Palestine Journal of Mathematics

Vol. 11(4)(2022) , 293–299

ON JORDAN $*$-DERIVATION AND GENERALIZED JORDAN $*$-DERIVATION

Somaye Grailoo Tanha, Abolfazl Niazi Motlagh and Abasalt Bodaghi

Communicated by Mohammad Ashraf

MSC 2010 Classifications: Primary 47A20; Secondary 47B47.

Keywords and phrases: Jordan$*$-derivation, numerical range , maximal numerical range, Hilbert space.

Abstract Let $\mathcal{H}$ be a complex infinite-dimensional Hilbert space and $\mathrm{A}, \mathrm{B} \in B(\mathcal{H})$, the Banach algebra of bounded operators on $\mathcal{H}$. In this paper, for the generalized Jordan $*$-derivation $\Delta_{\mathrm{A,B}}(X) := \mathrm{A}X - X^*\mathrm{B}$, we show that $\overline{R(\Delta_{\mathrm{A,B}})}^{\tau} \cap \ker(\Delta_{\mathrm{A}^*,\mathrm{B}^*}) = \{0\}$, where $\overline{R(\Delta_T)}^{\tau}$ denotes closure of $R(\Delta_T)$ with respect to the norm topoplogy or the weak operator topology. We also investigate the norm of operator $\delta_{S,T} : X \mapsto SXT - TX^*S$, as an extension of generalized Jordan $*$-derivation $\Delta_{S,T}$, acting on $B(\mathcal{H})$. Furthermore, we obtain a lower bound for the operator $\delta_{S,T}(X)$.

1 Introduction and Preliminaries

Let $\mathcal{A}$ be a $*$-algebra. A Jordan $*$-derivation on $\mathcal{A}$ is a linear mapping $E : \mathcal{A} \longrightarrow \mathcal{A}$ which satisfies

$$E(a^2) = aE(a) + E(a)a^*$$

for all $a \in \mathcal{A}$. Note that for a fixed $a \in \mathcal{A}$, the mapping $\Delta_a(x) := ax - x^*a$ is a Jordan $*$-derivation; such Jordan $*$-derivations are called inner.

Here, we present some results from [1] and [8] as follows:

(i) Every Jordan $*$-derivation on complex $*$-algebra with identity is inner.

(ii) Every Jordan $*$-derivation on algebra of all bounded linear operators on a real Hilbert space $\mathcal{H}$ (dim $\mathcal{H} > 1$), is inner.

(iii) Every Jordan $*$-derivation on the quaternion algebra is inner.

Let $\mathcal{H}$ be a complex infinite-dimensional Hilbert space and let $B(\mathcal{H})$ be the Banach algebra of all bounded linear operators on $\mathcal{H}$. For operators $\mathrm{A}, \mathrm{B} \in B(\mathcal{H})$ we define generalized Jordan $*$-derivation $\Delta_{\mathrm{A,B}}$ by

$$\Delta_{\mathrm{A,B}}(X) := \mathrm{A}X - X^*\mathrm{B},$$

for all $X \in B(\mathcal{H})$. Note that if $\mathrm{A} = \mathrm{B}$, then $\Delta_{\mathrm{A}} := \Delta_{\mathrm{A,A}}$ is a Jordan $*$-derivation. The kernel and range of $\Delta_{\mathrm{A,B}}$ are denoted by $R(\Delta_{\mathrm{A,B}})$ and $\ker(\Delta_{\mathrm{A,B}})$, respectively.

