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On k-Firm and (Weakly) Firm Commutative Semirings with their Total Graphs

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Abstract We generalize the notion of the (weakly) firm commutative rings to k-firm and (weakly) firm commutative semirings. A [ring, semiring] is said to be (weakly) firm if it contains an (weakly) essential prime ideal and the zero-component of each (weakly) essential prime ideal is (weakly) essential. An (weakly) essential ideal is one with nonzero intersection with every nonzero (prime) ideal. For a prime ideal P of a commutative [ring, semiring] with identity, we denote (as usual) by O_P its zero-component; that is, the set of members of P that are annihilated by non-members of P. We study semirings in which O_P is an (weakly) essential ideal whenever P is an (weakly) essential prime [resp. k-prime] ideal [resp. for the k-firmness case]. We will study some algebraic properties of k-firm and (weakly) firm semirings. We will prove that a semiring is not weakly firm (consequently, not firm) if its zero divisor set is an ideal and by an example show that the class of firm semirings is properly contained in the class of weakly firm semirings. Finally, we also observe some connections between these type of semirings and their total graphs via the set of their zero divisors.

1 Introduction

The main goal of this paper is to extend the work of the (authors [10]) and Dube [4] on *(weakly) firm commutative rings* to *(weakly) firm and k-firm* Commutative semirings (Definitions 4.3 and 5.2). We introduce the notion of a (weakly) firm and a k-firm semiring, which are the generalization of a (weakly) firm ring, and easily show (Theorem 5.8) that any semiring R is not weakly firm (consequently, not firm) provided that $Z(R)$, its set of zero divisors, is an ideal of R. Furthermore, since the definition of *the total graph of a semiring* R is mainly based on $Z(R)$, Thus, it is a natural approach to relate the *total graph* of a commutative semiring to (weakly) firm semirings via Theorem 5.8 and we will apply, in the last section (Section 6), some of the results (related to the *total graph* of a semiring) from [5] in this context.

• In [10], besides the investigation related to some algebraic properties of weakly firm rings, we study the firmness and weakly firmness of a (finite) commutative ring by applying some known results related to the zero-divisor [resp. total] graphs of commutative rings that are taken from [2] and [9] [resp. [1]], respectively.

• Notice that, besides using many results from semiring theory (which are stated in Section 2 of this paper as background), the general pattern of the proofs in this paper

($*$) are (almost) parallel to the ring case without any extra (major) assumptions on the semiring R;

($*$) are (somewhat) parallel to the ring case by assuming that (maximal, prime) ideals are subtractive and R is a [*Gelfand (in particular, simple), multiplicatively cancellative*] semiring.

Throughout the paper all [rings, semirings] are commutative with identity $1 \neq 0$, unless the contrary is explicitly stated, with $0a = 0$ for all $a \in R$. Also, $Z(R)^* = Z(R) \setminus \{0\}$ denotes the set of all nonzero zero divisors of R.

Recall that an essential ideal of a ring is one with nonzero intersection with every nonzero ideal. One of the interesting things about these ideals is that the socle of a ring, which is "built from below" by taking the union of all minimal prime ideals and then generating an ideal, can also be "built from above" by intersecting all essential ideals. In [6], the authors study the ideal obtained by intersecting all essential maximal ideals of a semi-primitive ring. They then characterize those Tychonoff spaces X for which the socle of $C(X)$ is the intersection of the essential maximal ideals.

• The work of Dube [4] was in part motivated by reading [6] which part of it is a characterization of those Tychonoff spaces X for which the socle of $C(X)$ is the intersection of the essential maximal ideals. In his work [4], besides many interesting examples, he defines (strongly) firm rings and characterizes them in terms of the lattices of their radical ideals provided that the rings have no nonzero nilpotent elements. It is shown that any proper ideal of a firm reduced ring, when viewed as a ring in its own right, is firm [resp. the classical ring of quotients of any ring (not necessarily reduced) of this kind is itself of this kind, direct products of (finitely many) rings of this kind are themselves of this kind, the ring of real-valued continuous functions on a Tychonoff space is of this kind precisely when the underlying set of the space is infinite]. It is also shown that for some (different) classes of rings, firm and strongly firm coincide.

• The organization of this paper is as follows: In Section 2, we collect some facts about commutative semirings and (undirected) graphs that are relevant to our discussion in this paper. Section 3 is devoted on some properties of the *zero-component* and *the pure part of an ideal* of a semiring analogous to the ring case. We will show, in contrast to the ring case, that the pure part of a *maximal ideal* of a semiring need not be equal to its zero-component in general and could be properly contained in its zero-component (see Examples 3.2 and 3.3). In the fourth section, we focus only on *firm and k-firm* semirings (Definition 4.3) and *compare* some of their properties with the firm rings. In Section 5, we introduce the notion of the *weakly firm* semirings (Definition 5.2) and study some of their algebraic properties. The *key result* in this section (paper) is Theorem 5.8 that excludes a class of semirings R of being (weakly) firm when $Z(R)$ is an ideal of R. Finally, the last section (Section 6) is devoted to (non)(weakly) firmness of a semiring R that are related (mainly via Theorem 5.8) to some *graph-theoretic* properties of the total graph of R.

2 Preliminaries: Semirings and Graphs

This section consists of two parts that will be relevant for our discussion, where the first and second part, respectively, provide some facts about commutative semirings and (undirected) graphs.

2.1 Semirings

Here we collect a few facts about (commutative) semirings that will be relevant for our discussion and mainly follow Golan in [7]. Notice that results in Remark 2.3 are basic in our discussion (especially when we assume ideals are subtractive) and will be used frequently in the sequel (implicitly).

