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ON PRINCIPALLY μ- LIFTING MODULES

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Abstract Let R be an arbitrary ring with identity and let M be a left R - module. In this paper, I introduce a class of modules analogous to that of principally lifting modules. The module M is called principally μ-lifting module if for every cyclic submodule A of M, there exists a submodule D of A such that $M = D \oplus D', D' \leq M$ and $A \cap D' <_\mu D'$. I also introduce a generalization of μ-hollow modules, namely, a module M is said to be principally μ-hollow module if every proper cyclic submodule of M is μ-small in M. I prove some results of principally lifting modules can be extended to principally μ lifting modules for this general

1 Introduction

Through this paper, R denotes an associative ring with identity, and all modules are unital left R - modules. A submodule A of M is called μ-small in M if, for every [2] submodule K of M, the equality $M = A + K$ such that $\frac{M}{K}$ is cosingular implies $M = K$. Clearly that every small submodule is μ - small in M. A module M is called μ-hollow module if every proper submodule of M is μ-small in M, see [1]. A submodule P is a μ-supplement of N in M if $M = P + N$ and $P \cap N$ is μ-small in P, while M is called μ supplemented if every submodule of M has a μ-supplement in M, see [2]. A module M is called μ-lifting module if for every submodule A of M, there exists a submodule D of A such that $M = D \oplus D', D' \leq M$ and $A \cap D' <_\mu D'$, see [3]. In this paper, we define principally μ-hollow modules as a generalization of μ-hollow modules. Also as generalization of μ-lifting modules , I introduce the concept of principally μ-lifting module as follows: An R-module M is called principally μ-lifting module if for every cyclic submodule A of M, there exists a submodule D of A such that $M = D \oplus D', D' \leq M$ and $A \cap D' \ll_\mu D'$.

This paper is organized as follows: Section two is devoted to define principally $\mu-$ hollow modules and the basic properties of these modules that will be used in the sequel.

In section three, the notion of principally μ-lifting modules is introduced. It is shown that every μ-lifting module is principally μ-lifting. I give an example to show that the reverse implication need not hold in general.

In what follows, by Z, Q and $\frac{Z}{nZ}$ we denote, respectively, integers, rational numbers, and the ring of integers and the Z - module of integers modulo n.

2 Principally μ-hollow modules.

In this section, I introduce the concept of principally $\mu - 0$ hollow modules and investigate the basic properties of these modules.

Definition 2.1. A nonzero R-module M is called principally μ-hollow module if every proper cyclic submodule of M is μ-small in M.

1- Z_4 as Z-module is principally μ-hollow module.

2- Z_6 as Z-module is not principally μ-hollow, since $\{\overline{0}, \overline{3}\}$ and $\{\overline{0}, \overline{2}, \overline{4}\}$ are not μ small in Z_6.

3- Z as Z-module is not principally μ-hollow because nZ is not μ-small in Z for each positive integer n.

4- Every simple module is a principally μ-hollow. For example Z_2 as Z-module.

5- It is clear that every μ-hollow module is principally μ-hollow. But the converse is not true in general. For example: since every cyclic submodule of Q as Z - module is small in Q and hence μ-small, then Q is principally μ-hollow but Q is not μ hollow since Q is not supplemented Z-module.

Proposition 2.2. *A nonzero epimorphic image of principally μ-hollow is principally μ-hollow.*

Proof. Let $f : M \to M'$ be a nonzero epimorphism and let M be a principally μ-hollow module, we have to show that M' is principally μ-hollow, let A be a proper cyclic submodule of M', then $f^{-1}(A)$ is a proper cyclic submodule of M, if $f^{-1}(A) = M$, then $A = M'$, which is a contradiction. Since M is principally μ-hollow, then $f^{-1}(A) \ll_\mu M$, and hence $A \ll_\mu M'$. □

Corollary 2.3. *Let M be a principally μ-hollow and let A be a submodule of M. Then $\frac{M}{A}$ is principally μ-hollow*

Proof. Let $\pi : M \to \frac{M}{A}$ be the natural epimorphism. Since M is principally μ-hollow, then by previous proposition $\frac{M}{A}$ is principally μ-hollow. □

Note In general, the converse of previous corollary is not true as the following example shows: Consider Z as Z-module, note that $\frac{Z}{4Z} \cong Z_4$ is principally μ-hollow, but Z is not principally μ-hollow.