The purpose of the second section is to study of the relation between $\overline{R(\Delta_{\mathrm{A}_i,\mathrm{A}_j})} \cap \ker(\Delta_{\mathrm{A}_i,\mathrm{A}_j})$ for $i, j = 1, 2$ and $\overline{R(\Delta_T)} \cap \ker(\Delta_T)$ in which $T = \mathrm{A}_1 \oplus \mathrm{A}_2 := \begin{bmatrix} \mathrm{A}_1 & 0 \\ 0 & \mathrm{A}_2 \end{bmatrix}$,

The numerical range of an operator A in $B(\mathcal{H})$ is defined by $W(\mathrm{A}) := \{\langle \mathrm{A}x, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$ and the numerical radius of A is defined via $w(\mathrm{A}) :=\sup\{|\lambda| : \lambda \in W(\mathrm{A})\}$ where $\langle ., . \rangle$ and $\|.\|$ stand, respectively, for the inner product on $\mathcal{H}$ and the norm associated with it. It was shown in [4] that $\overline{W(\mathrm{A})}$ is a compact convex subset of $\mathbb{C}$ and $\sigma(\mathrm{A}) \subseteq \overline{W(\mathrm{A})}$. Recall that the spectrum of an operator A, $\sigma(\mathrm{A})$, consists of those complex numbers λ such that $\mathrm{A} - \lambda I$ is not invertible and the spectral radius is given by $r(\mathrm{A}) =\sup\{|\lambda| : \lambda \in \sigma(\mathrm{A})\}$. The relationship between the numerical range and spectra has been studied by several authors; see for instance [3] and [4].

The concept of maximal numerical range was introduced by Stampfli [9] for proving the norm of a derivation.

Definition 1.1. The maximal numerical range of A $\in B(\mathcal{H})$, denoted by $W_0(\mathrm{A})$ is the set

$$W_0(\mathrm{A}) = \{\lambda \in \mathbb{C} : \exists \{x_n\} \subseteq \mathcal{H}, \langle \mathrm{A}x_n, x_n \rangle \to \lambda, \|x_n\| = 1, \|\mathrm{A}x_n\| \to \|\mathrm{A}\|\}.$$

It was also shown in [9, Lemma 2] that $W_0(\mathrm{A})$ is convex, and is contained in the closure of $W(\mathrm{A})$, that is $W_0(\mathrm{A}) \subset \overline{W(\mathrm{A})}$.

The next results which are useful for us, have been proved by the second author in [5].

Theorem 1.2. *Let $\mathcal{H}$ be a Hilbert space and $T \in B(\mathcal{H})$. If $\lambda \in W_0(T)$, then*

$$\|\Delta_T\| \geq 2(\|T\|^2 - |\lambda|^2)^{\frac{1}{2}}.$$

Proposition 1.3. *Let $\mathcal{H}$ be a Hilbert space and $T \in B(\mathcal{H})$. Then, $\|\Delta_T\| = 2\|T\|$ if and only if $0 \in W_0(T)$.*

Theorem 1.4. *Let $\mathcal{H}$ be a Hilbert space and $T \in B(\mathcal{H})$. Then,*

$$\|\Delta_T\| \geq 2 \sup_{\lambda \in W_0(T)} |Im(\lambda)|.$$

Proposition 1.5. *If $i\|T\| \in W_0(T)$, then $\|\Delta_T\| = 2\|T\|$, where $i^2 = -1$.*

Definition 1.6. A normal operator on a complex Hilbert space $\mathcal{H}$ is a continuous linear operator $N : \mathcal{H} \longrightarrow \mathcal{H}$ that commutes with its hermitian adjoint N^*, that is, $NN^* = N^*N$.

Definition 1.7. Two operators A and B on a Hilbert space $\mathcal{H}$ is called *similar* if there exists a bounded operator C on $\mathcal{H}$ having a bounded inverse such that the equality $A = C^{-1}BC$ hold. If C is a unitary operator, then A and B are said to be *unitarily* equivalent.

