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The Domination Number of Idempotent Divisor Graphs of Commutative Rings

Mohammed N. Authman, Husam Q. Mohammad and Nazar H. Shuker

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Abstract The idempotent divisor graph of a commutative ring R is a graph with vertices set in $R^* = R - 0$, and for any distinct vertices x and y, x adjacent with y if and only if $xy = e$ for some non-unit idempotent element $e = e^2 \in R$ and is denoted by $\pi(R)$. The purpose of this work is to study the fundamental properties of dominating sets of $\pi(R)$, and to find the domination number and the connected domination number of $\pi(R)$.

1 Introduction

Throughout this papare R is a finite commutative ring with multiplication non-zero identity 1. We denote $U(R)$, $Z(R)$, and $I(R)$ the sets of unit elements, zero divisors, and idempotent elements of the ring R respectively. In 1988 [5], Beck gave the relationship between ring theory and graph theory when he studied the coloring of a commutative ring. Later [2] in 1999, Anderson and Livingston developed this model by studying the zero divisor graph $\Gamma(R)$ with vertices z_1, $z_2 \in Z(R)^* = Z(R) - \{0\}$, $\{z_1, z_2\}$ is an edge in $\Gamma(R)$ if and only if $z_1.z_2 = 0$. Many other authors studied this notion, for examples see [9, 10, 11] and [13] . Also there are many other definition that connect these theories of graph and ring, for example see [1, 3] and [14]. In 2022 [8], Mohammad and Shuker, gave the concept of the idempotent divisor graph of commutative ring R with unit $1 \neq 0$, it is a graph denoted by $\pi(R)$ which has vertices set in $R^* = R - \{0\}$, and for any distinct vertices x and y, x adjacent with y if and only if $x.y = e$ for some non-unit idempotent element $e^2 = e \in R$. If R is a local ring, the only non-unit idempotent element in R is zero, they assumed that $V(\pi(R)) = Z(R)^*$, this means that $\pi(R) = \Gamma(R)$. Also they presented some fundamental properties of this graph, they proved this graph is a connected with diameter less than or equal three. In 2022 [4], we found the idempotent divisor graphs of degrees n, for $7 \leq n \leq 14$ and the rings whose correspond to them, also we gave some properties of the idempotent divisor graph and we classified it by its diameter.

In a graph theory," for the connected graph G, the eccentricity of a vertex $v \in (G)$, denoted by $e(v)$ is the distance between v and a vertex furthest from v. The diameter of G is equal to $max_{v \in V(G)} e(v)$, and denoted by $diam(G)$ If U is nonempty subset of $V(G)$ of a graph G, then the subgraph $< U >$ of G induced by U is the graph having vertex set U and whose edge set consists of those edges of G incident with two elements of U. For the vertex v in $V(G)$, $N_{G(v)}$ denoted the set of all vertices that adjacent with v. A dominating set for a graph G is a set $D \subseteq V(G)$ such that every vertex of $V(G)$ either lie in D or adjacent to at least one vertex of D. The set which has the minimum cardinality of a dominating set is called the minimum dominating set. The domination number $\gamma(G)$ is the number of vertices in a minimum dominating set for G. A minimal dominating set in a graph G is a dominating set that contains no dominating set as a proper subset. Two vertices that are not adjacent in a graph G are said to be independent. A set S of vertices is an independent if every two vertices of S are independent, if S is a dominating set for G then S is called an independent dominating set for G. A subset S of vertices is a connected dominating set if S is a dominating set and $<S>$ is connected subgraph of G. The connected domination number of G, denoted by $\gamma_c(G)$, is the minimum cardinality of a connected dominating set ". For more details see [7] and [12].