By a semiring $(R, +, \cdot)$, we will mean a nonempty set R with two binary operations of addition and multiplication defined on R such that $(R, +)$ and $(R, \cdot)$ are commutative monoids with identity elements 0 and 1, respectively, where Multiplication distributes over addition (from either side) and $0a = 0$ for all $a \in R$ and $1 \neq 0$.

• A nonempty subset I of a semiring R will be called an *ideal* if $a, b \in I$ and r in R implies $a + b$ in I and ra in I.

• A *prime ideal* of a commutative semiring R is a *proper ideal* P of R in which $x \in P$ or $y \in P$ whenever $xy \in P$ for all $x, y \in R$ (see also [7, Corollary, 6.5]).

We adhere to the convention that prime ideals are assumed to be proper ideals. In general, by "ideal" (in contrast to Golan, see [7, Chapter 5] on ideals) we do not necessarily mean a proper ideal. We shall thus always say "proper ideal" when we mean a proper ideal.

Recall that a semiring R is called *reduced* if it has no nilpotent elements apart from 0. The symbols $U(R)$ and $\mathrm{Spec}(R)$ have their usual meanings; namely, the sets of units and prime, ideals of R, respectively. We shall frequently write the zero ideal simply as 0, unless it becomes necessary to write it as $\{0\}$. The annihilator of a set I will be written as $\mathrm{Ann}(I)$, and $\mathrm{Ann}(a)$ abbreviates $\mathrm{Ann}(\{a\})$.

Definition 2.1. A *subtractive ideal (= k-ideal)* I of a semiring S is an ideal such that if $a, a+b \in I$, then $b \in I$. An ideal I of S is said to be a *strong ideal* (= a *strongly k-ideal*) if and only if $a + b \in I$ implies that $a \in I$ and $b \in I$.

Remark 2.2. From the above definition, it is clear that (0) is a k-ideal of S. Also, every strongly k-ideal of a semiring S is a k-ideal of S. But the converse need not be true in general. For example, the set $2N$ of all nonnegative even integers is a subtractive ideal of the semiring of all nonnegative integers. But it is not a strongly k-ideal since $3 + 5 \in 2N$ while neither 3 nor 5 belong to $2N$. Note that in [7], Golan uses the term "subtractive ideal", [resp. strong] for a k-ideal [resp. strongly k-ideal] but in the literature of semirings, authors use equivalently the term "k-ideal" [resp. strongly k-ideal] as well. Throughout this work, we mainly follow Golan in [7]. Also, for some examples of *nonsubtractive ideals* in a semiring, see Chapter 5 of [7].

We now, for the sake of convenience, state some key facts about semirings which will use them frequently in the sequel (especially when we assume ideals are subtractive) and use them implicitly or refer to the following remark directly.

Remark 2.3. The following results in a commutative semiring hold:

(a) Any ideal in a semiring is contained in a maximal ideal ([7, Proposition 5.47].

(b) Any maximal ideal in a semiring is prime ([7, Corollary 6.11]).

(c) Any subtractive ideal in a semiring is contained in a maximal subtractive ideal ([11, Corollary 2.2]).

(d) A maximal subtractive ideal is prime ([11, Proposition 3.1]).

We now recall some definitions from [7] and write them here for the sake of completeness as follows.

• A semiring R is *zerosumfree* if and only if $r + r' = 0$ implies that $r = r' = 0$. A semiring with no nonzero zero divisors is called an *entire (= semidomain)*, i.e., for any $a, b \in R$ with $ab = 0$, then either $a = 0$ or $b = 0$. A *semifield* is a semiring in which every nonzero element has a *multiplicative inverse*. A semiring R is said to be *simple* if $1 + r = 1$ for each $r \in R$. Let R be a semiring and $G(R) = \{r \in R \mid 1 + r \in U(R)\}$. A semiring R is called a *Gelfand semiring* when $G(R) = R$. Clearly, every simple semiring is Gelfand. Of course, bounded distributive lattices are among Gelfand semirings. But the class of the Gelfand semirings is quite wider as [7, Example 3.38] shows.

2.2 Graphs

Let G be a simple graph. We say that G is *connected* if there is a *path* between any two distinct vertices of G. For vertices x and y of G, we define $d(x, y)$ to be the length of a shortest path from x to y ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such path). The *diameter* of G is $\mathrm{diam}(G) = \sup\{d(x, y) \mid x \textit{ and } y \text{ are vertices of } G\}$. The diameter is 0 if the graph consists of a single vertex and a connected graph with more than one vertex has diameter 1 if and only if it is *complete*; i.e., each pair of distinct vertices forms an *edge*. The *girth* of G, denoted by gr(G), is the length of a shortest *cycle* in G (gr$(G) = \infty$ if G contains no cycles). We denote the complete graph on n vertices by K_n and the *complete bipartite graph* on m and n vertices by $K_{m,n}$ (we allow m and n to be infinite cardinals). We will sometimes call a $K_{1,n}$ a *star graph*.

A general reference for graph theory is [3] or any standard text in graph theory. Also, the reader can refer to [2, 1, 5] for all necessary definitions that are related to graphs in this paper.

3 The Zero-component and Pure Part of an Ideal

In this section we study some properties of the zero-component and the pure part of an ideal of a semiring analogous to the ring case. We will show, in contrast to the ring case, that the pure part of a maximal ideal of a semiring need not be equal to its zero-component in general and could be properly contained in its zero-component (see Examples 3.2 and 3.3). Note that the notion of the pure part of an ideal in a ring was defined in [4] in order to study *strongly firm rings*, but we won't discuss strong firmness of a semiring in this paper.