Let $T = \begin{bmatrix} \mathrm{A} & \mathrm{C} \\ \mathrm{D} & \mathrm{B} \end{bmatrix}$ be an operator on $\mathcal{H} \oplus \mathcal{H}$. The adjoint of T is defined by $T^* = \begin{bmatrix} \mathrm{A}^* & \mathrm{D}^* \\ \mathrm{C}^* & \mathrm{B}^* \end{bmatrix}$. Set

$$M_{\mathrm{A,B}} := R(\Delta_{\mathrm{A,B}}) \cap \ker(\Delta_{\mathrm{A}^*,\mathrm{B}^*}) \text{ and } \overline{M_{\mathrm{A,B}}}^{\tau} := \overline{R(\Delta_{\mathrm{A,B}})}^{\tau} \cap \ker(\Delta_{\mathrm{A}^*,\mathrm{B}^*}),$$

where $\overline{R(\Delta_{\mathrm{A,B}})}^{\tau}$ denotes closure of $R(\Delta_{\mathrm{A,B}})$ with respect to the norm topoplogy or the weak operator topology. Note that if A = B, then we put $M_{\mathrm{A,A}} := M_{\mathrm{A}}$ and $\overline{M_{\mathrm{A,A}}}^{\tau} := \overline{M_{\mathrm{A}}}^{\tau}$.

It is easily verified that if w-$\lim_n X_n = A$ and w-$\lim_n Yn = B$, then $w-\lim_n \begin{bmatrix} X_n & 0 \\ 0 & Y_n \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. In other words,

$$\begin{aligned} \lim_n \left\langle \begin{bmatrix} X_n & 0 \\ 0 & Y_n \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right\rangle &= \lim_n \left\langle \begin{bmatrix} X_n(a) \\ Y_n(b) \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right\rangle \\ &= \lim_n \left\langle X_n(a), c \rangle + \lim_n \langle Y_n(b), d) \right\rangle \\ &= \langle A(a), b \rangle + \langle B(c), d \rangle \\ &= \left\langle \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right\rangle. \end{aligned}$$

In this paper, under some conditions we prove that

$$\overline{R(\Delta_{\mathrm{A,B}})}^{\tau} \cap ker(\Delta_{\mathrm{A}^*,\mathrm{B}^*}) = \{0\}.$$

We also describe some classes of operators A and B for which the equality $\overline{R(\Delta_{\mathrm{A}})} \cap \ker(\Delta_{\mathrm{A}}) = \{0\}$ implies $\overline{R(\Delta_{\mathrm{B}})} \cap \ker(\Delta_{\mathrm{B}}) = \{0\}$. Moreover, we obtain a lower bound for operator $\delta_{A,B}(X) = AXB - BX^*A$. Indeed, we show that

$$\|\delta_{S,T}\| \geq \sup_{\lambda \in W_0(ST-TS)} \frac{|\lambda|^2}{\|ST-TS\|} \text{ if } ST \neq TS,$$
$$\|\delta_{S,T}\| \geq 2 \sup_{\lambda \in W_0(ST)} |Im(\lambda)|, \text{ if } ST = TS.$$

2 On the Range and Kernel of Jordan $*$-derivations

In this section, we study the range and kernel of Jordan $*$-derivations with different conditions.

Theorem 2.1. *Let* $\mathrm{A}, \mathrm{B} \in B(\mathcal{H})$. *Set* $T = \begin{bmatrix} \mathrm{A} & 0 \\ 0 & \mathrm{B} \end{bmatrix}$ *on* $\mathcal{H} \oplus \mathcal{H}$.

(i) *If* $\overline{M_T}^{\tau} = \{0\}$, *then* $\overline{M_{\mathrm{A}}}^{\tau} \cap \overline{M_{\mathrm{B}}}^{\tau} = \{0\}$;

(ii) *If* $M_T = \{0\}$, *then* $M_{\mathrm{A}} \cap M_{\mathrm{B}} = \{0\}$.