In a ring theory, " Let R be a finite commutative ring with unit $1 \neq 0$, R is said to be local if it contains a single maximal ideal. The ring R is direct product of the rings R_i for

$i = 1, 2, ..., n$, if $R \cong R_1 \times R_2 \times ... \times R_n = \{(r_1, r_2, ..., r_n) : r_i \in R_i\}$. If R be a finite non local then R is direct product of local rings. A ring R is said to be reduced ring if and only if R contains no non-zero nilpotent elements, that is, $x^2 = 0$ implies $x = 0$ for any $x \in R$. If R be a finite commutative reduced ring with 1, then there is an integer number n with $n \geq 1$ such that $R \cong K_1 \times K_2 \times ... \times K_n$ where K_i are finite fields for $i = 1, 2, ..., n$. For an element r in R, $ann(r) = \{x \in R : xr = rx = 0\}$. F_p we denoted a field of order p, where p is a power of prime number " See [6]. In section two we showed cases in which some subsets of R^* are dominating, minimal dominating, connected dominating, or independent dominating sets for $\pi(R)$, as $Z(R)^*$, $U(R)$, and others. In the third section we found the domination number, and the connected domination number of $\pi(R)$ for any ring R.

2 Dominating Sets of Idempotent Divisor Graph of Commutative Ring

We begin this section with the following main results.

Proposition 2.1. *Let R be a ring with $R \cong R_1 \times R_2 \times ... \times \times R_n$ where R_i are local rings for all $i = 1, 2, ..., n$, for some positive integer $n \geq 2$. Then the set $D = \{(0_1, 0_2, ..., d_j, 0_{j+1}, ..., 0_n), d_j \in R_j - \{0_j\}$, for $j = 1, 2, ..., n\}$ is a dominating set for $\pi(R)$. In particular, if R_j is a field, then the set $D_1 = \{(0_1, 0_2, ..., d_j, 0_{j+1}, ..., 0_n), d_j \in U(R_j)$, for $j = 1, 2, ..., n\}$ is a minimal dominating set of $\pi(R)$.*

Proof. Let $r = (r_1, r_2, ..., r_j, r_{j+1}, ..., r_n) \in R^* - D$. For $j \in \{1, 2, ..., n\}$, we have three cases:

Case1: If $r_j = 0_j$, then $rx = 0$ for all $x \in D$.

Case2: If $r_j \in Z(R_j)^*$, then R_j is a non-field local ring in this case, there exists an element $a_j \in Z(R_j)^*$ such that $a_j \cdot x_j = 0_j$ for all $x_j \in Z(R_1)^*$. Hence r is adjacent with $(0_1, 0_2, ..., 0_{j-1}, a_j, 0_{j+1}, ..., 0_n) \in D$.

Case3: If r_j is a unit in R_j, then is there is $y_j \in R_j$ with $r_j y_j = 1_j$. Clearly r is adjacent with $(0_1, 0_2, 0_3, ..., 0_{j-1}, y_j, 0_{j+1}, ..., 0_n) \in D$.

From the cases above we conclude that for any $r \in R^* - D$ there is a vertex in D that adjacent with r. Therefore D is a dominating set for $\pi(R)$. To show that D_1 is a minimal, if $R_j = F_2$, then the proof is clear. Now, let $w = (0_1, 0_2, 0_3, ..., 0_{j-1}, w_j, 0_{j+1}, ..., 0_n) \in D_1$ such that $D_1 - \{w\}$ is a dominating set, and let $s = (s_1, s_2, ..., s_j, s_{j+1}, ..., s_n) \in R^* - \{D_1 - \{w\}\}$ with $s_i \in R_i$ for $i = 1, 2, ..., n$ such that $s_j \in R_j - \{0_j, w_j^{-1}\}$. Then $w.s = (0_1, 0_2, ..., 0_{j-1}, w_j s_j, 0_{j+1}, ..., 0_n) \notin I(R) - \{1\}$, thus $D_1 - \{w\}$ is not dominating set which is a contradiction. Therefore D_1 is a minimal dominating set for $\pi(R)$. □

Proposition 2.2. *Let R be a ring then $Z(R)^*$ is connected dominating set for $\pi(R)$, but it is not minimal dominating set.*

