**Palestine Journal of Mathematics**

**Vol. 11(4)(2022)** , 325–330 

# CLOSED WEAK RAD-SUPPLEMENTED MODULES

S. K. Choubey[1], Laba K. Das[2] and M. K. Patel[3]

Communicated by Mohammad Ashraf

MSC 2020 Classification: Primary 16D10; Secondary 16D80, 16D99, 16N80.

Keywords and phrases: Closed submodule, $rs$-module, $wrs$-module, $cwrs$-module.

**Acknowledgement:** Authos would like to thank the referee for their valuable time and useful comments, which helps to improve the presentation of paper. M. K. Patel wishes to thank National Board for higher Mathematics, with file No.: 02211/3/2019 NBHM $(R.P.)$ RDII/1439, for financial assistantship.

**Abstract** We introduce an idea of closed weak Rad-supplemented module (briefly, $cwrs$) as a proper generalization of both extending and weak Rad-supplemented (briefly, $wrs$) modules, by weakening the extending condition that is *closed submodule is a direct summand by the closed submodule has a* $wrs$. It observed that $cwrs$ module is not inherited by direct sum and homomorphic image, examples are provided in Remark 2.6 and Remark 2.16. In this regard we proved several results under some conditions which showed that $cwrs$ is closed under finite direct sum and homomorphic image. Couple of them are listed as: $i)$ A local distributive (or duo or distributive) module $M = M_1 \oplus M_2$ is $cwrs$ if and only if each components $M_1$ and $M_2$ are $cwrs$. $ii)$ Every non-singular homomorphic image of a $cwrs$ module is again a $cwrs$.

## 1 Introduction

Recall from books [5] and [9], a submodule $B$ of $M$ is called a supplement (weak supplement) of submodule $A$ of $M$, if $A + B = M$ and $A \cap B \ll B(A \cap B \ll M)$. A module $M$ is called supplemented (weak supplemented (briefly, $ws$)) if each submodule of $M$ has a supplement (weak supplement) in $M$. It is clear that every direct summands are supplement and every supplements are weak supplement. An Artinian module is an example of supplemented as well as $ws$. $M$ is called amply supplemented (briefly, $as$ ) if for every submodules $A$ and $B$ of $M$ with, $A + B = M$ such that supplement of $A$ in $M$ lies in side $B$, thus we can say that every hollow modules are $as$ and every $as-$modules are supplemented. Details on supplemented, $ws$ and $as-$modules are given in [5]. After that, several authors have conducted detailed studies on the various generalizations of supplemented and $ws-$modules. Xue [14], weakening the concepts of supplement condition to introduce new concept generalized supplemented also known as Rad-supplemented (briefly, $rs$) if every submodule of $M$ has Rad-supplement in $M$, where submodule $B$ is Rad-supplement of $A$ if $A + B = M$ and $A \cap B \subseteq RadB$. Cleary, lifting module and supplemented module are lies in the class of $rs-$module, which are also studied by many other authors in a series of papers ([2], [3], [10], & [11]). An $R-$module $M$ is called weak Rad-supplemented (briefly, $wrs$) if every submodule of $M$ has a weak Rad-supplement in $M$ where submodule $B$ is weak Rad-supplement of $A$ if $M = A + B$ and $A \cap B \subseteq RadM$. The set of rational number $\mathbb{Q}$ is $rs$ as well as $wrs$ but not supplemented module over the set of integers $\mathbb{Z}$.

A submodule $A \neq 0$ of $M$ is an essential submodule (represented by, $A \subseteq_e M$) if for any submodule $B \neq 0$ of $M$, $A \cap B \neq 0$, every submodule of uniform module like, $M = \mathbb{Z}_{\mathbb{Z}}$ is essential in $M$. A submodule $A \subseteq M$ is closed or a complement (represented by $A \subseteq_c M$) if it has no proper essential extension in $M$ i.e., if $A \subseteq_e B$ for some submodule $B \subseteq M \Rightarrow B = A$. It is clear that direct summands are closed submodule in $M$. $M$ is an extending (or $CS$ ) module if, every closed submodule of $M$ is a direct summand or equivalently every submodule of $M$ is essential in a direct summand of $M$. The class of extending module includes uniform, semisimple, (quasi-) injective modules and finitely generated torsion-free $\mathbb{Z}-$modules.