The following proposition gives a condition for the converse of corollary (2.4).

Proposition 2.4. *If A is μ-small submodule of M and $\frac{M}{A}$ is principally μ-hollow, then M is principally μ-hollow.*

Proof. Let $m \in M$ and assume that $Rm + N = M$ for some submodule N of M with $\frac{M}{N}$ cosingular. Then $R(m + A) = \frac{Rm+A}{A}$ is a cyclic submodule of $\frac{M}{A}$ and $\frac{M}{A} = \frac{(Rm+A)}{A} + \frac{N+A}{A}$ and $\frac{M}{A+N}$ is cosingular. Since $\frac{M}{A}$ is principally μ-hollow, then $M = N + A$. But A is μ-small submodule of M, therefore $M = N$. Thus, M is principally μ-hollow. □

Proposition 2.5. *Every direct summand of μ-hollow is μ-hollow.*

Proof. Clear from Prop. (2.3).

A direct sum of μ-hollow modules need not be μ-hollow as the following example shows. The Z-modules Z_2 and Z_8 are μ-hollow, but $Z_8 \oplus Z_2$ is not μ-hollow module.

Let M be an R-module. Recall that a submodule A of M is called a fully invariant if g$(A) \leq A$, for every $g \in \text{End}(M)$ and M is called duo module if every submodule of M is fully invariant. See [4].

Now, we give conditions under which the direct sum of μ-hollow modules is a μ hollow. □

Proposition 2.6. *Let $M = M_1 \oplus M_2$ be a duo module , then M is principally μ-hollow if and only if M_1 and M_2 are principally μ-hollow, provided that $A \cap M_i$ is proper cyclic submodule of $M_i, i = 1, 2, \forall A \subset M$.*

Proof. ($\Rightarrow$) Clear by Prop. (2.6).
($\Leftarrow$) Let mR be a proper submodule of M. Since M is a duo module, then $mR = (mR \cap M_1) \oplus (mR \cap M_2)$. Hence each of $mR \cap M_1, mR \cap M_2$ is a proper submodule of M_1 $mR \ll_\mu M$. Thus M is μ-hollow.

Recall that an R-module M is called distributive if for all A, B and $C \leq M, A \cap (B + C)$ $= (A \cap B) + (A \cap C)$. See [5].

By similar argument, one can easily prove the following proposition. □

Proposition 2.7. *Let M_1 and M_2 be R-modules and let $M = M_1 \oplus M_2$ be a distributive module. Then M is principally μ-hollow if and only if M_1 and M_2 are principally μ hollow, provided that $A \cap M_i$ is proper cyclic submodule of $M_i, i = 1, 2, \forall A \subset M$.*

3 Principally $\mu-$lifting modules

Motivated by the generalization of μ-lifting modules, I introduce principally $\mu-$ lifting modules. This section is devoted to investigating some properties of this class of modules.

Definition 3.1. An R-module M is called principally μ-lifting module if for every cyclic submodule A of M, there exists a submodule D of A such that $M = D \oplus D', D' \leq M$ and $A \cap D' <<_\mu D'$

(1) Every principally μ-hollow module is principally μ-lifting module. The converse is not true in general, for example Z_6 as Z - module is principally μ-hollow but not principally μ-hollow module.

(2) Every semisimple module is principally μ-lifting module. The converse is not true in general, for example Z_4 as Z - module.

(3) It is clear that every μ-lifting module is principally μ-lifting. The converse is not true in general, for example: Consider Q as Z - module. Since Q is principally μ hollow, then Q is principally μ-lifting Z-module which is not μ-lifting.

(4) Let p be a prime integer, and n any positive integer Then the Z-module $M = \frac{Z}{Zp^n}$ is a principally μ-lifting module.

(5) Z as Z-module is not principally μ-lifting.

(6) $Z_8 \oplus Z_2$ as Z-module is not principally μ-lifting, since it is not principally lifting cosingular module.

(7) Principally μ-lifting modules are closed under isomorphisms.