proof. Let $\mathrm{C} \in \overline{M_{\mathrm{A}}}^{\tau} \cap \overline{M_{\mathrm{B}}}^{\tau}$. Then, there exists two sequences $\{X_n\}, \{Y_n\} \subseteq B(\mathcal{H})$ such that $\tau - \lim \Delta_{\mathrm{A}}(X_n) = \tau - \lim \Delta_{\mathrm{B}}(Y_n) = \mathrm{C}$. Therefore, we have $\tau - \lim \mathrm{A}X_n - X_n^*\mathrm{A} = \tau - \lim \mathrm{B}Y_n - Y_n^*\mathrm{B} = \mathrm{C}$. Moreover, $\Delta_{\mathrm{A}^*}(\mathrm{C}) = \Delta_{\mathrm{B}^*}(\mathrm{C}) = 0$ implies that $\mathrm{A}^*\mathrm{C} = \mathrm{C}^*\mathrm{A}^*$ and $\mathrm{B}^*\mathrm{C} = \mathrm{C}^*\mathrm{B}^*$. Now, set $\mathrm{S} := \begin{bmatrix} \mathrm{C} & 0 \\ 0 & \mathrm{C} \end{bmatrix}$ and $Z_n := \begin{bmatrix} X_n & \\ 0 & Y_n \end{bmatrix}$ on $\mathcal{H} \oplus \mathcal{H}$. Thus

$$\begin{aligned} TZ_n - Z_n^*T &= \begin{bmatrix} \mathrm{A} & 0 \\ 0 & \mathrm{B} \end{bmatrix} \begin{bmatrix} X_n & 0 \\ 0 & Y_n \end{bmatrix} - \begin{bmatrix} X_n^* & 0 \\ 0 & Y_n^* \end{bmatrix} \begin{bmatrix} \mathrm{A} & 0 \\ 0 & \mathrm{B} \end{bmatrix} \\ &= \begin{bmatrix} \Delta_{\mathrm{A}}(X_n) & 0 \\ 0 & \Delta_{\mathrm{B}}(Y_n) \end{bmatrix}. \end{aligned}$$

Suppose that $\tau = w$ is weak operator topology. Since $w\text{-}\lim \mathrm{A}X_n - X_n^*\mathrm{A} = w\text{-}\lim \mathrm{B}Y_n - Y_n^*\mathrm{B} = \mathrm{C}$, we get

$$w - \lim \begin{bmatrix} \Delta_{\mathrm{A}}(X_n) & 0 \\ 0 & \Delta_{\mathrm{B}}(Y_n) \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} = S.$$

It follows from the relations above that

$$\begin{aligned} w - \lim_n (TZ_n - Z_n^*T) &= w - \lim_n \begin{bmatrix} \Delta_{\mathrm{A}}(X_n) & 0 \\ 0 & \Delta_{\mathrm{B}}(Y_n) \end{bmatrix} \\ &= \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} = S. \end{aligned}$$

Next, we show the above results for the norm topology. Let $\lim_{\|\cdot\|} \mathrm{A}X_n - X_n^*\mathrm{A} = \lim_{\|\cdot\|} \mathrm{B}Y_n - Y_n^*\mathrm{B} = \mathrm{C}$. Then

$$\begin{aligned} &\lim_n \left\| \begin{bmatrix} \Delta_{\mathrm{A}}(X_n) & 0 \\ 0 & \Delta_{\mathrm{B}}(Y_n) \end{bmatrix} - \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \right\| \\ &= \lim_n \left\| \begin{bmatrix} \Delta_{\mathrm{A}}(X_n) - C & 0 \\ 0 & \Delta_{\mathrm{B}}(Y_n) - C \end{bmatrix} \right\| \\ &= \lim_n \|\Delta_{\mathrm{A}}(X_n) - C\| + \lim_n \|\Delta_{\mathrm{B}}(Y_n) - C\| \\ &= 0. \end{aligned}$$

The relations above necessitate that

$$\lim_{\|\cdot\|}\begin{bmatrix} \Delta_{\mathrm{A}}(X_n) & 0 \\ 0 & \Delta_{\mathrm{B}}(Y_n) \end{bmatrix} = \begin{bmatrix} \mathrm{C} & 0 \\ 0 & \mathrm{C} \end{bmatrix}.$$