• Let P be a prime ideal of a semiring R. The *zero-component* of P, denoted O_P, is defined by

$$O_P = \{a \in P \mid ab = 0 \text{for some } b \in R \setminus P\}.$$

Observe that O_P is an ideal *consisting entirely of zero-divisors*. Let $a, b \in O_P$. Thus $ax = 0$ and $by = 0$ for some $x, y \in R \setminus P$. Clearly, O_P is an ideal of R since $(a + b)xy = 0$ and $(ra)x = r(ax) = 0$ for any $r \in R$ (note that $xy \notin P$ since P is prime). Actually, O_P is a subtractive ideal of R since $a + b$ and b in O_P implies $(a + b)x = 0$ and $by = 0$ for some $x, y \notin P$, and hence $(a + b)(xy) = 0 = a(xy)$ which implies $a \in O_P$.

• The *pure part* of an ideal I of a semiring R, denoted mI, is the ideal

$$mI = \{a \in I \mid a + ab = 0 \text{ for some} b \in I\} \subseteq \bigcup\{\text{Ann}(1 + x) \mid x \in I\},$$

and *equality holds for subtractive ideals*.

Remark 3.1. From the above definition, it is clear that the pure part of any ideal in a zerosumfree [resp. Gelfand (in particular, simple)] semiring is zero.

It is not difficult to show that mI is an ideal of R. Let $a, b \in mI$, thus $a + ax = 0$ and $b + by = 0$ for some $x, y \in I$. Now $(a + b) + (a + b)(x + y + xy) = 0$ and $(ra) + (ra)x = 0$ for any $r \in R$.

Observe that the containment $mP \subseteq O_P$ holds for every subtractive prime ideal P since for any $a, b \in P$, $a + ab = a(1 + b) = 0$, we have $1 + b \notin P$ by subtractiveness of P. On the other hand, for any maximal ideal M of a ring we have $mM = O_M$, but this is not true for semirings in general (see Examples 3.2 and 3.3). Indeed (as shown in Section 2.1 of [4]) for the case of a ring R, let $a \in O_M$, and take $b \notin M$ such that $ab = 0$. Since M is a maximal ideal, there exist $c \in M$ and $d \in R$ such that $1 = c + db$. Then $a = a(c + db) = ac$, which shows that $a \in mM$.

We now, in contrast to the ring case, define a semiring R to show that mM is properly contained in O_M for a maximal (subtractive) ideal M of R. Note that by Remark 2.3, every maximal (subtractive) ideal in a commutative semiring with identity is prime.

Example 3.2. (cf. [7, Example 5.31]) A semiring S is said to be idempotent if it is both additively and multiplicatively idempotent. Consider the idempotent semiring $S = \{0, 1, a\}$ in which $1 + a = a + 1 = a$. Let $R = S \times S$ be the direct product of the semirings. Clearly, $M = \{0, a\} \times S = \{(0, 0), (0, 1), (a, 1), (a, 0), (0, a), (a, a)\}$ is a non-subtractive ideal of R since $(1, 0) + (a, 0) = (a, 0) \in M$ but $(1, 0) \notin M$. Also M is a maximal ideal in R since for any ideal $N \supseteq M$, either $(1, 0) \in N$ or $(1, a) \in N$, which either $N = R$ or n is not an ideal when $1, 0) \notin N$ and $(1, a) \in N$. Clearly, a proper nonzero ideal I of R is not subtractive if it contains (a, a) since $(1, 1) + (a, a) = (a, a)$ but $(1, 1) \notin I$. On the other hand, a nonzero subtractive ideal

I of R must contain either $(1,0)$ or $(0,1)$, otherwise, $I = (0,0)$. In this case, I contains $(a,0)$ or $(0,a)$. Now, it is vacuously true that $B = \{0\} \times S$ and $C = S \times \{0\}$ are the only maximal subtractive ideals of R. Clearly, $mB = (0,0)$ is properly contained in $O_B = B$. Also for the maximal ideal M, we have $mM = \{(0,0)\}$ and $O_M = \{(0,0),(0,1),(0,a)\}$.

We end this section by constructing a semiring with a unique nonsubtractive maximal ideal [resp. maximal subtractive ideal] whose both the zero-component and pure part equal to zero.

Example 3.3. Let R be the collection of elements of the form $a + b\alpha + c\beta$, where $a,b,c \in B = \{0,1\}$ (the Boolean semiring, i.e. $1+1=1$, which is different from $\mathbb{Z}_2$) and $\alpha\beta = \beta\alpha = \alpha^2 = \beta^2 = 0$. Clearly R contains eight elements and is a local semiring with the unique nonsubtractive maximal ideal $M = R \setminus \{1\}$ and $J = \{0,\alpha,\beta,\alpha+\beta\}$, which is the only maximal subtractive ideal of R. Now it is easy to see that $mJ = O_J = 0$ and $mM = 0 = O_M$.

4 Firm and k-Firm Semirings

In this section, we extend some of the results in [4] (which is the study of *firm Commutative rings*) to *Commutative semirings in which zero-components of essential (subtractive) primes are essential*, which is the study of *firm and k-firm Commutative semirings* [Definition 4.3].

Recall that an ideal of a ring R is said to be *essential* if it has nonzero intersection with every nonzero ideal of R. If I is an ideal of R and $\mathrm{Ann}(I) = 0$, then I is essential. For reduced rings, an ideal is essential if and only if its annihilator is 0. We will show that the similar result is also true for reduced commutative semirings as well (Proposition 4.2).

