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A NOTE ON FEDOROV'S THEOREM

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Abstract The main purpose of this paper is to present a new characterization of cyclic groups using only undergraduate-level group theory.

1 Introduction

Recall that a group G is said to be *cyclic*, if there exists some $g \in G$ such that $G = \langle g \rangle = \{g^m | m \in \mathbb{Z}\}$. It is well-known that G is a (nontrivial) cyclic group if and only if G is either isomorphic to $\mathbb{Z}_n$ (the additive group of integers modulo n), for some integer $n \geq 2$, or to the additive group $\mathbb{Z}$ (cf. [4, 6]). Cyclic groups are very important in group theory. This is why we find several characterizations of such groups in the literature (see for instance, [1, 2, 3, 5, 7]). Recall also that if G is a group and H is a subgroup of G, then the *index* $[G : H]$ of H in G is simply the cardinality of the set G/H of left cosets of H in G; more compactly, $[G : H] = |G/H| = |\{gH|g \in G\}|$. In 1951, Fedorov [2] has proved the following theorem:

Fedorov's Theorem. An infinite group G is isomorphic to the group of integers $\mathbb{Z}$ if and only if every nontrivial subgroup of G has finite index.

An elementary proof of the above theorem was provided in [5] by Lanski. Our purpose here is to improve Fedorov's Theorem, by presenting new characterizations of cyclic groups via subgroup indices. We emphasize that our proofs are elementary and simple.

We let $|X|$ denote the cardinality of a set X. For any element x in a group G, we let $o(x)$ (resp., $\langle x \rangle$) denote the order of x (resp., the subgroup of G generated by x). Any undefined terminology is standard as in [4].

2 Main results

We start our investigations with the following result.

Lemma 2.1. *Let G be a group and let H be a subgroup of G. Then the following conditions hold true.*

(i) $|G| = [G : H] \times |H|$.

(ii) If G is uncountable and H is countable, then $|G| = [G : H]$.

Proof. (i) Lagrange's Theorem is most often stated for finite groups, but it has a natural formation for infinite groups too. Pick a representative of each left coset of H in G, and then consider the following mapping $\theta : (G/H) \times H \to G$ defined by $\theta(gH, h) = gh$, for any $(gH, h) \in (G/H) \times H$. One can easily check that θ is bijective. Thus $|(G/H) \times H| = |G|$; or equivalently, $[G : H] \times |H| = |G|$.
(ii) As G is uncountable and H is countable, it follows from basic cardinal arithmetics (cf. [8, App. 2, Corollary 3.8]) that $[G : H] = |G|$, as desired. This completes the proof. □

Theorem 2.2. *Let G be an infinite group. Then the following statements are equivalent:*

(i) G is isomorphic to the group of integers $\mathbb{Z}$.

(ii) Every nontrivial subgroup of G has finite index.

(iii) Every nontrivial subgroup of G has index less that $|G|$.

(iv) Distinct subgroups of G have distinct indices.

Proof. (i)⇔(ii) See [2] or [6].
(ii)⇒(iii) Trivial.
(iii)⇒(ii) It follows from Lemma 2.1 that G is countably infinite. If H is a nontrivial subgroup of G, then, by assumption, $[G : H] < |G| = \aleph_0$. Therefore, $[G : H]$ is finite.
(i)⇒(iv) Trivial.
(iv)⇒(ii) G is countable by virtue of Lemma 2.1. Now, let H be a nontrivial subgroup of G. As $[G : H] \leq |G| = [G : \{e\}]$ and $[G : H] \neq [G : \{e\}]$, then it follows that $[G : H] < |G| = \aleph_0$. Hence, $[G : H]$ is finite. This completes the proof. □

The next theorem gives an analog of Theorem 2.2, but for finite groups.

Theorem 2.3. *Let G be a nontrivial finite group. Then the following statements are equivalent:*

(i) G is isomorphic to the group of integers modulo n *for some integer* $n \geq 2$.

(ii) Distinct subgroups of G have distinct indices.

Proof. (i)⇒(ii) Trivial.
(ii)⇒(i) Let $n = |G|$. Two cases may occur:
Case 1. $n = p^k$ for some prime number p and some positive integer k.
In this case, let $x \in G$ with maximal order, say p^s, where $s \leq k$. We claim that $G = \langle x \rangle$. To this end, let $y \in G$ and write $o(y) = p^l$. As $o(y) \leq o(x)$, then $l \leq s$. Now, let $a \in \langle x \rangle$ with order p^l (such element exists since $\langle x \rangle$ is cyclic and p^l divides p^s). Notice that $\langle y \rangle$ and $\langle a \rangle$ are two subgroups of G with the same order. Thus, by assumption, $\langle y \rangle = \langle a \rangle$. Hence, $y \in \langle x \rangle$. This shows that $G = \langle x \rangle$ is cyclic, as claimed.
Case 2. $n = p_1^{k_1} p_2^{k_2} \cdots p_s^{k_s}$, where $s \geq 2$ and $p_1, \cdots, p_s$ are distinct prime numbers.
As the Sylow p_i-subgroups of G have the same indices, it follows that for any i, there exists a unique Sylow p_i-subgroup of G, say H_i. Moreover, G is isomorphic to the direct product $H_1 \times \cdots \times H_s$. It is not difficult to check that any subgroup of G inherits the property described in assertion (ii). Thus, it follows from case 1 that $H_i \cong \mathbb{Z}_{p_i^{k_i}}$ for any i. Therefore, $G \cong \mathbb{Z}_{p_1^{k_1}} \times \cdots \times \mathbb{Z}_{p_s^{k_s}} \cong \mathbb{Z}_n$. This completes the proof. □

References

[1] M. Deaconescu, R. Khazal, *A characterization of the finite cyclic groups*, An. Univ. Vest Timis. Ser. Mat.-Inform. 32 (1994), 37-40.

[2] Y. Fedorov, *On infinite groups of which all nontrivial subgroups have a finite index*, Uspekhi Mat. Nauk. 6 (1951), 187-189.

[3] K. Kovács, *On a characterization of cyclic groups by sums and differences*, Studia Sci. Math. Hungar. 36 (2000), 307-311.

[4] S. Lang, *Algebra*, Third edition. Springer, New York, 2002.

[5] C. Lanski, *A characterization of infinite cyclic groups*, Math. Mag. 74 (2001), 61-65.

[6] W. R. Scott, *Group Theory*, Dover, New York, 1987.

[7] G. Walls, *A characterization of finite cyclic groups*, An. Univ. Vest Timis. Ser. Mat.-Inform. 42 (2004), 141-149.

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