Motivated by above notions, we are introducing a new notion as proper generalization of $wrs-$module is called $cwrs$ by weakening the condition of extending module that is *closed sub-*

*module is a direct summand by the closed submodule has a $wrs$.* Thus $cwrs-$module is the generalization of both extending module and $wrs-$module. The class of $cwrs-$module includes hollow, local, uniform, semisimple, (quasi-) injective, extending and $wrs-$modules. We proved that the property of module being $cwrs$ is inherited by direct summands and factor modules, however observed that, it is not inherited by direct sums and homomorphic images, examples are provided in Remark 2.6 and Remark 2.16. Thus, our main concerned is to investigate the conditions, which ensure the property of module being $cwrs$ is inherited by direct sum and homomorphic image. In this regard we proved several results under some conditions, which showed that $cwrs$ is closed under finite direct sum and homomorphic image. A couple of results among them are listed as; $i)$ A local distributive (or duo or distributive) $R-$module $M = M_1 \oplus M_2$ is $cwrs$ if and only if each components $M_1$ and $M_2$ are $cwrs$, $ii)$ Every non-singular homomorphic image of a $cwrs-$module $M$ is again a $cwrs$.

Throughout this article, $R$ will denote an associative ring with identity and all modules are unitary right $R-$modules. Consider $M$ be an $R-$module, then the representation $A \subseteq M$, $A \subseteq_e M$ and $A \subseteq_c M$ means that $A$ is a submodule, essential and closed submodule of $M$ respectively. $A$ is called small in $M$ (represented by $A \ll M$) if there is no any proper submodule $B \subseteq M$ such that $A + B = M$. Module $M$ is hollow, if every proper submodule is small in $M$. The sum of all small submodules of $M$ or the intersection of all maximal submodules of $M$ is known as radical of $M$ denoted by $RadM$. A is fully invariant submodule of $M$ if $f(A) \subseteq A$ for each $f \in S = End(M_R)$. If every submodule of $M$ is fully invariant, then $M$ is called duo module.

Now we are listing some well know property of closed submodule (Lemma 1.1), small submodule (Lemma 1.2) and radical of module $M$ i.e. $RadM$ (Lemma 1.3), which will be used in proving ensuing results.

**Lemma 1.1.** *[6, 1.10] Let the submodules $A$, $B$ and $C$ of an $R-$module $M$ such that, $A \subseteq B$, then we have;*
*(1) There exists, $K \subseteq_c M$, such that $C \subseteq_e K$.*
*(2) If $K \subseteq_c M_1 \oplus M_2$, then $K \cap M_1 \subseteq_c M_1$ and $K \cap M_2 \subseteq_c M_2$, where $M_1 \oplus M_2$ is duo module.*
*(3) If $B \subseteq_c M$ then $B/A \subseteq_c M/A$.*
*(4) $A \subseteq_c B$ and $B \subseteq_c M$ then $A \subseteq_c M$.*
*(5) Let $N$ be a non-singular, $C \subseteq_c N$ and an epimorphism $f : M \longrightarrow N$, then $f^{-1}(C) \subseteq_c M$.*

**Lemma 1.2.** *[5, 2.2 & 2.3] Let the submodules $A$, $B$ and $C_i (1 \leq i \leq n)$ of an $R-$module $M$ such that $A \subseteq B$, then we have;*
*(1) $B \ll M$ if and only if $A \ll M$ and $B/A \ll M/A$.*
*(2) $C_i \ll M (1 \leq i \leq n)$ if and only if $C_1 + C_2 + ... + C_n \ll M$.*
*(3) $A \ll M$ if and only if $A \ll B$, for a direct summand $B$ of $M$.*
*(4) If $A \ll M$ then $M$ is finitely generated if and only if $M/A$ is finitely generated.*
*(5) $A_1 \oplus A_2 \ll B_1 \oplus B_2$ if and only if $A_1 \ll A_2$ and $B_1 \ll B_2$.*
*6) If $A \ll M$ and module homomorphism $f : M \longrightarrow N$, then $f(A) \ll M$.*