Definition 4.1. An ideal I of a commutative semiring R is *essential* if it has nonzero intersection with every nonzero ideal of R.

We now write the condition for the essentialness of an ideal in a reduced semiring.

Proposition 4.2. *Let R be a commutative semiring, then The following conditions hold:*

(a) If I is an ideal of R and $Ann(I) = 0$, then I is essential.

(b) For reduced semirings, an ideal is essential if and only if its annihilator is 0.

Proof. (a) Suppose that I is not essential. Then there exists a nonzero ideal J such that $I \cap J = 0$. Thus $IJ = 0$ since $IJ \subseteq I \cap J$ and hence I has a nonzero annihilator, yielding a contradiction.

(b) Suppose that I is essential and $\mathrm{Ann}(I) \neq 0$. In this case, $0 \neq a \in I \cap \mathrm{Ann}(I)$ implies $a^2 = 0$ and hence by assumption $a = 0$, yielding a contradiction. □

Definition 4.3. A commutative semiring R is *firm* [resp. *k-firm*] if it has an essential [resp. subtractive] prime ideal, and O_P is essential whenever P is an essential [resp. subtractive] prime ideal in R. On the other hand, we say R is *anti-firm* [resp. *anti k-firm*] if it has an essential [resp. subtractive] prime ideal P for which O_P is not essential. A semiring can of course fail to have an essential prime ideal (for instance any semifield), so whenever we assert that a particular semiring is firm [resp. k-firm] we will need to demonstrate that it actually does have an essential [resp. subtractive] prime ideal.

We now write an example of an anti-firm semiring (see Examples 3.2 and 3.3).

Example 4.4. In Example 3.2, M is an essential maximal ideal of R (and hence prime (by Remark 2.3 since it is maximal) but $O_M = B$, which is not essential since $B \cap C = (0,0)$.

Remark 4.5. If R is a firm semiring which contains an essential subtractive prime ideal, then R is also k-firm since an essential subtractive prime ideal is also an essential prime ideal. On the other hand, R is anti-firm if it is anti k-firm. Note that firm and k-firm coincide for the case of a ring since each ideal in a ring is subtractive.

If we assume the Axiom of Choice (as we shall do whenever we need it), then a semiring has an essential [resp. subtractive] prime ideal if and only if it has an essential [resp. subtractive] maximal ideal (see Remark 2.3). A semiring with no essential [resp. subtractive] prime ideal is neither firm nor anti-firm [resp. neither k-firm nor anti k-firm].

Our next result (Proposition 4.8) shows that every (subtractive) ideal in a reduced (k-) firm semiring is (k-) firm when viewed as a semiring in its own right. Incidentally, this will be the only instance where the presence of the identity is not assumed. Note that in [7], Golan uses the term "hemiring" for a semiring with no multiplicative identity.

We provide the following two lemmas for the proof of the next result (Proposition 4.8).

Lemma 4.6. (cf. [8, Theorem 5.1]) *Let I be any proper ideal of a semiring R, then* $\mathrm{Spec}(I) = \{Q \cap I \mid Q \in \mathrm{Spec}(R) and I \nsubseteq Q\}$.

Moreover, if I is a proper subtractive ideal of R, then $\mathrm{Spec}_k(I) = \{Q \cap I \mid Q \in \mathrm{Spec}_k(R)$ *and* $I \nsubseteq Q\}$, *where $Spec_k(I)$ and $Spec_k(R)$ are the sets of the subtractive prime ideals of I and R, respectively.*

Proof. It suffices to show $\mathrm{Spec}(I) \subseteq \{Q \cap I \mid Q \in \mathrm{Spec}(R)$ and $I \nsubseteq Q\}$ [resp. $\mathrm{Spec}_k(I) \subseteq \{Q \cap I \mid Q \in \mathrm{Spec}_k(R)$ and $I \nsubseteq Q\}$] since the reverse inclusion is clear (note that the intersection of subtractive ideals is subtractive). Let $P \in \mathrm{Spec}(I)$ and $Q = \{a \in R \mid Ia \subseteq P\}$. We show that Q is a prime ideal in R (i.e. $Q \in \mathrm{Spec}(R)$) and $P = Q \cap I$. Let $a, b \in Q$. Then $I(a + b) \subseteq Ia + Ib \subseteq P$, whence $a + b \in Q$. For every $x \in R$, we have $Ixa \subseteq Ia \subseteq P$, which implies $xa \in Q$. Therefore Q is an ideal of R. Now let $c, d \notin Q$ be arbitrary. Then there exist $i, j \in I$ such that $ic, jd \notin P$. Since $P \in \mathrm{Spec}(I)$, and $ic, jd \in I$ such that $(ic)(jd) = (ij)cd \notin P$, thus Icd not subseteq P, so cd notin Q. Hence $Q \in \mathrm{Spec}(R)$. Finally, if $p \in P$, then $Ip \subseteq P$, so $p \in Q \cap I$ implies $P \subseteq Q \cap I$. Now if $k \in Q \cap I$ and $k \in I \setminus P$, then $Ik \subseteq P$ implies $I \subseteq P$ since $P \in \mathrm{Spec}(I)$, yielding a contradiction. Thus, $P = Q \cap I$. For the proof of "moreover part", it suffices to show that Q is subtractive, which easily follows since $P \in \mathrm{Spec}_k(I)$ is subtractive. □

