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A SIMPLE PROOF OF THE FERRAND-OLIVIER CLASSIFICATION OF THE MINIMAL RING EXTENSIONS OF A FIELD

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Abstract We present a simple proof of the Ferrand-Olivier result that if K is a field, then a nonzero K-algebra S is a minimal ring extension of K if and only if S is K-algebra isomorphic to (exactly one of) a minimal field extension of K, $K \times K$ or $K[X]/(X^2)$, where X is an indeterminate over K.

1 Introduction

All rings and algebras considered in this note are commutative with identity; all subrings, inclusion of rings, ring extensions, ring homomorphisms, and modules are unital. Recall from [1] that a ring extension $R \subset S$ is said to be *minimal* (or that S is a *minimal ring extension* of R) if there does not exist a ring T such that $R \subset T \subset S$. (As usual, $\subset$ denotes proper inclusion). An important first step toward the classification of minimal ring extensions was taken by Ferrand-Olivier, who demonstrated the following result (cf. [1, Lemme 1.2]), which we label as Theorem 1.

Theorem 1 (Ferrand-Olivier). Let K be a field and S a nonzero K-algebra. View S as a ring extension of K by means of the injective structural ring homomorphism $K \longrightarrow S$. Then S is a minimal ring extension of K if and only if S is K-algebra isomorphic to (necessarily exactly one) a minimal field extension of K, $K \times K$ or $K[X]/(X^2)$, where X is an indeterminate over K.

Ferrand and Olivier have published a sketch of a proof of Theorem 1 (cf. [1, page 462]). On the other hand, at the request of a number of students and colleagues, Dobbs [2, Section 2] has published a detailed proof of the above mentioned result. Both proofs are based mainly on the structure theorem of commutative Artinian rings [3, Theorem 3, page 205], which states that every commutative Artinian ring can be identified as a finite product of commutative Artinian (quasi-)local rings. The aim of this short note is to provide a very simple proof of this result using very simple algebra tools.

2 Proof of Theorem 1

For the "Only if" part, note that $S = K[\alpha]$ for some $\alpha \in S \setminus K$ since $K \subset S$ is a minimal extension. Moreover, α is algebraic over K since otherwise, we get the following inclusion relations: $K \subset K[\alpha^2] \subset S$, contradicting the minimality of $K \subset S$. If S is an integral domain, then $S = K(\alpha)$ is a field. Assume now that S is not an integral domain and pick two nonzero elements $a, b \in S$ such that $ab = 0$. Clearly, $a, b \notin K$ since K is a field. By minimality of $K \subset S$, we get $S = K[a] = K[b]$. As b is algebraic over K, we let $f(X)$ denote the minimal polynomial of b over K and we let $n := \deg(f)$. Clearly, $n \geq 2$ since $b \notin K$. Moreover, $K[b] = K + Kb + ... + Kb^{n-1}$. Thus, $a = a_0 + a_1 b + ... + a_{n-1} b^{n-1}$ for some elements $a_0, a_1, ..., a_{n-1} \in K$. Multiply both sides of the latter equation with a, we get immediately $a^2 = a_0 a$. Therefore, the minimal polynomial of a over K is $g(X) = X^2 - a_0 X$. On the other hand, $S = K[a] \cong K[X]/(g(X))$. If $a_0 = 0$, then $S \cong K[X]/(X^2)$. If $a_0 \neq 0$, then according to the chinese remainder theorem, we obtain $S \cong K[X]/(X) \times K[X]/(X - a_0) \cong K \times K$.

The "If" part is clear. Indeed, if S is a minimal field extension of K then we are done. Now, if $S \cong K \times K$ or $S \cong K[X]/(X^2)$, then as a K-vector space, S has dimension 2. Thus, $K \subset S$ is a minimal ring extension since any intermediate ring of $K \subset S$ is a K-subspace of S. This completes the proof. □

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