The following proposition gives a condition under which principally μ-hollow and principally μ-lifting modules are equivalent.

Proposition 3.2. *Let M be an indecomposable module. Then M is principally μ hollow if and only if M principally is μ-lifting.*

Proof. Let M be an indecomposable principally μ-lifting and let A be a propoer cyclic submodule of M, there exists a submodule D of A such that $M = D \oplus D', D' \leq M$ and $A \cap D' <<_\mu D'$. But M is indecomposable, therefore $D = 0$ and $D' = M$, hence we conclude that $A \ll_\mu M$. Thus M is principally μ-hollow. The converse is obvious. □

The following propositions give characterizations of μ-lifting modules.

Proposition 3.3. *Let M be an R-module. Then M is principally μ-lifting if and only if for every cyclic submodule A of M, there exists a submodule D of A such that $M = D \oplus D', D' \leq M$ and $A \cap D' <_\mu M$.*

Proof. Clear. □

Proposition 3.4. *Let M be an R-module. The following statements are equivalent.*

(1) M is principally μ-lifting module.

(2) Every cyclic submodule A of M can be written as $A = D \oplus S$, where D is a direct summand of M and $S \ll_\mu M$.

(3) For every cyclic submodule A of M, there exists a direct summand D of M such that $D \leq A$ and $D \leq_{\mu\, ce}\ A$ in M.

Proof. (1) $\Rightarrow$ (2) Suppose that M is a principally μ-lifting module and let A be cyclic a submodule of M, then there exists a submodule D of A such that $M = D \oplus D', D' \leq M$ and $A \cap D' \ll_\mu M$, by proposition (3.4). Now, $A = A \cap M = A \cap (D \oplus D') = D \oplus (A \cap D')$. Thus we get the result.
(2) $\Rightarrow$ (3) Let A be a cyclic submodule of M. By (2), $A = D \oplus S$, where D is a direct summand of M and $S <<_\mu M$. We have to show that $\frac{A}{D} \ll_\mu \frac{M}{D}$, let $\frac{M}{D} = \frac{A}{D} + \frac{U}{D}$, $\frac{M}{U}$ is cosingular, then $\frac{M}{D} = \frac{D+S}{D} + \frac{U}{D}$ and hence $M = D + S + U = S + U$. But $S <<_\mu M$, therefore $M = U$.
(3) $\Rightarrow$ (1) Let A be a cyclic submodule of M. By (3), there exists a direct summand D of M such that $D \leq A$ and $D \leq \mu$ ce A in M. We want to show that $A \cap D' <_\mu D'$, let $D' = (A \cap D') + U$, $\frac{D'}{U}$ is cosingular. Since $M = D + D' = D + (A \cap D') + U$, then $\frac{M}{D} = \frac{D+(A\cap D')+U}{D} = \frac{D+(A\cap D')}{D} + \frac{U+D}{D}$. Since $D \leq D + (A \cap D') \leq A$ and $D \leq_{\mu ce} A$ in M, then $D \leq_{\mu \text{ ce}} D + (A \cap D')$ in M, by [, proposition (2.5)] and $\frac{M}{U+D} = \frac{D+D'}{U+D} = \frac{(D+U)+D'}{U+D} \cong \frac{D'}{D'\cap(U+D)} = \frac{D'}{U}$ which is cosingular, hence $\frac{M}{D} = \frac{U+D}{D}$, implies that $M = U + D$ and clearly that $U \cap D = 0$, then $M = U \oplus D$, that is $U = D'$. Thus M is a principally μ-lifting module. □

Theorem 3.5. *Let M be an R-module. The following statements are equivalent.*

(1) M is principally μ-lifting module.

(2) Every cyclic submodule A of M has a μ-supplement B in M such that $A \cap B$ is a direct summand of A.