Lemma 4.7. *Let I be any proper (subtractive) ideal of a reduced semiring R. If K is an essential (subtractive) ideal in R, then $I \cap K$ is an essential (subtractive) ideal in I.*

Proof. Since I is a reduced semiring (hemiring), it suffices to show that the only element of I that annihilates $I \cap K$ is 0. The reason is that in a reduced semiring, an ideal is essential if and only if it is annihilated by 0 only (Proposition 4.2). So suppose u is an element of I such that $ut = 0$ for every $t \in I \cap K$. Let $k \in K$. Then $uk \in I \cap K$, and so $u(uk) = 0$, which implies $(uk)^2 = 0$ and hence $uk = 0$ since R is reduced. Thus, $uK = \{0\}$, which implies $u = 0$ since K is an essential ideal in a reduced semiring (Proposition 4.2). The proof for "subtractive case" follows directly from this result since the intersection of two subtractive ideals is subtractive. □

Proposition 4.8. *Let R be a reduced firm [resp. k-firm] semiring. If I is an [resp. subtractive] ideal of R, then I is firm [resp. k-firm] when viewed as a semiring.*

Proof. It is clear from 4.6 that

$$\mathrm{Spec}_{[k]}(I) = \{P \cap I \mid P \in \mathrm{Spec}_{[k]}(R) \text{ and } I \nsubseteq P\},$$

where $\mathrm{Spec}_{[k]}$ stands for "Spec" or "Spec_k", respectively. Observe that, by Lemma 4.7, if K is an essential (subtractive) ideal in R, then $I \cap K$ is an essential (subtractive) ideal in I since R is reduced. So, I does have an essential (subtractive) ideal since R has (by hypothesis). Now let Q be an essential (subtractive) prime ideal in I. Pick $P \in \mathrm{Spec}(R)$ [resp. $P \in \mathrm{Spec}_k(R)$] such that $I \nsubseteq P$ and $Q = P \cap I$. We claim that P is essential in R. Let J be an ideal of R with $J \cap P = 0$. Then $Q \cap (I \cap J) = P \cap I \cap J = 0$, which implies $I \cap J = 0$ since it is an ideal of I with zero intersection with the essential ideal Q of I. Since P is a prime ideal in R and $I \cap J \subseteq P$, it follows that $J \subseteq P$ since $I \nsubseteq P$. Thus, $J = 0$, showing that P is essential. So, by hypothesis, O_P is an essential ideal in R, making $O_P \cap I$ essential in I (Lemma 4.7). We claim that $O_P \cap I \subseteq O_Q$. Let $x \in O_P \cap I$. Take $u \in R \setminus P$ with $ux = 0$. Since $I \nsubseteq P$, there exists $v \in I \setminus P$. Since P is a prime ideal of R missing both u and v, we have $uv \in I \setminus Q$. Since

$(uv)x = 0$, it follows that $x \in O_Q$. Thus, the essential ideal $O_P \cap I$ of I is contained in O_Q, which makes O_Q essential in I (note that $A \subseteq B$ implies $A \cap C \subseteq B \subseteq C$ for all sets A, B, and C). Therefore I is firm [resp. k-firm]. □

Corollary 4.9. *If R is a reduced firm semiring containing an essential subtractive prime ideal, then every subtractive ideal of R is k-firm when viewed as a semiring.*

Proof. The proof follows directly from the above proposition and Remark 4.5 which states that R is a k-firm semiring. □

The following proposition excludes the firmness of a class of semirings, namely the multiplicatively cancellative semirings. A semiring R is called *multiplicatively cancellative* if whenever $rs = rt$ for elements $r, s, t \in R$ with $r \neq 0$, then $s = t$.

Proposition 4.10. *A multiplicatively cancellative semiring is not firm.*

Proof. Suppose to the contrary that R is a firm semiring with an essential prime ideal P. Thus, by Remark 2.3, P is contained in a maximal ideal M. Clearly M is an essential ideal of R since P is essential. Hence $0 \neq a \in O_M$ implies $ab = 0$ for some $b \in R \setminus M$. Thus $R = (M, b)$ implies $1 = m + rb$ for some $m \in M$ and $r \in R$, which implies $a = am + 0$. Consequently, by multiplicatively cancellative property of R, $1 = m \in M$, yielding a contradiction. □

Remark 4.11. The above proposition can be proved directly since for any nonzero $a \in M$ and $b \notin M$, $ab = 0 = a0$ implies $b = 0$, yielding a contradiction. On the other hand, $O_M = 0$ is not essential. Note that this result is immediate since a multiplicatively cancellative semiring is entire (= semidomain).

We now write two examples of multiplicatively cancellative semirings that consequently, by the above proposition, are not firm.

Example 4.12. The following two examples are taken from [7].

(a) (cf. [7, Example 3.28]) The semiring N of nonnegative integers is a multiplicatively cancellative semiring which is not a division semiring. Indeed, $U(N) = \{1\}$.

(b) (cf. [7, Example 3.29]) If R is a Noetherian commutative integral domain then the additively-idempotent semiring ideal(R) is multiplicatively cancellative if and only if R is a Prufer domain. More generally, a commutative integral domain R is a Prufer domain if and only if every finitely-generated nonzero ideal of R is multiplicatively cancelable.

In the following two propositions, an easy criterion shows that in order to check whether a semiring is firm [resp. k-firm] we need only limit to maximal [resp. subtractive] ideals.

Proposition 4.13. *Consider the following equivalent conditions on a semiring R which has an essential ideal.*

(a) R is firm.

(b) O_M is essential for every essential maximal ideal M of R.