**Lemma 1.3.** *[9, 21.6] For an $R-$module $M$, we have;*
*(1) $RadM$ is fully invariant in $M$.*
*(2) $M$ is finitely generated if and only if $RadM \ll M$.*
*(3) $RadA = A \cap RadM$ holds only for supplement (or $rs$) submodule $A$ of $M$.*
*(4) If $M = \oplus_{i \in I} M_i$, then $RadM = \oplus_{i \in I} RadM_i$ and $M/RadM = \oplus_{i \in I} M_i/RadM_i$.*
*(5) $f(RadM) \subset RadN$ for any module homomorphism $f : M \longrightarrow N$.*

## 2 Closed weak Rad-supplemented modules

**Definition 2.1.** A right $R-$module $M$ is closed weak Rad-supplemented (briefly, $cwrs$) if there exists a submodule $B$ of $M$ such that $M = A + B$ and $A \cap B \subseteq RadM$, corresponding to any closed submodule $A$ of $M$.

**Example 2.2.** (1) Every local and hollow module are $cwrs$, because every local is hollow and every hollow is $rs-$module.
(2) Every $wrs-$module is $cwrs$.

(3) Every extending module and every uniform module are $cwrs$.
(4) $M = \mathbb{Z}_{\mathbb{Z}}$ is $CS$ and hence it will be $cwrs$, but $M$ is not $wrs$ as for all $n \geq 2$, $n\mathbb{Z}$ has not $wrs$ in $\mathbb{Z}$. Consider $A = 2\mathbb{Z}$, then $B = 3\mathbb{Z}$ is only submodule of $\mathbb{Z}_{\mathbb{Z}}$ such that $M = 2\mathbb{Z} + 3\mathbb{Z}$ and $2\mathbb{Z} \cap 3\mathbb{Z} = 6\mathbb{Z} \nsubseteq RadM = 0$.
(5) The set of rational number $\mathbb{Q}$ is $rs$ as well as $wrs$ but not supplemented module over the set of integers $\mathbb{Z}$ .

Thus we have the following implications, whose converse is not necessarily true:

$$rs \Rightarrow wrs \Rightarrow cwrs$$

In the following results we are going to prove that the property of module being cwrs is closed under direct summands, factor modules and finite direct sums.

**Lemma 2.3.** *Let the submodules $A$ and $B$ of module $M$ such that $A \subseteq B \subseteq M$. If $C$ is $wrs$ of a closed submodule $B$ in $M$, then $(C + A)/A$ is $wrs$ of $B/A$ in $M/A$.*

**Proof.** Proof is similar to Lemma 1.1.5 [10]. □

**Proposition 2.4.** *Every quotient (factor) module of a $cwrs-$module is again a $cwrs$.*

**Proof.** Suppose $M$ is $cwrs$ and $B \subseteq_c M$. By Lemma 1.1(3), $B/A \subseteq_c M/A$ for every submodule $A \subseteq B \subseteq M$. As $M$ is $cwrs$, then there exists a $wrs$ $C$ of $B$ in $M$. Then by Lemma 2.3, $(C + A)/A$ will be $wrs$ of $B/A$ in $M/A$ and hence $M/A$ is $cwrs$. □

**Corollary 2.5.** *Every direct summands of a $cwrs-$module is again a $cwrs$.*

**Proof.** Suppose $A$ is any direct summand of a $cwrs-$module $M$ and $B \subseteq_c A$. Since each direct summand is closed, so $A \subseteq_c M$ and hence $B \subseteq_c M$ by Lemma 1.1(4). As $M$ is $cwrs$, then there exists a submodule $C$ of $M$ with the property that $M = B + C$ and $B \cap C \subseteq RadM$ thus we get $A = (A \cap C) + B$. Since $A$ is a direct summand, then $A \cap C \cap B \subseteq A \cap RadM = RadA$ i.e., $A \cap (B \cap C) \subseteq RadA$. Therefore, $A$ is $cwrs$.
Alternatively; Let $A$ be any direct summand of a $cwrs-$module $M$ i.e. for some submodule $B$ of $M$, we write $M = A \oplus B$. By Proposition 2.4, $M/B = (A \oplus B)/B \cong A$ is also $cwrs$. □

**Remark 2.6.** It is observed by many authors that, the direct sum of $CS-$modules is not necessarily $CS$ ([1], Example 2.4). Now consider the polynomial ring $R = \mathbb{Z}[x]$ is a commutative Noetherian domain with an indeterminate $x$, then $R_R$ is $CS-$module while module $M = R \oplus R$ is not $CS$. Also, ([6], 7.6) for any prime $p$, $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/p^3\mathbb{Z}$ are $CS-$modules, however $M = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}$ is not $CS$. As every $CS-$module are $cwrs$, so these examples work directly to conclude that direct sum of $cwrs-$modules need not be $cwrs$. □

Now, we investigate the conditions which ensure the property of module being cwrs is inherited by sum and direct sum.