Proof. If R is a local ring, then $V(\pi(R)) = Z(R)^*$, thus $\pi(R) = \Gamma(R)$. Since $\Gamma(R)$ is a connected graph then $Z(R)^*$ is connected dominating set for $\pi(R)$ in this case, since R is a local ring, then there is a vertex r, that adjacent with every other vertieces in $\pi(R) = \Gamma(R)$, thus $\{r\} \subset Z(R)^*$ be a dominating set for $\pi(R)$. Therefore $Z(R)^*$ is not minimal dominating set. Now, let R be a non local ring, and $u \in R^* - Z(R)^* = U(R)$, then u is adjacent with $eu^{-1} \in Z(R)^*$ for any $e \in I(R) - \{0\}$, hence $Z(R)^*$ is dominating set for $\pi(R)$, also since $< Z(R)^* >$ is an induced subgraph of $\Gamma(R)$, and since $\Gamma(R)$ is connected then $Z(R)^*$ is connected dominating set for $\pi(R)$. On the other hand, since R be a non local ring, then there is an integer $n \geq 2$ such that $R \cong R_1 \times R_2 \times \cdots \times R_n$ where R_i are local rings for all $i = 1, 2, ..., n$, let $D = \{(d_1, 0_2, 0_3, ..., 0_n) : d_1 \in R_1 - \{0_1\}\}$, by 2.1 we get D be a dominating set for $\pi(R)$, but D is a proper subset of $Z(R)^*$. Therefore $Z(R)^*$ is not minimal dominating set □

In the following result we show that there is a minimum dominating set for $\pi(R)$ not contain any unit of the ring R.

Proposition 2.3. *Let R be a ring and D is a minimum dominating set for $\pi(R)$, then there is a minimum dominating set S without units such that $|S| = |D|$.*

Proof. If R is a local ring, then the proof is obvious. Let R be a non local ring then there is an $n \in Z^+$ such that $R \cong R_1 \times R_2 \times ... \times R_n$ where R_i are local rings for all $i = 1, 2, ..., n$. Let $u = (u_1, u_2, ..., u_n)$ be a unit in D. Clearly $v = (u_1, 0_2, 0_3, ..., 0_n)$ adjacent with every vertices of $N_{\pi(R)}(u)$, it means that $N_{\pi(R)}(u) \subseteq N_{\pi(R)}(v)$, then $S = \{v\} \cup D - \{u\}$ be a minimum dominating set for $\pi(R)$. Since u is arbitry in D, therefore D without units and hence $|S| = |D|$. □

Proposition 2.4. *Let R be a reduced ring. Then $U(R)$ is an independent minimal dominating set for $\pi(R)$.*

Proof. Since R is a reduced ring, then there is an $n \in Z^+$ such that $R \cong K_1 \times K_2 \times ... \times K_n$ where K_i are fields for all $i = 1, 2, ..., n$. Let $x \in R^* - U(R) = Z(R)^*$, then x has unique representation as $x = (x_1, x_2, ..., x_n)$ with $x_i \in K_i$ for $i = 1, 2, ..., n$ such that there is at least one of x_i equal to zero for some $i = 1, 2, ..., n$, and there is at least one of x_i not equal to zero since $x \neq 0$. Let $y = (y_1, y_2, ..., y_n) \in U(R)$ with

$$y_i = \left\{ \begin{array}{c} 1_i : if x_i = 0_i, for 1 \le i \le n \\ x^{-1} : if x_i \neq 0_i, for 1 \le i \le n \end{array} \right\}$$

then we get $xy = (x_1y_1, x_2y_2, ..., x_ny_n)$ where

$$x_iy_i = \left\{ \begin{array}{c} 1_i \cdot 0_i : if x_i = 0_i, for 1 \le i \le n \\ x_i^{-1} \cdot x_i : if x_i \neq 0_i, for 1 \le i \le n \end{array} = \left\{ \begin{array}{c} 0_i : if x_i = 0_i, for 1 \le i \le n \\ 1_i : if x_i \neq 0_i, for 1 \le i \le n \end{array} \right\} \right\}, then xy \in I(R) - \{1\}$$

Hence $U(R)$ is a dominating set for $\pi(R)$, also since any two units in R^* are not adjacent then $U(R)$ is an independent dominating set for $\pi(R)$. Now, let $U(R)$ contains a dominating proper subset D, then there is at least $u \in U(R)$ with $u \notin D$, since D be a dominating set by assumption then there is $x \in D \subseteq U(R)$ such that u adjacent with x this is impossible because each of u and x are units, thus $U(R)$ contains no dominating set as a proper subset. Therefore $U(R)$ is a minimal dominating set for $\pi(R)$. □

If R is not reduced ring then the Prpoposition 2.4 is not true in general, as the following example shows:

Example 2.5. Let $R \cong Z_4 \times Z_4$, then $U(R) = \{(1, 1), (1, 3), (3, 1), (3, 3)\}$, and $\pi(R)$ has the following form

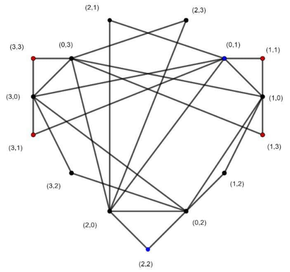

Figure 1. $\pi(Z_4 \times Z_4)$

Observe that $(2, 2) \in R^* - U(R) = Z(R)^*$ is not adjacent with any vertex of $U(R)$, so $U(R)$ is not dominating set for $\pi(R)$.

Proposition 2.6. *Let R be a non local ring and $e \in I(R) - \{0, 1\}$. Then $eR^* = eR - \{0\}$ be a dominating set for $\pi(R)$.*

Proof. Since R be a non local then $R \cong R_1 \times R_2 \times \cdots \times R_n$ where R_i are local rings for all $i = 1, 2, ..., n, n \in Z^+$. We can write $eR = \{(e_1r_1, e_2r_2, ..., e_nr_n) : r_i \in R_i, e_i = 0_i, 1_i$, for $i = 1, 2, ..., n\}$, since $e \neq 0$ then there is $e_j \neq 0_j$, that is $e_j = 1_j$ for some $j = 1, 2, ..., n$, then the set $\{(0_1, 0_2, ..., r_j, 0_{j+1}, ..., 0_n) : r_j \in R_j, for j = 1, 2, ..., n\}$ is a subset of eR, then the set $D = \{(0_1, 0_2, ..., r_j, 0_{j+1}, ..., 0_n) : r_j \in R_j - \{0_j\}$, for $j = 1, 2, ..., n\} \subset eR^*$, by Proposition 2.1 we have D be a dominating set for $\pi(R)$, therefore eR^* be dominating set for $\pi(R)$. □

3 Domination and connected domination numbers of the idempotent divisor graph of commutative ring

The present section is developed to find the domination and the connected domination numbers of $\pi(R)$ for any ring R. We start this section with the following well-known result.

Lemma 3.1 (4). *Let R be a finite ring , then*

(i) $diam(\pi(R)) = 0$ *if and only if* $R \cong Z_4 or Z_2[X]/(X^2)$.

(ii) $diam(\pi(R)) = 1$ *if and only if R is a local ring with* $(Z(R))^2 = 0$*, or R is Boolean ring.*

(iii) $diam(\pi(R)) = 2$ *if and only if R is local with* $(Z(R))^2 \neq 0$*, or* $R \cong F_2 \times R_1 \times R_2 \times \cdots \times R_m$ *where R_i is a local ring for all* $i \in \{1, 2, ..., m\}$ *with* $R_i \neq F_2$ *for at least one of* $i \in \{1, 2, ..., m\}$*, and F_2 is a field of order 2.*

(iv) For all cases except (1), (2) and (3), $diam(\pi(R)) = 3$.

Next, we shall give the following main results.

Theorem 3.2. *Let R be a ring with* $diam(\pi(R)) \leq 2$*. Then* $\gamma(\pi(R)) = \gamma_c(\pi(R)) = 1$.

Proof. If $diam(\pi(R)) = 1$ then $\pi(R)$ is complete graph, hence $\gamma(\pi(R)) = 1$. If R is a local, then there is a vertex, say x, which adjacent with every other vertices in $\Gamma(R) = \pi(R)$. Therefore $D = \{x\}$ is the minimum dominating set for $\Gamma(R) = \pi(R)$. Hence $\gamma(R) = 1$. Now, if R be a non local, and $diam(\pi(R)) = 1$, then $\pi(R)$ is a complete graph, thus $\gamma(R) = 1$, on the other hands if $diam(\pi(R)) = 2$, then by Lemma3.1, $R \cong F_2 \times R_1 \times R_2 \times \cdots \times R_m$ where R_i are local rings for all $i = 1, 2, ..., m, m \in Z^+$, and F_2 is a field of order 2, thus the vertex $y = (1_1, 0_2, 0_3, ..., 0_n)$ adjacent with every other vertices in $\pi(R)$. Hence $D = \{y\}$ is the minimum dominating set for $\pi(R)$. Therefore $\gamma(\pi(R)) = \gamma_c(\pi(R)) = 1$. □

Theorem 3.3. *Let R be a reduced ring with* $R \cong K_1 \times K_2 \times \cdots \times K_n$ *where K_i are fields for all* $i \in \{1, 2, ..., n\}, n \in Z^+$*. Then* $\gamma(\pi(R)) = min_{(1 \leq i \leq n)}(|K_i|) - 1$.