Proof. $(a) \Leftrightarrow (b)$: The left-to-right implication is trivial because our blanket assumption is that all semirings have the identity, so that maximal ideals are prime and every ideal is contained in a maximal ideal (Remark 2.3). Conversely, suppose O_M is essential for every essential maximal ideal M. Let P be an essential prime ideal of R. Pick a maximal ideal M with $M \supseteq P$. Then M is essential, and hence O_M is essential by the present hypothesis. But $O_M \subseteq O_P$, so O_P is essential. Therefore R is firm. □

We now discuss the above results for the essential (maximal) subtractive ideals.

Proposition 4.14. *Consider the following equivalent conditions on a semiring R which has an essential subtractive ideal.*

(a) R is k-firm.

(b) O_M is essential for every essential maximal subtractive ideal M of R.

Proof. $(a) \Leftrightarrow (b)$: The left-to-right implication is trivial because our blanket assumption is that all semirings have the identity, so that maximal subtractive ideals are prime and every subtractive ideal is contained in a maximal subtractive ideal (Remark 2.3). Conversely, suppose O_M is essential for every essential maximal subtractive ideal M. Let P be an essential subtractive prime ideal of R. By Remark 2.3, pick a maximal subtractive ideal M with $M \supseteq P$. Then M is essential, and hence O_M is essential by the present hypothesis. But $O_M \subseteq O_P$, so O_P is essential. Therefore R is k-firm. □

Let us now give some examples of k-firm and anti-firm semirings.

Example 4.15. (a) We define a commutative semiring R to be von Neumann regular if and only if for any $a \in R$, there exists $b \in R$ such that $a + a^2 b = 0$. Now, by this definition, every von Neumann regular semiring with at least one essential subtractive ideal is k-firm because every subtractive ideal of R is contained in a maximal subtractive ideal (Remark 2.3) and $O_P = P$ for any subtractive prime ideal P since $(1 + ab) \notin P$ for any $a \in P$. Note that a maximal [resp. subtractive] ideal in a semiring is also prime by Remark 2.3.

(b) A semidomain R is never firm [resp. k-firm] since $O_P = 0$ for any prime ideal P of R. It is anti-firm [resp. anti k-firm] if and only if it is not a semifield. Clearly, by Remark 2.3, R (not a field) has a nontrivial maximal [resp. subtractive] ideal M, which is essential (by Proposition 4.2) since $aM = 0$ (i.e., $\mathrm{Ann}(M) = 0$) and hence $a = 0$; and obviously $O_M = 0$.

(c) In [7, Proposition 5.49], it states that an element a of a semiring R is a unit if and only if a belongs to no maximal ideal of R. Thus, the local [resp. k-local] semiring in Example 3.3 is anti-firm [resp. anti k-firm]. A semiring is local [resp. k-local] if it has a unique maximal [resp. subtractive] ideal. Clearly, a local semiring is never firm ($O_M = 0$ for the unique maximal ideal M); and it is anti-firm if and only if it has at least one nonzero nonunit.

• We now show that a McCoy semiring is not firm. Similar to the literature in commutative rings, a semiring R is said to be *McCoy [resp, countably McCoy]* if each finitely [resp, countably] generated ideal $I \subseteq Z(R)$ has a nonzero annihilator.

Proposition 4.16. *Let R be a reduced McCoy [resp, countably McCoy] semiring. If R contains an essential prime ideal P such that O_P is finitely [resp, countably] generated ideal, then R is not firm (it is actually anti-firm).*

Proof. The result follows since $O_P \subseteq Z(R)$ has a nonzero annihilator by the assumption and hence is not essential, by Proposition 4.2, since R is reduced by hypothesis. □

We end this section with the following remark which is related to the anti-firmness of the product of two semirings.

Remark 4.17. See Proposition 4.2 for the assertion. If R is a reduced anti-firm [resp. anti k-firm] semiring, then $R \times S$ is anti-firm [resp. anti k-firm] for any semiring S. To see this, let M be an essential maximal [resp. subtractive] ideal of R for which O_M is non-essential. Then $M \times S$ is an essential maximal [resp. subtractive] ideal of $R \times S$, but for any nonzero $a \in R$ that annihilates O_M, the nonzero element $(a, 0)$ of $R \times S$ annihilates $O_{M \times S}$.

5 Weakly Firm Semirings

The main purpose in this section (Theorem 5.8), is to provide an example to exclude a class of semirings R of being (weakly) firm when the set of zero divisors of R is an ideal of R. In the next section, we will use this result to relate the (weakly) firm semirings to their total graphs. We also show that the class of firm semirings is properly contained in the class of weakly firm semirings (Examples 5.3 and 5.4).

Definition 5.1. An ideal I of a commutative semiring R is *weakly essential* if it has nonzero intersection with every nonzero prime ideal of R.

Definition 5.2. A semiring R is weakly firm if it has a weakly essential prime ideal and the zero-component of every weakly essential prime is weakly essential. On the other hand, we say R is *weakly anti-firm* if it has a weakly essential prime ideal P for which O_P is not weakly essential.

Clearly, each firm semiring is weakly firm by definition. In the following two examples we show that the class of firm semirings is properly contained in the class of weakly firm semirings.

Example 5.3. Let $R = B \times B \times B$, where each B is a Boolean semiring. Then R is a weakly firm semiring which is not firm. Let $M_1 = B \times B \times \{0\}$, $M_2 = B \times \{0\} \times B$, $M_3 = \{0\} \times B \times B$, $I_1 = \{0\} \times \{0\} \times B$, $I_2 = \{0\} \times B \times \{0\}$, and $I_3 = B \times \{0\} \times \{0\}$. Clearly, M_i's are the only maximal (weakly essential prime) ideals of R and neither of I_i's is prime or essential. Note that $Z(R)$ is not an ideal of R.