**Lemma 2.7.** *Let the submodules $A$ and $B$ of an $R-$module $M$ such that $A + B$ has a $wrs$ $C$ in $M$ and $A \cap (B + C)$ has a $wrs$ $D$ in $A$. Then $C + D$ is a $wrs$ of $B$ in $M$.*

**Proof.** Proof is similar to Lemma 3.1.4 [10]. □

**Proposition 2.8.** *The direct sum of $cwrs-$module $M_1$ and $M_2$ i.e. $M = M_1 \oplus M_2$ is cwrs if, $M_1 \cap (M_2 + A) \subseteq_c M_1$ and $M_2 \cap (A + B) \subseteq_c M_2$ for any $A \subseteq_c M$, where $B$ is a $wrs$ $M_1 \cap (M_2 + A)$ in $M_1$.*

**Proof.** Consider $M = M_1 + (M_2 + A)$ for any $A \subseteq_c M$ has a $wrs$ 0 in $M$. Since $M_1$ is $cwrs$ and $M_1 \cap (M_2 + A) \subseteq_c M_1$, then there exists a submodule $B \subseteq M_1$, such that $M_1 = B + M_1 \cap (M_2 + A)$ and $B \cap [M_1 \cap (M_2 + A)] = B \cap (M_2 + A) \subseteq RadM_1$. By Lemma 2.7, $B$ is a $wrs$ of $M_2 + A$ in $M$, so $M = B + (M_2 + A)$. Since $M_2$ is $cwrs$ and $M_2 \cap (A + B) \subseteq_c M_2$, then $M_2 \cap (A + B) \subseteq_c M_2$ has a $wrs$ $C$ in $M_2$. Again by above Lemma 2.7, $B + C$ is a $wrs$ of $A$ in $M$ and hence $M$ is $cwrs$. □

**Proposition 2.9.** *Let $M_1$ be $cwrs$ and $M_2$ be any $R-$module. Construct $M = M_1 + M_2$ and let $A \cap M_1 \subseteq_c M_1$ for any $A \subseteq_c M$. Then module $M$ is $cwrs$ if and only if every $A \subseteq_c M$ with $M_2 \not\subset A$ has a $wrs$ in $M$.*

**Proof.** Necessary part is obvious. Conversely, suppose that $A \subseteq_c M$ with $M_2 \not\subset A$, then by assumption $A$ has a $wrs$. Assume, $M \subseteq A$, then $M = M_1 + M_2 = M_1 + A$ and $M_1 + A$ has $wrs$ $B = 0$ in $M$. As $A \cap M_1 \subseteq_c M_1$ and $M_1$ is $cwrs$, $A \cap M_1$ has a $wrs$ $C$ in $M_1$. By using Lemma 2.7, $B + C = C$ is a $wrs$ of $A$ in $M$. □

Recall from [9], a module element $m \in M$ is called singular if $r_R(m) = \{r \in R | m.r = 0\} \subseteq_e R$. Module $M$ is non-singular if it has no non-trivial singular elements, otherwise it is singular. For example $\mathbb{Z}_p$, for any prime $p$ is a singular while the set of integers $\mathbb{Z}$ is non-singular as a module over $\mathbb{Z}$.

**Corollary 2.10.** *Let a nonsingular module $M = M_1 + M_2$, where $M_1$ is $cwrs$ and $M_2$ be any module. Then module $M$ is $cwrs$ if and only if every $A \subseteq_c M$ with $M_2 \not\subset A$ has a $wrs$ in $M$.*

**Proof.** Proof is clear in the light of Proposition 2.9. □

An $R-$module $M$ is amply weak Rad-supplemented (briefly, $awrs$) if $M = A + B$ implies $B$ contains a $wrs$ of $A$ in $M$, for every submodules $A$ and $B$ of $M$.