Hence, $S = \begin{bmatrix} \mathrm{C} & 0 \\ 0 & \mathrm{C} \end{bmatrix} \in \overline{R(\Delta_T)}^{\tau}$. On the other hand,

$$S^*T^* = \begin{bmatrix} \mathrm{C}^* & 0 \\ 0 & \mathrm{C}^* \end{bmatrix}\begin{bmatrix} \mathrm{A}^* & 0 \\ 0 & \mathrm{B}^* \end{bmatrix} = \begin{bmatrix} \mathrm{C}^*\mathrm{A}^* & 0 \\ 0 & \mathrm{C}^*\mathrm{B}^* \end{bmatrix}$$

and

$$T^*S = \begin{bmatrix} \mathrm{A}^* & 0 \\ 0 & \mathrm{B}^* \end{bmatrix}\begin{bmatrix} \mathrm{C} & 0 \\ 0 & \mathrm{C} \end{bmatrix} = \begin{bmatrix} \mathrm{A}^*\mathrm{C} & 0 \\ 0 & \mathrm{B}^*\mathrm{C} \end{bmatrix}.$$

Using the equalities $\mathrm{A}^*C = \mathrm{C}^*\mathrm{A}^*$ and $\mathrm{B}^*C = \mathrm{C}^*\mathrm{B}^*$, one can conclude that $S^*T^* = T^*S$. Furthermore, $S \in \overline{R(\Delta_T)}^{\tau} \cap \ker(\Delta_{T^*}) = \{0\}$. Therefore, $\mathrm{C} = 0$. This completes the proof of (i). The proof (ii) follows immediately from replacing $\overline{R(\Delta_T)}^{\tau}$ by $R(\Delta_T)$. □

Theorem 2.2. *Let* $\mathrm{A}_1, \mathrm{A}_2 \in B(\mathcal{H})$. *Suppose that* $\overline{M_{\mathrm{A}_i,\mathrm{A}_j}} = \{0\}$ *for* $i, j = 1, 2$. *Then,* $\overline{M_T} = \{0\}$ *in which* $T = \mathrm{A}_1 \oplus \mathrm{A}_2 = \begin{bmatrix} \mathrm{A}_1 & 0 \\ 0 & \mathrm{A}_2 \end{bmatrix}$.

proof Let $C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \in \overline{M_T} = \overline{R(\Delta_T)} \cap \ker(\Delta_T)$. Then, there exists a sequence $\{X_n\} \subseteq B(\mathcal{H} \oplus \mathcal{H})$ such that $\lim_{\|\cdot\|} \Delta_T(X_n) = \mathrm{C}$. Therefore, we have $\lim_{\|\cdot\|} TX_n - X_n^*T = \mathrm{C}$. Furthermore, $\mathrm{C} \in \ker(\Delta_T)$ implies that $T\mathrm{C} = \mathrm{C}^*T$. Set $X_n = \begin{bmatrix} X_{11n} & X_{12n} \\ X_{21n} & X_{22n} \end{bmatrix}$. We obtain

$$\begin{aligned}
\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} &= \lim_{\|\cdot\|} TX_n - X_n^*T \\
&= \lim_{\|\cdot\|} \begin{bmatrix} \mathrm{A}_1 & 0 \\ 0 & \mathrm{A}_2 \end{bmatrix}\begin{bmatrix} X_{11n} & X_{12n} \\ X_{21n} & X_{22n} \end{bmatrix} \\
&- \begin{bmatrix} X_{11n}^* & X_{21n}^* \\ X_{12n}^* & X_{22n}^* \end{bmatrix}\begin{bmatrix} \mathrm{A}_1 & 0 \\ 0 & \mathrm{A}_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{A}_1X_{11n} - X_{11n}^*\mathrm{A}_1 & \mathrm{A}_1X_{12n} - X_{21n}^*\mathrm{A}_2 \\ \mathrm{A}_2X_{21n} - X_{12n}^*\mathrm{A}_1 & \mathrm{A}_2X_{22n} - X_{22n}^*\mathrm{A}_2 \end{bmatrix}.
\end{aligned}$$