Example 5.4. Suppose $R = R_1 \times R_2 \times \cdots \times R_n$, where each R_i is a semifield ($1 \le i \le n$). Then

(a) R is not firm for any finite $n \ge 1$.

(b) R is not (weakly) firm if $n = 1$ or 2.

(c) R is a nonfirm weakly firm semiring if $n \ge 3$ (see also Example 5.12).

The following lemma will be used in the sequel implicitly whenever $Z(R)$ is not an ideal of R.

Lemma 5.5. *Let R be a semiring such that $Z(R)$ is not an ideal of R. Then there are distinct $r, r' \in Z(R)^*$ such that $r + r' \in \mathrm{Reg}(R)$ and hence $|Z(R)| \ge 3$.*

Proof. It is enough to show that $Z(R)$ is closed under scalar multiplication of its elements by elements of R since $Z(R)$ is not an ideal by hypothesis. Let $a \in Z(R)$ and $r \in R$. There is a non-zero element $s \in R$ with $sa = 0$; hence $s(ra) = r(sa) = 0$. Thus ra is in $Z(R)$. This completes the proof. □

For the proof of the next theorem, we need the following two lemmas, where the theorem provides an example of a class of nonweakly firm semirings, which obviously is a class of non-firm semirings. Namely, those semirings whose each set of zero divisors is an ideal. Note that $Z(R)$, the set of zero divisors of a semiring R, is an ideal of R when it is closed under addition).

Lemma 5.6. *Let $Z(R)$ be the set of zero divisors of a semiring R. Then $xy \in Z(R)$ for $x, y \in R$ implies $x \in Z(R)$ or $y \in Z(R)$. So if $Z(R)$ is an ideal of R, then $Z(R)$ is actually a prime ideal of R.*

Proof. Suppose $xy \in Z(R)$ and $y \ne 0$. Thus, by definition, there exists $0 \ne z \in R$ such that $(xy)z = x(yz) = 0$. Now the result follows when $yz \ne 0$ or $yz = 0$. □

Lemma 5.7. *Let R be a semiring whose set of zero divisors $Z(R) \ne 0$ is a nonzero ideal of R and suppose that I is a nonzero ideal of R. Then I and $Z(R)$ have a nonzero intersection and hence $Z(R)$ is essential.*

Proof. Suppose that I is not contained in $Z(R)$ and $b \in I \setminus Z(R)$. Clearly, if $I \cap Z(R) = 0$, then $ab = 0$ for each nonzero $a \in Z(R)$, which implies b is a zero divisor and hence yielding a contradiction. □

We are now in a position to prove our main result in this section, which has an epistemological value and easily with a simple argument provides a sharp classification of commutative semirings related to their nonweakly firmness.

Theorem 5.8. *Let $Z(R)$ be the set of zero divisors of a commutative semiring R such that $P = Z(R)$ is an ideal of R. Then R is not weakly firm and consequently, not firm. Further, if $Z(R) \ne 0$, then R is weakly anti-firm.*

Proof. Obviously, if R is a semidomain, then the result is immediate since $O_P = 0$ for any prime ideal of R. Now, the proof follows directly from the above two lemmas since $Z(R)$ is a prime ideal by hypothesis and thus has a nonzero intersection with any nonzero prime ideal and hence is weakly essential prime by definition. Clearly, $O_P = 0$, which is obviously not (weakly) essential and for the further part, see Definition 5.2. □

Remark 5.9. Note that the converse of the above theorem need not be true in general. That is, there are some examples of (nonweakly firm) nonfirm semirings whose set of zero divisors is not an ideal (see for example, Examples 5.3 and 5.4). Also, in order to check that $Z(R)$ is not an ideal of R, it suffices to show That $x + y \notin Z(R)$ for some distinct elements $x, y \in Z(R)$ (i.e., $x + y$ is a regular element of R) since $Z(R)$ is always closed under multiplication by elements of R (see Lemma 5.5). For example, the set of the zero divisors of the direct product of unital semirings with more than one factor is not an ideal.

It is possible for a semiring not to have an (weakly) essential prime ideal (see Examples 5.3 and 5.4). If we assume the Axiom of Choice (as we shall do whenever we need it), then a semiring has an (weakly) essential prime ideal if and only if it has an (weakly) essential maximal ideal (see Remark 2.3). A semiring with no (weakly) essential prime ideal is neither (weakly) firm nor (weakly) anti-firm.

• In contrast to [4, Proposition 3.1] that states every ideal in a reduced firm ring is firm (when viewed as a ring in its own right), we show in the following example that this is not true for reduced weakly firm semirings in general.

Example 5.10. (cf. [4, Proposition 3.1]) Let $R = R_1 \times R_2 \times R_3$ be a finite reduced semiring as defined in Example 5.4, where each R_i is a semifield $(1 \leq i \leq 3)$ and let $I = R_1 \times R_2 \times \{0\}$. Clearly, by Example 5.4, R is weakly firm but I is not.

The following easy criterion shows that in order to check whether a semiring is (weakly) firm we need only limit to maximal ideals.

Proposition 5.11. (cf. [4, Proposition 3.2]) *The following two conditions are equivalent for a semiring R which has an (weakly) essential ideal.*

(a) A is (weakly) firm.