**Proposition 2.11.** *Let submodules $A$ and $B$ of an $R-$module $M$ with the property $A \cap B \subseteq_c M$, where $B \subseteq_c M$. If $B$ is $cwrs$, then $M$ is $awrs$, where $M = A + B$.*

**Proof.** Assume that $M = A+B$ and $A\cap B \subseteq_c M$ so $A\cap B \subseteq_c B$. By assumption $B$ is $cwrs$, then there exists $C \subseteq B$ such that $B = (A\cap B)+C$ and $(A\cap B)\cap C = A\cap C \subseteq RadB$, so $B \subseteq A+C$, hence $M = A + B \subseteq A + C$. Thus we have $M = A + C$ and $A \cap C \subseteq RadB \subseteq RadM$, so $C$ is $wrs$ of $A$ in $B$. Therefore, $M$ is $awrs-$module. □

**Corollary 2.12.** *Let submodules $A$ and $B$ of a semisimple $R-$module $M$, such that $M = A+B$, then $M$ is $awrs$.*

Recall from [13], a module $M$ is said to be distributive, if intersection distribute over addition or addition distribute over intersection for every submodule $A$, $B$ and $C$ of $M$, i.e. $A\cap(B+C) = (A \cap B) + (A \cap C)$ or $A + (B \cap C) = (A + B) \cap (A + C)$. The class of distributive modules includes all duo modules.

**Proposition 2.13.** *A distributive $R-$module $M = M_1 \oplus M_2$ is $cwrs$ if and only if each components $M_1$ and $M_2$ are $cwrs$.*

**Proof.** Assume that $M = M_1 \oplus M_2$ is $cwrs$, then $M_1$ and $M_2$ are $cwrs$ by Corollary 2.5. Conversely, assume that $A \subseteq_c M$ and $M_1$, $M_2$ are $cwrs-$module. As every duo module is distributive applying Lemma 1.1(2), we get, $A \cap M_1 \subseteq_c M_1$ and $A \cap M_2 \subseteq_c M_2$. By assumption there exists submodule $B_i \subseteq M_i$, $i = 1, 2$ such that $M_i = (A \cap M_i) + B_i$ and $(A \cap M_i) \cap B_i = A \cap B_i \subseteq RadM_i$. Take $C = B_1 \oplus B_2$, as $M$ is distributive, we can write $M = M_1 \oplus M_2 = [(A\cap M_1)+B_1]\oplus[(A\cap M_2)+B_2] = (B_1\oplus B_2)+[(A\cap M_1)\oplus(A\cap M_2)] = C+[A\cap(M_1\oplus M_2)] = C+(A\cap M) = C+A$. Also, $A\cap C = A\cap(B_1\oplus B_2) = (A\cap B_1)\oplus(A\cap B_2) \subseteq RadM_1\oplus RadM_2 = RadM$ by Lemma 1.3(4). Thus $A \subseteq_c M$ has $wrs$ $C$ in $M$ and hence $M$ is $cwrs$. □

Define, a module $M$ to be a local distributive, if for every closed submodule $A$, $B$ and $C$ of $M$ such that $A \cap (B + C) = (A \cap B) + (A \cap C)$ or $A + (B \cap C) = (A + B) \cap (A + C)$. The class of local distributive modules includes all duo modules and distributive modules but the converse need not be true for example; Consider the ring of integers $\mathbb{Z}$ as module over itself, then $M = \mathbb{Z} \oplus \mathbb{Z}$ is local distributive but not distributive module.

**Corollary 2.14.** *A local distributive $R-$module $M = M_1 \oplus M_2$ is $cwrs$ if and only if each component $M_1$ and $M_2$ of $M$ are $cwrs$.*

**Corollary 2.15.** *Let every closed submodule of an $R-$module $M = M_1 \oplus M_2$ is fully invariant, then $M$ is $cwrs$ if and only if each component $M_!$ and $M_2$ of $M$ are $cwrs$.*

**Remark 2.16.** The homomorphic image of $CS-$modules is not necessarily be $CS$ ([1], Example 2.3). Thus we conclude that homomorphic image of $cwrs-$module need not be $cwrs$. Now, we investigate the conditions which ensure that the property of module being $cwrs$ is inherited by homomorphic image. □

A right $R-$module $M$ is relatively $c-$Rickart to a right $R-$module $N$, if $Kerf \subseteq_c M$ for every homomorphism $f : M \longrightarrow N$. Every simple and semisimple modules are relatively $c-$Rickart.