Proof. If $diam\pi(R)) \leq 2$, then the proof is directly by Theorem 3.2. Now, if $diam(\pi(R)) = 3$, by appling Lemma 3.1 we have $R \cong K_1 \times K_2 \times \cdots \times K_n$ where K_i are fields with $K_i \ncong F_2$ for all $i \in \{1, 2, ..., n\}, n \in Z^*$. Without lose the generality, assume that K_1 be a minimum order field of the fields K_i for $i = 1, 2, ..., n$, that is $|K_1| = min_{(1 \leq i \leq n)}(|K_i|)$, and let $D = \{(d, 0_2, 0_3, ..., 0_n) : d \in K_1 - \{0_1\}\}$, clearly $|D| = |K_1| - 1$, and by Proposition 2.1 we get that D is a minimal dominating set for $\pi(R)$.

Next, we shall show that D is a minimum dominating set for $\pi(R)$, and we prove this by contradiction. Suppose that M be a minimum dominating set for $\pi(R)$ with $|M| < |D|$, then by Proposition 2.3 there is a minimum dominating set D_1 without units such that $|M| = |D_1|$. Let $D_1 = \{(d_1, d_2, ..., d_n) : d_i \in S_i, for, i = 1, 2, ..., n\}$ where $S_i \subseteq K_i$ for $i = 2, 3, ..., n$ such that $R^* - D_1 = \prod_{i=1}^n (K_i - S_i)$,

and $D_1 = \prod_{i=1}^n S_i$. Since D_1 without units then $U(R) \subseteq \prod_{i=1}^n (K_i - S_i)$. Let $x_i \in K_i - \{0_i\}$, then $x_i^{-1} \in K_i - \{0_i\}$ for all $i = 1, 2, ..., n$, thus $x = (x_1^{-1}, x_2^{-1}, ..., x_n^{-1}) \in U(R) \subseteq \prod_{i=1}^n (K_i - S_i) = R^* - D_1$, since $x \notin D_1$ then there is $d = (d_1, d_2, ..., d_n) \in D_1$ such that d adjacent with x, since $x \in U(R)$ and $d \neq 0$ then $xd \neq 0$, also there is $d_i \neq 0_i$ for at least one of i for $i = 1, 2, ..., n$, assume that $d_j \neq 0_j$ for one element j of the set $\{1, , 2, \ldots, n\}$, then we get $x_j^{-1} d_j = 1_j$, implies $x_j = d_j \in S_j$, hence $K_j - \{0_j\} \subseteq S_j$, then we get $|D| \leq |K_j| - 1 \leq |D_1| = |M|$ which is a contradiction. Therefore D is the minimum dominating set for $\pi(R)$, and $\gamma(\pi(R)) = min_{1 \leq i \leq n}(|K_i|) - 1$. □

Corollary 3.4. *Let R be a reduced ring with* $diam(\pi(R)) = 3$ *such that* $R \cong K_1 \times K_2 \times \cdots K_n$ *where K_i are fields for all* $i = 1, 2, ..., n, n \in Z^*$*. Then* $\gamma_c(\pi(R)) = min_{1 \leq i \leq n}(|K_i|)$.