Thus, we conclude that $C_{ij} \in \overline{R(\Delta_{\mathrm{A}_i,\mathrm{A}_j})}$ for $i, j = 1, 2$. In addition, by applying

$$\begin{bmatrix} \mathrm{A}_1 & 0 \\ 0 & \mathrm{A}_2 \end{bmatrix}\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} C_{11}^* & C_{21}^* \\ C_{12}^* & C_{22}^* \end{bmatrix}\begin{bmatrix} \mathrm{A}_1 & 0 \\ 0 & \mathrm{A}_2 \end{bmatrix},$$

we arrive at $C_{ij} \in \ker(\Delta_{\mathrm{A}_i,\mathrm{A}_j})$, $i, j = 1, 2$. Hence, $C_{ij} \in \overline{R(\Delta_{\mathrm{A}_i,\mathrm{A}_j})} \cap \ker(\Delta_{\mathrm{A}_i,\mathrm{A}_j}) = \{0\}$ for $i, j = 1, 2$ and now the proof is finished. □

Theorem 2.3. *Let* A *and* B *be unitary equivalent bounded operators on a Hilbert space* $\mathcal{H}$ *such that* $\mathrm{B} = \mathrm{UAU}^*$ *holds for some unitary operator* U. *Then,* $\overline{M_{\mathrm{A}}} = \{0\}$ *if and only if* $\overline{M_{\mathrm{B}}} = \{0\}$.

proof Let $C \in \overline{M_{\mathrm{B}}}$. Then, there exists a sequence $\{X_n\} \subseteq B(\mathcal{H})$ such that $\lim \Delta_{\mathrm{B}}(X_n) = \mathrm{C}$, and so $\lim_n \mathrm{B}X_n - X_n^*\mathrm{B} = \mathrm{C}$. Moreover, $\Delta_{\mathrm{B}^*,\mathrm{B}^*}(\mathrm{C}) = 0$ implies that $\mathrm{B}^*\mathrm{C} = \mathrm{C}^*\mathrm{B}^*$. Set $\mathrm{D} = \mathrm{UCU}^*$ and $Y_n = \mathrm{UX}_n\mathrm{U}^*$. Then, we see that $\mathrm{A}^*\mathrm{D} = \mathrm{D}^*\mathrm{A}^*$ and $\lim_n \mathrm{A}Y_n - Y_n^*\mathrm{A} = \mathrm{D}$. Thus, we conclude $D \in \overline{R(\Delta_{\mathrm{A}})} \cap \ker(\Delta_{\mathrm{A}})$ and therefore $\mathrm{D} = 0$ that implies $C = 0$. □

Before stating the next corollary, we need the following theorem of Putnam [6].

Theorem 2.4. *Let* $\mathrm{A}, \mathrm{B} \in B(\mathcal{H})$ *be normal operators and* B *be similar to* A *such that* $\mathrm{B} = \mathrm{C}^{-1}\mathrm{AC}$ *for some bounded invertable operator* C*. If* $\mathrm{C} = P\mathrm{U}$ *is the polar factorization of* C*, then* $\mathrm{B} = \mathrm{U}^*\mathrm{AU}$*, where* P *is a positive definite and* U *is a unitary operator.*

Corollary 2.5. *Let* $\mathrm{A}, \mathrm{B} \in \mathrm{B}(\mathcal{H})$ *be normal operators and* B *be similar to* A*. If* $\overline{R(\Delta_{\mathrm{A}})} \cap \ker(\Delta_{\mathrm{A}}) = \{0\}$*, then* $\overline{R(\Delta_{\mathrm{B}})} \cap \ker(\Delta_{\mathrm{B}}) = \{0\}$.

proof Since A and B are similar, By Theorem 2.4 there exists an unitary U such that $\mathrm{B} = \mathrm{U}^*\mathrm{AU}$. Now, Corollary 2.3 can be applied to obtain the desired result. □