**Corollary 2.17.** *If a module $M$ is $cwrs$ and relatively $c-$Rickart to a module $N$, then homomorphic image $Imf$ is $cwrs$ for any module homomorphism $f : M \longrightarrow N$.*

**Proof.** Suppose that $f : M \longrightarrow N$ be module homomorphism, then by assumptions $Kerf \subseteq_c M$. Applying Proposition 2.4, we get $M/Kerf \cong Imf$ is $cwrs$. □

Recall from [6], a right $R-$module $M$ is said to be torsion-free if $m.r \neq 0$ for any regular element $r \in R$ and $0 \neq m \in M$, otherwise it is called torsion. For example; the ring of integers $\mathbb{Z}$ is torsion-free while $\mathbb{Z}_p$ for any prime $p$, is torsion module over $\mathbb{Z}$. Also for $m \in M$, $r_R(m) = \{r \in R | m.r = 0\}$ is the right annihilator of $m$.

**Lemma 2.18.** *Let $A \subseteq_c N$ and $f : M \longrightarrow N$ be module homomorphism, then $A \cong B/Kerf$ for some submodule $B \subseteq M$. If $r_R(m) = r_R(f(m))$ for all $m \in M/Kerf$ or $N$ is torsion-free, then $B$ is closed in $M$.*

**Proposition 2.19.** *Let $M$ be a $cwrs$ and $f : M \longrightarrow N$ be an epimorphism with $r_R(m) = r_R(f(m))$ for all $m \in M/Kerf$ or $N$ is torsion-free, then $N$ is $cwrs-$module.*

**Proof.** By Lemma 2.18, for any $A \subseteq_c N$ there exists a $B \subseteq_c M$, such that $A \cong B/Kerf$. As $f : M \longrightarrow N$ be an epimorphism, then we can write $N \cong M/Kerf$ and by assumption $C$ be $cwrs$ of $B \subseteq_c M$ i.e. $M = B + C$. Applying Lemma 2.3, we get $N \cong M/Kerf = (B/Kerf) + (C + Kerf)/Kerf$ and hence $N$ is $cwrs-$module. □

A right $R-$module $M$ is small cover of a right $R-$module $N$ if there exists a small epimorphism $f : M \longrightarrow N$, i.e., $Kerf \ll M$.

**Proposition 2.20.** *Let $N$ be a $cwrs-$module and $f : M \longrightarrow N$ be a small epimorphism. If each $0 \neq A \subseteq_c M$ contains $Kerf$, then $M$ is $cwrs$.*

**Proof.** Let $0 \neq A \subseteq_c M$ and suppose that $f(A) \subseteq_c B \subseteq N$, as $f : M \longrightarrow N$ be a small epimorphism, then $A = A + Kerf = f^{-1}f(A) \subseteq_c f^{-1}(B)$. Hence $A = f^{-1}(B)$ and so, $f(A) = B$ is a closed submodule of $N$. By assumption $N$ is $cwrs$, so $f(A)$ has a $wrs$ in $N$. By Lemma 2.6 [10], $A$ has a $wrs$ in $M$ i.e., $M$ is $cwrs$. □

**Theorem 2.21.** *Every non-singular homomorphic image of a $cwrs-$module $M$ is again a $cwrs$.*

**Proof.** Assume that $f : M \longrightarrow N$ be a module epimorphism and $N = Imf$ is a non-singular module. Let, $A \subseteq_c N$ then by Lemma 1.1(5), $B = f^{-1}(A) \subseteq_c M$. Since $M$ is $cwrs$ then there exists a submodule $C$ of $M$ such that $M = B + C$ and $B \cap C \subseteq RadM$. Hence we can write $N = f(M) = f(B) + f(C) = A + f(C)$ and $A \cap f(C) = f(B) \cap f(C) = f(B \cap C)$. Since $Kerf = f^{-1}(0) \subseteq B$, then by Lemma 1.2(6) and Lemma 1.3(5) we get $A \cap f(C) = f(B) \cap f(C) = f(B \cap C) \subseteq f(RadM) \subseteq RadN$. Thus $f(C)$ is $wrs$ of $A$ in $N$ and hence module $N = Imf$ is $cwrs$ . □

**Lemma 2.22.** *Let submodules $A$ and $B$ of $M$ such that $B$ is a $wrs$ of a maximal submodule $C$ of $M$. If $A + B$ has a $wrs$ $D$ in $M$ then $A$ has a $wrs$ in $M$.*