Proof. (1) $\Rightarrow$ (2) Let M be principally μ-lifting module and let A be a cyclic submodule of M. By proposition (3.5), there exists a direct summand D of M such that $D \leq A$ and $D \leq_{\text{uce}} A$ in M. Now, $A = A \cap M = A \cap (D \oplus D') = D \oplus (A \cap D')$. Since $D \leq A$, then $M = A + D'$ and $A \cap D' <_\mu D'$. Hence D' is μ-supplement of A and $A \cap D'$ is a direct summand of A.
(2) $\Rightarrow$ (1) Let A be a cyclic submodule of M. By (2) A has a μ-supplement B in M such that $A \cap B$ is a direct summand of A. Then $M = A + B, A \cap B <_\mu B$ and $A = (A \cap B) \oplus Y, Y \leq A$. Since $M = A + B = (A \cap B) + Y + B = Y + B$ and $A \cap B \cap Y = B \cap Y = \{0\}$, then $M = B \oplus Y$. Thus M is principally μ-lifting module. □

The following proposition gives another characterization of μ-lifting module.

Proposition 3.6. *Let M be an R-module. Then M is principally μ-lifting module if and only if for every cyclic submodule A of M, there exists an idempotent $f \in \text{End}(M)$ such that $f(M) \leq A$ and $(I - f)(A) \ll_\mu (I - f)(M)$.*

Proof. ($\Rightarrow$) Assume that M is a principally μ-lifting module and let A be a cyclic submodule of M. By characterization (3.6) A has a μ-supplement B in M such that $A\cap$ B is a direct summand of A, then $M = A + B, A \cap B <<_\mu B$ and $A = (A \cap B) \oplus X, X \leq A$. Note $M = A + B = (A \cap B) + X + B = X + B$ and $A \cap B \cap X = B \cap X = \{0\}$, implies that $M = B \oplus X$. Now define the following map $f : M \to X$, it is clear that f is an idempotent and $f(M) \leq A$. It is sufficient to prove that $(I - f)(A) \ll_\mu (I - f)(M)$. One can easily show that $(I - f)(A) = A \cap (I - f)(M) = A \cap B \ll_\mu B = (I - f)(M)$.
($\Leftarrow$) Let A be a cyclic submodule of M. By our assumption, there exists an idempotent $f \in \text{End}(M)$ such that $f(M) \leq A$ and $(I - f)(A) \ll_\mu (I - f)(M)$, clearly that $M = f(M) \oplus (I - f)(M)$ and $A \cap (I - f)(M) = (I - f)(A) \ll_\mu (I - f)(M)$. Thus M is principally $\mu-$ lifting. □

Proposition 3.7. *Any direct summand of principally μ-lifting is principally μ-lifting.*

Proof. Let $M = M_1 \oplus M_2$ be a principally μ-lifting and let A be a cyclic submodule of M_1, then $A = D \oplus S$, where D is a direct summand of M and $S <<_\mu M$, by characterization (3.5). Since D is a direct summand of M contained in M_1, then D is a direct summand of M_1 and $S <_\mu M, S \leq M_1$ and M_1 is a direct summand of M, then $S <<_\mu M_1$. Thus M_1 is μ-lifting.

Next, we give some various conditions under which the quotient of μ-lifting module is μ-lifting. □

Proposition 3.8. *Let M be a principally μ-lifting R-module and let A be a cyclic submodule of M. Then $\frac{M}{A}$ is principally μ-lifting in each of the following cases.*

(1) *For every direct summand D of M, $\frac{D+A}{A}$ is a direct summand of $\frac{M}{A}$.*

(2) *M is distributive module.*

Proof. (1) Suppose that M is principally μ-lifting R-module and let $\frac{X}{A}$ be a cyclic submodule of $\frac{M}{A}$, then X is cyclic submodule of M, hence there exists $D \leq X$ such that $M = D \oplus D', D' \leq M$ and $D \leq_{\mu ce} X$ in M. By hypothesis, $\frac{D+A}{A}$ is a direct summand of $\frac{M}{A}$. Then $\frac{D+A}{A} \leq_{\mu \text{ ce}} \frac{X}{A}$ in $\frac{M}{A}$. Thus $\frac{M}{A}$ is principally μ-lifting.

(2) Suppose that M is distributive module, we use (1) to show that $\frac{M}{A}$ is principally μ-lifting. Let D be a direct summand of $M, M = D \oplus D', D' \leq M$, then $\frac{M}{A} = \frac{D+D'}{A} = \frac{D+A}{A} + \frac{D'+A}{A}$ and $\frac{D+A}{A} \cap \frac{D'+A}{A} = \frac{(D+A)\cap D'+[(D+A)\cap A]}{A} = \frac{(A\cap D')+(D\cap A)+A}{A} = A$. Hence $\frac{D+A}{A}$ is a direct summand of $\frac{M}{A}$. So, by (1) M is principally μ-lifting module.