3 A Lower Bound for Generalized Jordan $*$-derivations

Let $\mathcal{H}$ be a Hilbert space, $S, T \in B(\mathcal{H})$ and $\{x_n\} \subseteq \mathcal{H}$ be a sequence such that $\| x_n \| = 1$ and

$$\lim_n \| ST(x_n) - TS(x_n) \| = \| ST - TS \| .$$

Then

$$\| \delta_{S,T}(I)(x_n) \| = \| SIT(x_n) - TI^*S(x_n) \| = \| ST(x_n) - TS(x_n) \| .$$

Therefore

$$\| \delta_{S,T} \| \geq \lim_n \| ST(x_n) - TS(x_n) \| = \| ST - TS \| .$$

The following Theorem is an improvement of Theorem 1.4.

Theorem 3.1. *Let* $\mathcal{H}$ *be a Hilbert space and* $S, T \in B(\mathcal{H})$*. Then*

(i) $\|\delta_{S,T}\| \geq \sup_{\lambda \in W_0(ST-TS)} \frac{|\lambda|^2}{\|ST-TS\|}, if\ ST \neq TS;$

(ii) $\|\delta_{S,T}\| \geq 2\sup_{\lambda \in W_0(ST)} |Im(\lambda)|, provided\ that\ ST = TS.$

proof First suppose that $ST \neq TS$. Let $\lambda \in W_0(ST - TS)$. Then, there exists a sequence $\{x_n\} \subseteq \mathcal{H}$ such that $\| x_n \| = 1$ and

$$\lim_n \|ST - TS(x_n)\| = \|ST - TS\|, \quad \lim_n \langle ST - TS(x_n), x_n\rangle = \lambda.$$

Define the operator $(x_n \otimes ST - TS) \in B(\mathcal{H})$ by

$$(x_n \otimes ST - TS)y := |\langle ST - TS(x_n), x_n\rangle|^2 y, \qquad (y \in \mathcal{H}).$$

We have

$$\begin{aligned}
&\|\delta_{S,T}(x_n \otimes ST - TS)(x_n)\| \\
&= \|S(x_n \otimes ST - TS)T(x_n) - T(x_n \otimes ST - TS)^*S(x_n)\| \\
&= \left\||\langle ST - TS(x_n), x_n\rangle|^2 ST(x_n) - |\langle ST - TS(x_n), x_n\rangle|^2 TS(x_n)\right\| \\
&= |\langle ST - TS(x_n), x_n\rangle|^2\rangle \,\|ST - TS(x_n)\| .
\end{aligned}$$

On the other hand,

$$\|\delta_{S,T}(x_n \otimes ST - TS)(x_n)\| \leq \|ST - TS\|^2 \|\delta_{S,T}\| .$$

Thus,

$$|\langle (ST - TS)(x_n), x_n\rangle|^2 \|(ST - TS)(x_n)\| \leq \|ST - TS\|^2 \|\delta_{S,T}\| ,$$

and so

$$\lim_n |\langle (ST-TS)(x_n), x_n\rangle|^2 \,\|(ST-TS)(x_n))\| \leq \|ST-TS\|^2 \,\|\delta_{S,T}\|\,.$$

Hence,

$$\|\delta_{S,T}\|\,\|ST-TS\|^2 \geq |\lambda|^2\,\|ST-TS\|\,.$$

Therefore,

$$\|\delta_{S,T}\| \geq \sup_{\lambda\in W_0(ST-TS)} \frac{|\lambda|^2}{\|ST-TS\|}.$$

Now, suppose that $ST = TS$. Let $\lambda \in W_0(ST)$. Then, there exists a sequence $\{x_n\} \subseteq \mathcal{H}$ such that $\|x_n\| = 1$, $\lim_n \|ST(x_n)\| = \|ST\|$ and $\lim_n\langle ST(x_n), x_n\rangle = \lambda$. Consider the operator $(x_n \otimes ST) \in B(\mathcal{H})$ defined through

$$(x_n \otimes ST)y := \langle x_n, ST(x_n)\rangle y, \qquad (y \in \mathcal{H}).$$