**Proof.** Since $D$ is a $wrs$ of $A + B$ in $M$, then $M = A + B + D$ and $(A + B) \cap D \subseteq RadM$. If $B \cap (A + D) \subseteq B \cap C \subseteq RadM$, then $A \cap (B + D) \subseteq D \cap (A + B) + B \cap (A + D) \subseteq RadM$, hence $B + D$ is a $wrs$ of $A$ in $M$. Now suppose that $B \cap (A + D)$ is not contained in $B \cap C$, since $B/(B \cap C) \cong (B + C)/C = M/C$, $B \cap C$ is a maximal submodule of $B$. Therefore $B \cap C + B \cap (A + D) = B$ and since $B \cap C \subseteq RadM$, we have $M = A + D + B = A + D + [B \cap C + B \cap (A + D)] = A + D$. Since $A \cap D \subseteq (A + B) \cap D \subseteq RadM$, then $D$ is a $wrs$ of $A$ in $M$. □

**Theorem 2.23.** *Let for any submodule $A \subseteq M$, there is a $wrs$ $B$ of some maximal submodule $C$ of $M$ such that $A + B \subseteq_c M$. Then $M$ is $cwrs$ if and only if $M$ is a $wrs$.*

**Proof.** Proof is obvious in the light of the above Lemma 2.22. □

## References

[1] A. W. Chatters and S. M. Khuri, Endomorphism rings of modules over non-singular CS- rings, J. of London Math. Soc., 21; 434-444, 1980.

[2] E. Buyukasik, E. Mermut and S. Ozdemir, Rad-supplemented modules, Rend. Sem. Mat. Univ. Padova, 124; 157-177, 2010.

[3] E. Buyukasik and C. Lomp, On recent generalization of semiperfect rings, Bull. Aust. Math. Soc., 78(2), 317-325, 2008.

[4] I. Al-Khazzi and P. F. Smith, Modules with chain conditions on superfluous submodules, Comm. Algebra, 19(8), 2331-2351, 1991.

[5] J. Clark, C. Lomp, N. Vanaja and R. Wisbauer, Lifting Modules, Frontiers in Mathematics, Birkhaeuser Basel 2006.

[6] N. V. Dung, D. V. Huynh, P. F. Smith and R. Wisbauer, Extending Modules, CRC Press, 1994.

[7] R. Alizade, G. Bilhan and P. F. Smith, Modules whose maximal submodules have supplements, Comm. Algebra, 29(6); 2389-2405, 2001.

[8] R. Alizade and E. Buyukasik, Cofinitely weak supplemented modules, Comm. Algebra, 31(11); 5377-5390, 2003.

[9] R. Wisbauer, Foundations of Module and Ring Theory; A handbook for study and research, Gordon and Breach Science Publishers, 1991.

[10] S. K. Choubey, Some Generalizations of Supplemented and Lifting Modules and Their Properties, Ph D thesis, IIT-BHU, Varanasi, 2013.

[11] S. K. Choubey, M. K. Patel and V. Kumar, On weak$^*$ Rad-$\oplus$-supplemented modules, Maejo. Int. J. Sci. Technol., 11(03), 264-274, 2017.

[12] T. Y. Ghawi, Closed weak $G-$supplemented modules, J. Univ. Babylon for pure and App. Sci., 26(7), 2018.

[13] V. Erdogdu, Distributive modules, Canad. Math. Bull., 30 (2), 248-254, 1987.

[14] W. Xue, Characterizations of semiperfect and perfect rings, Publ. Mate., 40(1), 115- 125, 1996.

[15] Y. Zhou, Generalization of perfect, semiperfect and semiregular rings, Algebra Colloq., 7(3), 305-318, 2000.

[16] ZENG Qing-yi and SHI Mei-hua, On closed weak supplemented modules, J. of Zhejiang University SCI.-A, 7(2); 210-215, 2006.

## Author information

S. K. Choubey[1], Laba K. Das[2] and M. K. Patel[3], [1]Department of Mathematics, National Institute of Technology Sikkim, Ravangla, South Sikkim-737139, India; [2,3] Department of Mathematics, National Institute of Technology Nagaland, Dimapur-797103, India.
E-mail: choubeyitbhu2011@gmail.com, labakumardas74@gmail.com, mkpitb@gmail.com

Received: November 24, 2021.
Accepted: February 7, 2022.