Proof. Let $D = \{(d, 0_2, 0 - 3, ..., 0_n) : d \in K_1 - \{0_1\}\}$ be a minimum dominating set for $\pi(R)$ where K_1 is a field with $|K_1| = min_{1 \leq i \leq n}(|K_i|)$, and let D_1 be any connected minimum dominating set with $|D_1| = |D|$ where $D_1 = \{(d_1, d_2, ..., d_n) : d_i \in S_i, for, i = 1, 2, ..., n\}$ where $S_i \subseteq K_i$ for $i = 2, 3, ..., n$ such that $R^* - D_1 = \prod_{i=1}^n (K_i - S_i)$, and $D_1 = \prod_{i=1}^n S_i$, as a proof of Theorem 3.3 we have $K_j - \{0_j\} \subseteq S_j$ for some $j = 1, 2, ..., n$, as $D_1 = \prod_{i=1}^{n} S_i$ and $|D_1| = |D|$ then $K_j = K_1$ and $S_j = K_j - \{0_j\}$, implies $D_1 = \{(0_1, 0_2, ..., d_j, 0_{j+1}, ..., 0_n) : d_j \in S_j\}$, but $< D_1 >$ is not connected subgraph of $\pi(R)$ which is a contradiction, then there is no connected dominating set of order less than or equal $|D|$. Clearly $< D >$ is not connected subgraph of $\pi(R)$, and we see that the vertex $(0_1, 0_2, ..., 0_{n-1}, 1_n)$ is adjacent with every vertices of D, thus $< D \bigcup\{(0_1, 0_2, ..., 0_{n-1}, 1_n)\} >$ is connected subgraph. Therefore $\gamma_c(\pi(R)) = |D| + 1 = min_{1\leq i\leq n}(|K_i|)$. □

Example 3.5. Let $R \cong Z_3 \times Z_5$, then $\pi(R)$ has the form in Fig. 2

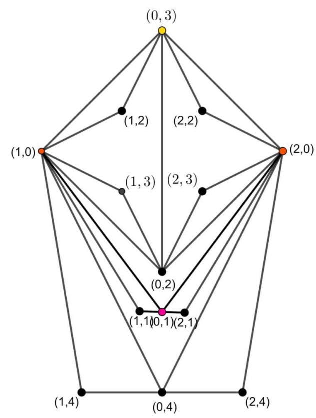

Figure 2. $\pi(Z_3 \times Z_5)$

Observe that $D = \{(1, 0), (2, 0)\}$ is the minimum dominating set for $\pi(R)$, and $\gamma(\pi(R)) = |Z_3| - 1 = 3 - 1 = 2$. Also we see that $D_1 = \{(1, 0), (2, 0), (0, 1)\}$ is the minimum connected dominating set for $\pi(R)$, thus $\gamma_c(\pi(R)) = |Z_3| = 3$.

Remark 3.6. If R is not reduced ring, then the Theorem 3.3 above is not true in general, to show this let $R \cong Z_7 \times Z_8$, we see that the set $D = \{(0, 1), (0, 3), (0, 4), (0, 5), (0, 7)\}$ is a dominating set for $\pi(R)$, because that for any $(a, b) \in R^* - D$, there is $(0, d) \in D$ with

$$d = \left\{ \begin{array}{c} 4 : if, b \in Z(Z_8), \\ b^{-1} : if, b \in U(Z_8) \end{array} \right\} \in Z_8,$$

and we have

$$(a, b)(0, d) = (0, bd) \left\{ \begin{array}{c} (0, 0) : if b \in Z(Z_8), \\ (0, 1) : if b \in U(Z_8) \end{array} \right\} \in I(R) - \{1\},$$

Theorem 3.7. *Let R be a ring with $R \cong R_1 \times R_2 \times \cdots \times R_n$ where R_i are non-fields local rings for all $i = 1, 2, ..., n, n \in Z^*$. Then $\gamma(\pi(R)) = min_{1\leq i\leq n}(|U(R_i)|) + 1$.*

