero of P. Hence $P^2 = \{0\}$.

Clearly $\mathcal{P}(R) \subseteq \rho(P)$. Since $P^2 = \{0\} \subseteq \mathcal{P}(R)$ and since $\mathcal{P}(R)$ is a semiprime ideal we have $P \subseteq \mathcal{P}(R)$. Hence $\rho(P) \subseteq \mathcal{P}(R)$ and we have $\mathcal{P}(R) = \rho(P)$. Now, since $P^2 = \{0\}$ we have $\sqrt{P} = \sum\{I \lhd R : I^n \subseteq P\} = \sum\{I \lhd R : I^{2n} \subseteq P^2 = \{0\}\} = \sum\{I \lhd R : I^m = \{0\}$ for some $m\} = \sqrt{\{0\}}$. □

Example 2.21. It should be noted that a proper ideal P with property that $P^2 = \{0\}$ need not be weakly principally right primary. Take $R = \begin{bmatrix} \mathbb{Q} & \mathbb{R} \\ \{0\} & \mathbb{Q} \end{bmatrix}$ and $P = \begin{bmatrix} \{0\} & \mathbb{R} \\ \{0\} & \{0\} \end{bmatrix}$. Clearly $P^2 = \{0\}$ yet P is not weakly principally right primary since

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbb{Q} & \mathbb{R} \\ \{0\} & \mathbb{Q} \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix} \subseteq P.$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \notin \begin{bmatrix} \{0\} & \mathbb{R} \\ \{0\} & \{0\} \end{bmatrix} \text{ and } \left(\left\langle \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix} \right\rangle\right)^n \nsubseteq P \text{ for every } n \in \mathbb{N}.$$

Corollary 2.22. *Let R be a ring and let P be an ideal of R. If $P^2 \neq \{0\}$ then P is principally right primary if and only if P is weakly principally right primary.*

Proof. This follows from Theorem 2.20. □

Corollary 2.23. *Let R be a semiprime ring and $P \neq \{0\}$ be a proper ideal of R. Then P is principally right primary if and only if P is weakly principally right primary.*

Proof. Suppose P is weakly principally right primary with $P \neq \{0\}$. If P is not principally right primary then $P^2 = \{0\} \subseteq \mathcal{P}(R) = \{0\}$. Since $\mathcal{P}(R)$ is a semiprime ideal we have $P \subseteq \mathcal{P}(R) = \{0\}$ a contradiction. Hence P is principally right primary. The converse is clear. □

In [4] Birkenmeier et al introduced the notion of a 2-primal ideal and a 2-primal ring.

Definition 2.24. [4, Definition 2.1] Let R be a ring and I an ideal of R. The ring R is 2-primal if the prime radical $\mathcal{P}(R)$ of R is equal to the set of nilpotent elements of R. The ideal I is 2-primal if the factor ring R/I is a 2-primal ring.

Proposition 2.25. *If P is a weakly principally right primary ideal which is not a principally right primary ideal of R, then R is 2-primal if and only if P is a 2-primal ideal.*

Proof. This follows from Theorem 2.20 and [4, Proposition 2.4]. □

Next, we state and prove a version of Nakayama's Lemma.

Theorem 2.26. *Let P be a weakly principally right primary ideal which is not a principally right primary ideal of R, then the following hold:*

(i) $P \subseteq \mathcal{P}(R) \subseteq J(R)$ where $J(R)$ is the Jacobson radical of the ring R;

(ii) If M is a right R-module and $MP = M$; then $M = \{0\}$;

(iii) If M is a right R-module and N is a submodule of M such that $MP + N = M$, then $M = N$.

Proof. (i) From Theorem 2.20 $P^2 = \{0\} \subseteq \mathcal{P}(R)$ and and since $\mathcal{P}(R)$ is semiprime we have $P \subseteq \mathcal{P}(R) \subseteq J(R)$.