□

Lemma 3.9. $[23, lemma 5-4]$: *Let M be an R-module, if $M = M_1 \oplus M_2$, then $\frac{M}{A} = \frac{A+M_1}{A} \oplus \frac{A+M_2}{A}$, for every fully invariant submodule A of M.*

Proposition 3.10. *Let M be a principally μ-lifting module if A is a fully invariant submodule of M, then $\frac{M}{A}$ is principally μ-lifting module.*

Proof. Let $\frac{X}{A}$ be a cyclic submodule of $\frac{M}{A}$. Since M is principally μ-lifting, there exists a submodule D of X such that $D \leq_{\mu ce} X$ in M and $M = D \oplus D', D' \leq M$. By lemma (3.10) we have $\frac{M}{A} = \frac{D+A}{A} \oplus \frac{D'+A}{A}$, let $f : \frac{M}{D} \to \frac{M}{D+A}$ be a map defined by $f(m + D) = m + D + A, \forall m \in M$, it is clear that f is an epimorphism. Now, since $D \leq_{\mu \text{ ce}} X$ in M, $\frac{X}{D} \ll_\mu \frac{M}{D}$ and $f\left(\frac{X}{D}\right) <_\mu f\left(\frac{M}{D}\right)$, by [1] which implies that $\frac{X}{D+A} \ll_\mu \frac{M}{D+A}$, hence $D + A \leq_{\mu ce} X$ in M and hence $\frac{D+A}{A} \leq_{\mu ce} \frac{X}{A}$ in $\frac{M}{A}$, by [1]. Thus $\frac{M}{A}$ is principally $\mu-$ lifting module. □

Remark 3.11. A direct sum of principally μ-lifting modules need not be principally $\mu-$ lifting module as the following example shows.

Let $M = Z_8 \oplus Z_2$ as Z-module. It is clear that Z_8 and Z_2 are principally μ-lifting Z modules, but M is not principally μ-lifting module.

Proposition 3.12. *Let $M = M_1 \oplus M_2$ be a duo module such that M_1 and M_2 are principally μ-lifting modules, then M is principally μ-lifting.*

Proof. Let $M = M_1 \oplus M_2$ be a duo module and let mR be a submodule of M, then mR is a fully invariant submodule of M. Hence $mR = mR \cap M = mR \cap (M_1 \oplus M_2) = (mR \cap M_1) \oplus (mR \cap M_2)$. Since M_1 and M_2 are principally μ-lifting modules, then $mR \cap M_1 = A_1 \oplus A_2$ and $mR \cap M_2 = A_3 \oplus A_4$, where A_1 and A_3 are direct summands of M_1 and M_2 respectively and A_2, A_4 are μ-small submodules of M_1 and M_2 respectively, by Prop. (3.5). It is clear that $A_1 \oplus A_3$ is a direct summand of M and $A_2 \oplus A_4$ is μ-small submodule of M. Thus M is principally μ-lifting. □

By the same argument, one can easily prove the following proposition.

Proposition 3.13. *Let $M = M_1 \oplus M_2$ be a distributive module such that M_1 and M_2 are principally μ-lifting modules, then M is principally μ-lifting.*

We end this section by the following diagram.

References

[1] W. Khalid , E. M. Kamil 2018. *On a generalization of small sub modules*, Sci. Int. (Lahore), : 359-365.

[2] E. M. Kamil, and W. Khalid, 2018. *On μ-supplemented and cofinitely μ- supplemented*, Sci. Int. (Lahore), : 567-572.

[3] E. M. Kamil, and W. Khalid, 2019. *On μ- lifting modules Iraqi Journal of Science 60* 359-365.

[4] C. Ozcan, A. Harmanci and P. F. Smith, " *Duo Modules*", Glasgow Math. Journal Trust, 48, 533-545, (2006).

[5] V. Erdogdu , 1987. *Distributive Modules* ,Canada Math. Bull, 30, pp:248-254.

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