Proof. By Lemma 3.1 we get that $diam(\pi(R)) = 3$. Without lose the generality, assume that R_1 is a ring which has a minimum number of units of R_i for $i = 1, 2, ..., n$, that is $|U(R_1)| = min_{1\leq i\leq n}(|U(R_i)|)$, and let $D_1 = \{(d, 0_2, 0_3, ..., 0_n) : d \in U(R_1)\}$, since R_1 is a non-field local ring then there is $\zeta_1 \in Z(R_1)^*$ such that $\zeta_1 \cdot x_1 = 0_1$ for all $x_1 \in Z(R_1)^*$. Now, let $D = D_1 \bigcup\{(\zeta_1, 0_2, 0_3, ..., 0_n)\}$. First, we show that D is a dominating set for $\pi(R)$, let $r = (r_1, r_2, ..., r_n) \in R^* - D$, if $r_1 = 0_1$ then $rx = 0$ for all $x \in D$, then let $r_1 \neq 0_1$, if $r_1 \in U(R_1)$ then r adjacent with $(r_1^{-1}, 0_2, 0_3, ..., 0_n) \in D$, on the other hand if $r_1 \in Z(R_1)^*$ then r adjacent with $(\zeta_1, 0_2, 0_3, ..., 0_n) \in D$, then D is dominating set for $\pi(R)$. To show that D is not contains any dominating subset, let $w = (w_1, 0_2, 0_3, ..., 0_n) \in D$ such that $D - \{w\}$ is a dominating set, and let $s = (s_1, s_2, ..., s_n) \in R^* - \{D - \{w\}\}$ with $s_i \in R_i$ for $i = 1, 2, ..., n$ such that $s_1 \neq 0_1$,

w_1^{-1} where $w_1 \in U(R_1)$, and $s_1 = 1_1$ where $w_1 = \zeta_1$, we see that $ws = (w_1 s_1, 0_2, 0_3, ..., 0_n) \notin I(R) - \{1\}$. Thus $D - \{w\}$ is not dominating set which is a contradiction.

Second, we show that D is a minimum dominating set for $\pi(R)$ by contradiction, suppose that M be a dominating set for $\pi(R)$ with $|M| < |D|$, then by Proposition 2.3 there is a minimum dominating set D_1 without units such that $|M| = |D_1|$. Let $D_1 = \{(y_1, y_2, ..., y_n) : y_i \in S_i$, for $i = 1, 2, ..., n\}$ where $S_i \subseteq R_i$ for $i = 2, 3, ..., n$ such that $D_1 = \prod_{i=1}^{n} S_i = S_1 \times S_2 \times \cdots S_n$, and $R^* - D_1 = \prod_{i=1}^{n}(R_i - S_i)$. Since D_1 without units, then $U(R) \subseteq \prod_{i=1}^{n}(R_i - S_i)$. Now, let $x_i \subseteq U(R_i)$, then $x_i^{-1} \in U(R_i)$ for all $i = 1, 2, ..., n$, thus $x = (x_1^{-1}, x_2^{-1}, ..., x_n^{-1}) \in U(R) \subseteq \prod_{i=1}^{n}(R_i - S_i) = R^* - D_1$, since $x \neq D_1$ then there is $d = (d_1, d_2, ..., d_n) \in D_1$ such that d adjacent with x, since $x \in U(R)$ and $d \neq 0$ then $xd \neq 0$, also since $d \neq 0$ then there is $d_i \neq 0_i$ for at least one of i for $i = 1, 2, ..., n$, assume that $dj \neq 0_j$ for one element j of the set $\{1, 2, ..., n\}$, then we get $x_j^{-1} d_j = 1_j$, implies $x_j = d_j \in S_j$, hence $U(R_j) \subseteq S_j$, therefore

$$|U(R_j)| \leq |D_1| \tag{3.1}$$

Let $z_i \in Z(R_i)^*$, and let $z = (z_1, z_2, ..., z_n)$, if $z \in D_1$ then by (3.1) we get $|D| \leq |U(R_j)| + 1 \leq |D_1|$ which is a contradiction, then $z \in \prod_{i=1}^{n}(R_i - S_i) = R^* - D_1$, then there is $d = (d_1, d_2, ..., d_n) \in D_1$ such that d adjacent with z, since $z \in Z(R)^*$ and $d \neq 0$ then there is $d_i \in ann(z_i) - \{0_i\}$ for at least one of i for $i = 1, 2, ..., n$, since $u_i \notin ann(z_i) - \{0_i\}$ for any $u_i \in U(R_i)$ then by (3.1) we get $|D| \leq |U(R_j)| + 1 \leq |D_1| = |M|$ which is a contradiction. Therefore D is the minimum dominating set for $\pi(R)$, hence $\gamma(\pi(R)) = |D| = min_{1 \leq i \leq n}(|U(R_i)|) + 1$. □