(ii) Since $MP = M$ and $P^2 = \{0\}$ (by Theorem 2.20), we have $\{0\} = MP^2 = MP = M$.

(iii) Given $MP + N = M$ implies that $MP^2 + NP = MP$ i.e. $NP = MP$ which implies that $NP + N = MP + N$. Hence $N = MP + N = M$.

□

In general, the intersection of a family of weakly right primary ideals is not weakly right primary. Indeed, consider the ring $R = \mathbb{Z}_{12}$. Then $I = \langle 2 \rangle$ and $J = \langle 3 \rangle$ are clearly weakly right primary ideals of R, but $I \cap J = \{0, 6\}$ is not a weakly right primary ideal of R (since $0 \neq 3 \cdot 2 \in I \cap J$, but neither $3 \in I \cap J$ nor $2 \in \sqrt{I \cap J}$).

However, we have the following results:

Proposition 2.27. *Let $\{I_i : i \in \Lambda\}$ be a finite collection of weakly principally right primary ideals of a ring R such that $\sqrt{I_i} = \sqrt{I_j}$ for every distinct $i, j \in \Lambda$. Then $I = \bigcap_{i\in\Lambda} I_i$ is a weakly principally right primary ideal of R.*

Proof. Let $A, B \lhd R$ such that $\{0\} \neq AB \subseteq I$. Suppose $A \nsubseteq \bigcap_{i\in\Lambda} I_i$ say $A \nsubseteq I_j$. Now, since $\{0\} \neq AB \subseteq I_j$ and I_j is a weakly principally right primary ideal and $A \nsubseteq I_j$, we have $B \subseteq \sqrt{I_j}$. Since $\sqrt{I} = \sqrt{\bigcap_{i\in\Lambda} I_i} = \bigcap_{i\in\Lambda} \sqrt{I_i} = \sqrt{I_j}$ from [5, Lemma 1.2 (iii)], we have $B \subseteq \sqrt{I}$ and we are done. □

Theorem 2.28. *Let R be a ring with identity and let $\{P_i : i \in I\}$ be a family of weakly principally right primary ideals of R that are not principally right primary. Then $P = \bigcap_{i\in I} P_i$ is a weakly principally right primary ideal of R.*

Proof. We first show that $\sqrt{P} = \bigcap_{i\in I} \sqrt{P_i}$. Clearly, $\sqrt{P} \subseteq \bigcap_{i\in I} \sqrt{P_i}$. For the other inclusion suppose that $r \in \bigcap_{i\in I} \sqrt{P_i}$. Hence $r \in \bigcap_{i\in I} \sqrt{P_i} = \sqrt{\{0\}}$ from Theorem 2.20. Hence $\langle r\rangle^n = \{0\}$ for some n. From this it follows that $\langle r\rangle^n \subseteq \bigcap_{i\in I} P_i = P$. Hence $r \in \sqrt{P}$ and we have $\sqrt{P} = \bigcap_{i\in I} \sqrt{P_i}$. Now, let $a, b \in R$ such that $\{0\} \neq \langle a\rangle \langle b\rangle \subseteq P$ and suppose $\langle a\rangle \nsubseteq P$. Hence $\langle a\rangle \nsubseteq P_j$ for some $j \in I$. Since $\{0\} \neq \langle a\rangle \langle b\rangle \subseteq P_j$ and P_j weakly principally right primary, we have $\langle b\rangle \subseteq \sqrt{P_j}$. Hence $\langle b\rangle \subseteq \sqrt{P_j} = \sqrt{\{0\}} = \sqrt{P}$ since $\sqrt{P} = \bigcap_{i\in I} \sqrt{P_i} = \bigcap_{i\in I} \sqrt{\{0\}} = \sqrt{\{0\}}$. Hence $\langle b\rangle \subseteq \sqrt{P}$ and we are done. □