(b) O_M is (weakly) essential for every (weakly) essential maximal ideal M of R.

Proof. The proof is similar to the proof of [4, Proposition 3.2]. $(a) \Leftrightarrow (b)$: The left-to-right implication is trivial because our blanket assumption is that all rings have the identity, so that maximal ideals are prime. Conversely, suppose O_M is (weakly) essential for every (weakly) essential maximal ideal M. Let P be an (weakly) essential prime ideal of R (it exists by the assumption and the fact that any ideal is contained in a maximal (prime) ideal) and clearly, if $I \subseteq J$ is (weakly) essential, then J is (weakly) essential by definition. Pick a maximal ideal M with $M \supseteq P$. Then M is (weakly) essential, and hence O_M is (weakly) essential by the present hypothesis. But $O_M \subseteq O_P$, so O_P is (weakly) essential. Therefore R is (weakly) firm. □

We now end this section with three examples of nonweakly firm and anti-firm semirings.

Example 5.12. (cf. [4, Examples 3.1])

(a) A semidomain is never weakly firm.

(b) A local semiring R is never (weakly) firm since $O_M = 0$ for its unique maximal ideal M; and it is weakly anti-firm if and only if it has at least one nonzero nonunit element.

(c) The local semiring N of nonnegative integers with the unique maximal ideal $M = N \setminus \{1\}$ is not weakly firm and it is weakly anti-firm since M is (weakly) essential. Indeed, $U(N) = \{1\}$ is the only unit of N.

6 Total Graphs and (Weakly) Firmness

In this section, we study some graph-theoretic properties of a (finite, weakly) firm semiring R by applying some known results related to the total graphs of commutative semirings that are taken (mainly) from Section 3 of [5]. That is, we relate (apply) some of the results of Section 3 of [5] to (weakly) firm semirings when $Z(R)$ is not an ideal of R which is a consequence of the (weakly) firmness of R by Theorem 5.8 above.

The total graph of a commutative ring R, denoted by $T(\Gamma(R))$, has been introduced and studied by D. F. Anderson and A. Badawi in [1]. It is the (undirected) graph with all elements of R as vertices, and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in Z(R)$. Their work is (mainly) divided into two cases depending on whether or not $Z(R)$ is an ideal of R ([1, Sections 2 and 3]), respectively. They also study the three (induced) subgraphs Nil$(\Gamma(R))$, $Z(\Gamma(R))$, and Reg$(\Gamma(R))$ of $T(\Gamma(R))$, with vertices Nil(R), $Z(R)$, and Reg(R). Where Nil(R) is the ideal of nilpotent elements, $Z(R)$ is the set of zero-divisors, and Reg(R) is the set of regular elements of R, respectively.

• As a natural extension of the total graph of a commutative ring, S. Ebrahimi and F. Esmaeili [5] study the notion of the total graph of a commutative semiring R. Hence, the total graph of R, denoted by $T(\Gamma(R))$, is the (undirected) graph with all elements of R as vertices, and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in Z(R)$, where $Z(R)$ is the set of zero-divisors of R. Consequently, similar to the work of D. F. Anderson and A. Badawi in [1], the study of the total graph of a commutative semiring R breaks naturally into two cases depending on whether or not $Z(R)$ is an ideal of R.

We now begin with some examples of nonweakly firm semirings.

Example 6.1. Examples 4.4 and 4.5 in Section 4 of [5] are two examples of semirings R with $Z(R)$ an ideal of R and hence providing examples of nonweakly firm semirings by Theorem 5.8.

Example 6.2. Let R be a commutative semiring such that $Z(\Gamma(R))$ is complete. Then R is not weakly firm.

Proof. The result follows from Theorem 5.8 since $Z(R)$, as an implication of the hypothesis, is an ideal of R. □

• The rest of this section is devoted on the case when $Z(R)$ is not an ideal of R which is a consequence of weakly firmness of R. Since $Z(R)$ is always closed under multiplication by elements of R, this just means that there are distinct $x, y \in Z(R)^*$ such that $x + y \in$ Reg(R). In this case, $|Z(R)| \geq 3$ (see Lemma 5.5).

We next, for a (weakly) firm semiring R, determine the connectedness and diameter of $Z(\Gamma(R))$.

Theorem 6.3. (cf. [5, Theorem 3.4]) *Let R be a (weakly) firm commutative semiring. Then $Z(\Gamma(R))$ is connected with diam$(Z(\Gamma(R))) = 2$.*

Proof. The result follows from [5, Theorem 3.4] since $Z(R)$ is not an ideal of R by Theorem 5.8. □

We now conclude the paper with the following two results related to the girth of $Z(\Gamma(R))$ and Reg$(\Gamma(R))$, respectively, when R is (weakly) firm and hence $Z(R)$ is not an ideal of R. Recall that $|Z(R)| \geq 3$ if $Z(R)$ is not an ideal of R.

Theorem 6.4. (cf. [5, Theorem 3.10]) *Let R be a (weakly) firm commutative semiring. Then either gr$(Z(\Gamma(R))) = 3$ or gr$(Z(\Gamma(R))) = \infty$.*

Proof. The result follows from [5, Theorem 3.10] since $Z(R)$ is not an ideal of R by Theorem 5.8. □

Theorem 6.5. (cf. [5, Theorem 3.12]) *Let R be a (weakly) firm commutative semiring. Then gr(Reg$(\Gamma(R))) = 3$ or ∞.*

Proof. The result follows from [5, Theorem 3.12] since $Z(R)$ is not an ideal of R by Theorem 5.8. □

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