Corollary 3.8. *Let R be a ring with $R \cong R_1 \times R_2 \times \cdots \times R_n$ where R_i are non-fields local rings for all $i = 1, 2, ..., n, n \in Z^*$. Then $\gamma_c(\pi(R)) = min_{1 \leq i \leq n}(|U(R_i)|) + 2$.*

Proof. Similar to the proof of Corollary 3.4 □

In Example 2.5 we have have $D = \{(1, 0), (2, 0), (3, 0)\}$ is a minimum dominating set for $\pi(Z_4 \times Z_4)$, thus $\gamma(\pi(Z_4 \times Z_4)) = 3$, and $D_c = \{(1, 0), (2, 0), (3, 0), (0, 1)\}$ is a minimum connected dominating set for $\pi(Z_4 \times Z_4)$, therefore $\gamma_c(\pi(Z_4 \times Z_4)) = 4$.

Remark 3.9. If $R \cong R_1 \times R_2 \times \cdots \times R_n$ where R_i are local rings with at least one of R_i is a field for $i = 1, 2, ..., n, n \in Z^*$, then the Theorem 3.7 above is not true in general, to show this let $R \cong Z_5 \times Z_8$, we see that $|U(Z_5)| = |U(Z_8)| = 4$, but the set $D = \{(1, 0), (2, 0), (3, 0), (4, 0)\}$ is a dominating set for $\pi(R)$, because that for any $(a, b) \in R^* - D$, there is $(d, 0) \in D$ with

$$d = \left\{ \begin{array}{c} 1 : if a = 0, \\ a^{-1} : if a \neq 0 \end{array} \right\}$$

and we have

$$(a, b)(d, 0) = \left\{ \begin{array}{c} (0, 0) : if a = 0, \\ (0, 1) : if a \neq 0 \end{array} \right\}$$

Hence $\gamma(\pi(Z_5 \times Z_8) \leq 4 < 5 = min_{1 \leq i \leq n}(|U(R_i)|) + 1$.

Using Theorems (3.3 and 3.7), we will give the main result for domination number, for any non local ring.

Theorem 3.10. *Let R be a non local ring with $diam(\pi(R)) = 3$, such that $R \cong R_1 \times R_2 \times \cdots \times R_n \times K_1 \times K_2 \times \cdots \times K_m$ where R_i are non-field local rings for all $i = 1, 2, ..., n$ and K_i are fields for $i = 1, 2, ..., m$, for some $n, m \in Z^+$. Then $\gamma(\pi(R)) = min\{min_{1 \leq i \leq n}(|U(R_i)|) + 1, min_{1 \leq i \leq m}(|K_i|) - 1\}$, and $\gamma_c(\pi(R)) = min\{min_1 \leq i \leq n(|U(R_i)| + 2), min_{1 \leq i \leq m}(|K_i|)\}$.*

Proof. Without lose the generality assume that $|K_1| = min_{1 \leq i \leq n}(|K_i|)$, and $|U(R_1)| = min_{1 \leq i \leq n}(|U(R_i)|)$, then we have two cases:

Case 1: $|K_1| \leq |U(R_1)|$. Then as a proof of Theorem 3.2 we can get that $\gamma(\pi(R)) = |U(K_1)| = |K_1| - 1 = min_{1 \leq i \leq n}(|K_i|) - 1$.

Case 2: $|K_1| > |U(R_1)|$. Then as a proof of Theorem 3.7 we can get that $\gamma(\pi(R)) = |U(R_1)| + 1 = min_{1\leq i\leq n}(|U(R_i)|) + 1$.

Then by cases above we get that $\gamma(\pi(R)) = min\{min_{1\leq i\leq n}(|U(R_i)|) + 1, min_{1\leq i\leq n}(|K_i|) - 1\}$. Also as a proof of Corollary 3.4 we will get $\gamma_c(\pi(R)) = min\{min_{1\leq i\leq n}(|U(R_i)| + 2), min_{1\leq i\leq n}(|K_i|)\}$. □

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Author information

Mohammed N. Authman, Husam Q. Mohammad and Nazar H. Shuker, Department of Mathematics, University of Mosul,, Iraq.
E-mail: alghabshaa@gmail.com; husamqm@uomosul.edu.iq; nazarh_2013@yahoo.com

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