Proposition 2.29. *Let P be a proper ideal of R and let Ω be the set of all weakly principally right primary ideals of R contained in P. If for each chain $\mathcal{C}$ in $(\Omega, \subseteq)$ we have $\sqrt{\bigcap_{C\in\mathcal{C}} C} = \bigcap_{C\in\mathcal{C}} \sqrt{C}$, then any weakly principally right primary ideal of R contains a minimal weakly principally right primary ideal.*

Proof. Apply Zorn's Lemma to the family of weakly principally right primary ideals of R contained in P. It suffices to check that, for any chain of weakly principally right primary ideals $\{P_i : i \in I\}$ in P, the intersection $P' = \cap P_i$ is weakly principally right primary. Let A and B be ideals of R such that $\{0\} \neq AB \subseteq P'$. Suppose that $A \nsubseteq P'$ and $B \nsubseteq \sqrt{P'} = \cap\sqrt{P_i}$. Then there exist $a \in A \backslash P'$ and $b \in B \backslash \sqrt{P'}$ and we have $a \notin P_i$ and $b \notin \sqrt{P_j}$ for some $i, j \in I$. If, say $P_i \subseteq P_j$, then a is outside P_i and b outside of $\sqrt{P_i}$. Since P_i is weakly principally right primary, we have $\langle a\rangle \langle b\rangle = \{0\}$ or $\langle a\rangle \langle b\rangle \nsubseteq P_i$. Because $\langle a\rangle \langle b\rangle \subseteq AB \subseteq P' \subseteq P_i$ we must have $\langle a\rangle \langle b\rangle = \{0\}$. Hence (a, b) is a twin zero for P_i. It now follows from Theorem 2.11 that $AB = \{0\}$. This contradicts our assumption hence $A \subseteq P'$ or $B \subseteq \sqrt{P'}$ and therefore P' is a weakly principally right primary ideal. □

Proposition 2.30. *Let $I \subseteq P$ be proper ideals of R. Then the following hold:*

(i) If P is a weakly (principally) right primary ideal, then P/I is a weakly (principally) right primary ideal.

(ii) If I and P/I are weakly (principally) right primary then P is weakly (principally) right primary.

Proof. We will prove it for the weakly right primary case:

(i) Let A, B be ideals of R such that $\{0\} \neq [(A + I)/I][(B + I)/I] \subseteq P/I$. Hence $AB \subseteq P$. If $AB = \{0\}$, then $[(A + I)/I][(B + I)/I] = (AB + I)/I = \{0\}$ a contradiction, hence $\{0\} \neq AB \subseteq P$. Since P is a weakly right primary ideal we have $A \subseteq I$ or $B^n \subseteq I$ for some $n \in \mathbb{N}$. Hence $[(A + I)/I] \subseteq P/I$ or $[(B + I)/I]^n = [(B^n + I)/I] \subseteq P/I$ for some $n \in \mathbb{N}$.

(ii) Let A, B be ideals of R such that $\{0\} \neq AB \subseteq P$. Now $[(A+I)/I][(B+I)/I] = [(AB+I)/I] \subseteq P/I$. If $AB \subseteq I$ then since I is a weakly right primary ideal we have $A \subseteq I \subseteq P$ or $B^n \subseteq I \subseteq P$ for some $n \in \mathbb{N}$. If $AB \nsubseteq I$ then $\{0\} \neq [(A+I)/I][(B+I)/I] \subseteq P/I$ and because P/I is weakly right primary, we get $[(A+I)/I] \subseteq P/I$ or $[(B+I)/I]^n \subseteq P/I$ for some $n \in \mathbb{N}$. Hence $A \subseteq P$ or $B^n \subseteq P$ for some $n \in \mathbb{N}$ and we are done.

□

Theorem 2.31. *Let R and S be rings and $f : R \to S$ be a surjective ring-homomorphism. Then the following statements hold:*

(i) If I is a weakly (principally) right primary ideal of R and ker$(f) \subseteq I$, then $f(I)$ is a weakly (principally) right primary ideal of S.