We get

$$\begin{aligned}
&\|\delta_{S,T}(x_n \otimes ST(x_n))(x_n)\|\\
&= \|S(x_n \otimes ST(x_n))T(x_n) - T(x_n \otimes ST(x_n))^* S(x_n)\|\\
&= \|S\langle x_n, ST(x_n)\rangle T(x_n) - T\langle ST(x_n), x_n\rangle S(x_n)\|\\
&= |\langle x_n, ST(x_n)\rangle - \langle ST(x_n), x_n\rangle|\,\|ST(x_n)\|\,.
\end{aligned}$$

Since $\|\delta_{S,T}(x_n \otimes ST)(x_n)\| \leq \|ST\|\,\|\delta_{S,T}\|$, we find

$$|\langle x_n, ST(x_n)\rangle - \langle ST(x_n), x_n\rangle|\,\|ST(x_n)\| \leq \|ST\|\,\|\delta_{S,T}\|\,.$$

The last inequality implies that

$$\lim_n |\langle x_n, ST(x_n)\rangle - \langle ST(x_n), x_n\rangle|\,\|ST(x_n)\| \leq \|ST\|\,\|\delta_{S,T}\|\,.$$

Hence,

$$2|Im(\lambda)|\|ST\| \leq \|ST\|\,\|\delta_{S,T}\|\,.$$

Therefore, $2|Im(\lambda)| \leq \|\delta_{S,T}\|$, and $\|\delta_{S,T}\| \geq 2\sup_{\lambda\in W_0(ST)} |Im(\lambda)|$. □

Corollary 3.2. *If* $i \parallel ST \parallel \in W_0(ST)$, *then* $\parallel \delta_{ST} \parallel = 2 \parallel S \parallel\parallel T \parallel$, *where* $i^2 = -1$.

References

[1] M. Bresar and B. Zalar, On the structure of Jordan ∗-derivations, *Aequationes Math.* **27**, 76–86 (1984).

[2] H. G. Dales, Banach algebra and Automatic continuity, *Oxford university Press*, (2001).

[3] K. E. Gustafson and D. K. M. Rao, Numerical Range: the field of values of linear operators and matrices, *Springer-verlag, New York*, (1997).

[4] P. R. Halmos, A Hilbert space problem book, *Van Nostrand, Princeton*, (1970).

[5] A. Niazi Motlagh, On the norm of Jordan ∗-derivations, *Khayyam J. Math.* **6**, 104–107 (2020) .

[6] C. R. Putnam, On normal operators in Hilbert space, *Amer. Jour. Math.* **73**, 357-362 (1951).

[7] A. Seddik, On the numerical range and norm of elementary operators, *Linear Mult. Algebra.* **52**, 293-302 (2004).

[8] P. Šemrl, On Jordan ∗-derivations and application, *Colloq. Math.* **59**, 241-251 (1990).

[9] J. G. Stampfli, The norm of a derivation, *Pasific J. Math.* **33**, 737 -747 (1970).

[10] O. Toeplitz, Das algebraische Analogon zu einem Satze von Fejer, *Math. Z.* **2**, 187-197 (1918).

Author information

Somaye Grailoo Tanha, Esfarayen University of Technology, Esfarayen, North Khorasan, Iran.
E-mail: grailotanha@gmail.com

Abolfazl Niazi Motlagh, Department of Mathematics, Faculty of basic sciences, University of Bojnord, P. O. Box 1339, Bojnord, Iran.
E-mail: a.niazi@ub.ac.ir

Abasalt Bodaghi, Department of Mathematics, Garmsar Branch, Islamic Azad University, Garmsar, Iran.
E-mail: abasalt.bodaghi@gmail.com

Received: December 21, 2021.
Accepted: April 27, 2022.