(ii) If J is a weakly (principally) right primary ideal of S and ker(f) is a weakly (principally) right primary ideal of R, then $f^{-1}(J)$ is a weakly (principally) right primary ideal of R.

Proof. We will prove it for the right primary case:

(i) Since I is a weakly right primary ideal of R and $\ker(f) \subseteq I$, we conclude that $I/\ker(f)$ is a weakly right primary ideal of $R/\ker(f)$ by Proposition 2.30 1. Since $R/\ker(f)$ is ring-isomorphic to S, the result follows.

(ii) Let $L = f^{-1}(J)$. Then $\ker(f) \subseteq L$. Since $R/\ker(f)$ is ring-isomorphic to S, we conclude that $L/\ker(f)$ is a weakly right primary ideal of $R/\ker(f)$. Since $\ker(f)$ is a weakly right primary ideal of R and $L/\ker(f)$ is a weakly right primary ideal of $R/\ker(f)$, we conclude that $L = f^{-1}(J)$ is a weakly right primary ideal of R by Proposition 2.30 2.

□

Theorem 2.32. *Let R be a ring with identity. For a proper ideal P of R the following statements are equivalent:*

(i) P is a weakly principally right primary ideal of R;

(ii) For $x \in R - \sqrt{P}$, $(P : \langle x\rangle_l) = \{r \in R : r\langle x\rangle_l \subseteq P\} = P \cup (\{0\} : \langle x\rangle_l)$;

(iii) For $x \in R - \sqrt{P}$, $(P : \langle x\rangle_l) = P$ or $(P : \langle x\rangle_l) = (\{0\} : \langle x\rangle_l)$.

Proof. 1. $\Longrightarrow$ 2. Let $y \in (P : \langle x\rangle_l)$ where $x \in R - \sqrt{P}$. Now $y\langle x\rangle_l \subseteq P$ and hence $\langle y\rangle_l \langle x\rangle_l \subseteq P$ If $\langle y\rangle_l \langle x\rangle_l \neq \{0\}$, then since P is weakly principally right primary it follows from Proposition 2.19 that $y \in P$. If $\langle y\rangle_l \langle x\rangle_l = \{0\}$, then $y \in (\{0\} : \langle x\rangle_l)$. So $(P : \langle x\rangle_l) \subseteq P \cup (\{0\} : \langle x\rangle_l)$. As the reverse containment holds for any ideal P, we have equality.

2. $\Rightarrow$ 3. Suppose $(P : \langle x\rangle_l) = P \cup (\{0\} : \langle x\rangle_l)$ where $x \in R - \sqrt{P}$. Since P and $(\{0\} : \langle x\rangle_l)$ are both ideals, we have $(P : \langle x\rangle_l) = P$ or $(P : \langle x\rangle_l) = (\{0\} : \langle x\rangle_l)$.

3. $\Rightarrow$ 1. Let $x, y \in R$ such that $\{0\} \neq \langle y\rangle_l \langle x\rangle_l \subseteq P$. Suppose $x \in R - \sqrt{P}$, then $(P : \langle x\rangle_l) \neq (\{0\} : \langle x\rangle_l)$ and from 3. we have $(P : \langle x\rangle_l) = P$. Hence $y \in P$ and we are done.

□

3 Idealization

A mapping γ which assigns to each ring R an ideal $\gamma(R)$ is called an ideal- mapping; $\gamma(R)$ is called the radical of the ring R. If $f(\gamma(R)) \subseteq \gamma(S)$ for any surjective homomorphism $f : R \longrightarrow S$, then the ideal-mapping γ is called a preradical.

From [7] we recall: An ideal-mapping γ is a *summable radical* (*s-radical* for short) if there is a homomorphically closed class $\mathcal{M}$ such that $\gamma(R) = \sum\{I \lhd R | \, I \in \mathcal{M}\}$ for all rings R. Any s-radical is an idempotent preradical.

For a preradical γ and an ideal I of a ring R, $\gamma(R/I)$ is an ideal of R/I; hence it is of the from $\gamma(R/I) = \gamma^\star(I)/I$ for some uniquely determined ideal $\gamma^\star(I)$ of R with $\gamma(I) \subseteq I \subseteq \gamma^\star(I)$. This ideal $\gamma^\star(I)$ is called the radical of the ideal I and is not to be confused with the radical $\gamma(I)$ of the ring I.

For a ring R and an $R-R$-bimodule M, we can form the idealization $R \boxplus M$ of M which is a ring. If $I \boxplus N$ is a homogeneous ideal of $R \boxplus M$, then it can be checked that M/N is an $R/I-R/I$-bimodule and one can construct $R/I \boxplus M/N$. It can then be shown that $(R \boxplus M)/(I \boxplus N) \cong R/I \boxplus M/N$.

For an s-radical γ determined by the homomorphically closed class $\mathcal{M}$ and an ideal I of a ring R, we have $\gamma^{\star}(I)/I = \gamma(R/I) = \sum\{J/I \lhd R/I \mid J/I \in \mathcal{M}\}$ which gives $\gamma^{\star}(I) = \sum\{J \lhd R \mid I \subseteq J$ with $J/I \in \mathcal{M}\}$. Recall from [16], if K is an ideal in an idealization $R \boxplus M$, then $K \subseteq I_K \boxplus N_K \lhd R \boxplus M$ where $I_K = \{a \in R | (a,m) \in K$ for some $m \in M\}$ and $N_K = \{m \in M | (a,m) \in K$ for some $a \in R\}$.

From [16] we have the following:

Two conditions that a homomorphically closed class $\mathcal{M}$ may satisfy are:

$(I1)$ For any idealization $R \boxplus M$ with $J \lhd R$ and $J \in \mathcal{M}$, also $J \boxplus M \in \mathcal{M}$; and

$(I2)$ If K is an ideal in an idealization $R \boxplus M$, then $K \in \mathcal{M}$ implies $I_K \in \mathcal{M}$.

Then we have:

Proposition 3.1. *[16] Let γ be an s-radical determined by a homomorpically closed class $\mathcal{M}$ which fulfills conditions $(I1)$ and $(I2)$. Then:*

(i) For any $R-R$-bimodule M, $\gamma(R \boxplus M) = \gamma(R) \boxplus M$; and

(ii) For any $R-R$-bimodule M and ideal $I \boxplus N$ of $R \boxplus M$, $\gamma^{\star}(I \boxplus N) = \gamma^{\star}(I) \boxplus M$.

Example 3.2. [16] Let $\mathcal{M}$ be the class of all nilpotent rings with γ the associated s-radical; i.e. for any ring R, $\gamma(R) = \sum\{J \lhd R |\ J$ nilpotent$\}$ and for $I \lhd R$, $\gamma^{\star}(I) = \sum\{J \lhd R |\ J^n \subseteq I$ for some $n \geq 1\} = \sqrt{I}$. For any $R-R$-bimodule M and an ideal $I \boxplus N$ of $R \boxplus M$, $\sqrt{I \boxplus N} = \sqrt{I} \boxplus M$.

Proposition 3.3. *The following are equivalent:*

(i) I is principally right primary;

(ii) let $A, B \lhd R$. If $AB \subseteq I$ then $A \subseteq I$ or $B \subseteq \sqrt{I}$.

(iii) if $a, b \in R$ such that $aRb \subseteq I$, then $a \in I$ or $b \in \sqrt{I}$.

Proof. We only have to prove 1. $\Longleftrightarrow$ 2. since 1. $\Longleftrightarrow$ 3. follows from [10, Proposition 3.3].

1. $\Longleftrightarrow$ 2. Assume that I is principally right primary. Say $AB \subseteq I$, but $A \nsubseteq I$. Then there exists $a \in A$ such that $\langle a \rangle \nsubseteq I$. Let $b \in B$. Since $\langle a \rangle \langle b \rangle \subseteq I$, $\langle b \rangle^n \subseteq I$ for some positive integer n. As $B = \sum_{b \in B} \langle b \rangle$, we have that $B \subseteq \sqrt{I}$.

Conversely, let $A = \langle a \rangle$ and $B = \langle b \rangle$ be principal ideals of R such that $AB \subseteq I$. Then $\langle a \rangle \subseteq I$ or $\langle b \rangle \subseteq \sqrt{I}$ by assumption. If $\langle a \rangle \subseteq I$, then we done. So, assume that $\langle a \rangle \nsubseteq I$. Hence $\langle b \rangle \subseteq \sqrt{I}$ and by Lemma 2.7 $\langle b \rangle^n \subseteq I$ for some positive integer n. So I is principally right primary. □

Theorem 3.4. *Let R be a ring with identity and M an $R-R$-bimodule, with I a proper ideal of R. Then $I \boxplus M$ is a principally right primary ideal of $R \boxplus M$ if and only if I is a principally right primary ideal of R.*

Proof. Let $(a,m),(b,n) \in R \boxplus M$ such that $(a,m)R \boxplus M(b,n) \subseteq I \boxplus M$. Hence $aRb \subseteq I$ and since I is a principally right primary ideal of R, it follows from Proposition 3.3 that $a \in I$ or $b \in \sqrt{I}$, hence $(a,m) \in I \boxplus M$ or $(b,n) \in \sqrt{I} \boxplus M = \sqrt{I \boxplus M}$ and we are done. For the converse, let $a, b \in R$ such $aRb \subseteq I$. For any $m \in M$, we have $(a,m)R \boxplus M(b,m) \subseteq aRb \boxplus M \subseteq I \boxplus M$. Since $I \boxplus M$ is a principally right primary ideal of $R \boxplus M$ it follows from Proposition 3.3 that $(a,m) \in I \boxplus M$ or $(b,m) \in \sqrt{I \boxplus M} = \sqrt{I} \boxplus M$. Hence $a \in I$ or $b \in \sqrt{I}$ and we are done. □

Theorem 3.5. *Let R be a ring with identity and M an $R-R$-bimodule, with I a proper ideal of R. Then $I \boxplus M$ is a weakly principally right primary ideal of $R \boxplus M$ if and only if I is a weakly principally right primary ideal of R and for $a, b \in R$ with $aRb = \{0\}$ but $a \notin I$ and $b \notin \sqrt{I}$, we have $aM = Mb = \{0\}$.*

Proof. Suppose $I \boxplus M$ is a weakly principally right primary ideal of $R \boxplus M$. Let $\{0\} \neq aRb \subseteq I$ where $a, b \in R$. Now $\{(0,0)\} \neq (a,0)R \boxplus M(b,0) \subseteq I \boxplus M$ and since $I \boxplus M$ is a weakly principally right primary ideal it follows from Proposition 2.19 that $(a,0) \in I \boxplus M$ or $(b,0) \in \sqrt{I \boxplus M} = \sqrt{I} \boxplus M$. Hence $a \in I$ or $b \in \sqrt{I}$. So I is weakly principally right primary. Now suppose $aRb = \{0\}$ but $a \notin I$ and $b \notin \sqrt{I}$. We claim that $aM = Mb = \{0\}$. Assume say $aM \neq \{0\}$, so there exists $m \in M$ such that $am \neq 0$. Now we have $(0,0) \neq (a,0)(1,0)(b,m) \in (a,0)R \boxplus M(b,m) \subseteq aRb \boxplus M = \{0\} \boxplus M \subseteq I \boxplus M$. But $(a,0) \notin I \boxplus M$ and $(b,m) \notin \sqrt{I} \boxplus M = \sqrt{I \boxplus M}$ contradicting the fact that $I \boxplus M$ is a weakly principally right primary ideal.

Conversely, assume $(0,0) \neq (a,m)R \boxplus M(b,n) \subseteq I \boxplus M$ for $a,b \in R$ and $n,m \in M$. We have $aRb \subseteq I$. Two cases are possible:

Case 1: $\{0\} \neq aRb \subseteq I$. Now I a weakly principally right primary ideal of R gives $a \in I$ or $b \in \sqrt{I}$. Hence $(a,m) \in I \boxplus M$ or $(b,n) \in \sqrt{I} \boxplus M = \sqrt{I \boxplus M}$ as desired.

Case 2: $\{0\} = aRb \subseteq I$. We may assume $a \notin I$ and $b \notin \sqrt{I}$. Hence from assumption $aM = Mb = \{0\}$. Now $(a,m)R \boxplus M(b,n) \subseteq (aRb, aM + aMb + Mb) = \{(0,0)\}$ a contradiction. □

Example 3.6. [1, Example 20] Let $R = K[x,y]/(x^2,y^2)$, where K is a field and let $M = K[x,y]/(x,y) = K$ as an R module. $I = \{0\} \boxplus M$ is a weakly primary ideal of $R \boxplus M$ which is not a primary ideal.

4 Product of rings

Let $R = R_1 \times R_2$ where each R_i is a ring with identity. Then the following hold:

(i) If I_1 is an ideal of R_1, then $\sqrt{(I_1 \times R_2)} = \sqrt{I_1} \times R_2$.

(ii) If I_2 is an ideal of R_2, then $\sqrt{(R_1 \times I_2)} = R_1 \times \sqrt{I_2}$.

Theorem 4.1. *Let $R = R_1 \times R_2$ where each R_i is a ring with identity. Then the following hold:*

(i) If P_1 is a principally right primary ideal of R_1, then $P_1 \times R_2$ is a principally right primary ideal of R.

(ii) If P_2 is a principally right primary ideal of R_2, then $R_1 \times P_2$ is a principally right primary ideal of R.

(iii) If P is a weakly principally right primary ideal of R, then either $P = \{0\}$ or P is principally right primary.

Proof. (i) Let $(a,b)R(c,d) = aR_1c \times bR_2d \subseteq P_1 \times R_2$ where $(a,b),(c,d) \in R$. Since P_1 is a principally right primary ideal of R_1 it follows from Proposition 3.3 that either $a \in P_1$ or $c \in \sqrt{P_1}$. It follows that either $(a,b) \in P_1 \times R_2$ or $(c,d) \in \sqrt{P_1} \times R_2 = \sqrt{P_1 \times R_2}$. Thus $P_1 \times R_2$ is principally right primary.

(ii) Likewise, $R_1 \times P_2$ is a principally right primary ideal of R. This proof is similar to that in case 1. and we omit it.

(iii) Let $P = P_1 \times P_2$ be a weakly principally right primary ideal of R. We can assume that $P \neq \{0\}$. So there is an element (a,b) of P with $(a,b) \neq (0,0)$. Then $\{(0,0)\} \neq (a,1)R(1,b) \subseteq P$ gives either $(a,1) \in P$ or $(1,b) \in \sqrt{P}$. If $(a,1) \in P$, then $P = P_1 \times R_2$. We show that P_1 is principally right primary; hence P is principally right primary by 1. Let $cR_1d \subseteq P_1$, where $c,d \in R_1$. Then $\{(0,0)\} \neq (c,1)R(d,1) = (cR_1d) \times R_2 \subseteq P$, so either $(c,1) \in P$ or $(d,1) \in \sqrt{P} = \sqrt{P_1} \times R_2$ and hence either $c \in P_1$ or $d \in \sqrt{P_1}$. If $(1,b) \in \sqrt{P}$, then $(1,b^n) \in (\langle(1,b)\rangle)^n \subseteq P$ for some n, so $P = R_1 \times P_2$. By a similar argument, $R_1 \times P_2$ is principally right primary.

□

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