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New notion of sequence spaces of modulus function

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Abstract It was Shiue who have introduced the Cesàro spaces and was later studied by various authors as cited in the text. In regard of this, the main structure of this paper is to study and interpret the new space of the form $\mathfrak{C}_{(p)}(\mathcal{F},\theta)$ by employing lacunary sequences and sequences of strictly positive real numbers. Some basic interesting properties and inclusions relations will be determined.

1 Introduction

By Λ we mean the set of all real or complex sequences and by sequence space we mean of subspace of Λ. By $\mathbf{N}$, we represent the set $\{0, 1, 2, \cdots\}$; by $\mathbf{R}$ we represent the set of all real numbers and $\mathbf{C}$ will represent the set of all complex numbers. We denote bounded sequences by l_∞; convergent sequences by c and those sequences with limit as zero by c_0.

We call a space $\mathcal{Y}$ to be FK space if it is a complete metric space with continuous coordinated $p_r : \mathcal{Y} \to \mathbb{C}$ where $p_r(u) = u_r$ for all $u \in \mathcal{Y}$ and $r \in \mathbb{N}$. A normed FK space is called a BK space as defined in [12], [3] and others.

A modulus function $\mathfrak{f}$ is a function from $[0,\ \infty)$ to $[0,\ \infty)$ such that
(i) $\mathfrak{f}(\varsigma) = 0$ if and only if $\varsigma = 0$,
(ii) $\mathfrak{f}(\varsigma + \varrho) \leq \mathfrak{f}(\varsigma) + \mathfrak{f}(\varrho)\ \forall \varsigma, \varrho \geq 0$
(iii) $\mathfrak{f}$ is increasing,
(iv) $\mathfrak{f}$ if continuous from right at $\varsigma = 0$.

A modulus function may be bounded or unbounded. For example the function $\mathfrak{f}(\varsigma) = \frac{\varsigma}{\varsigma+1}$ is bounded but the function $\mathfrak{f}(\varsigma) = \varsigma^p$ for $0 < p \leq 1$ is unbounded. Ruckle, used this notion to define the space

$$L(\mathfrak{f}) = \{u = (u_i) \in \Lambda : \sum_{i=1}^{\infty} \mathfrak{f}(|u_i|) < \infty\}.$$

For $\mathfrak{f}(u) = u^p$, then $L(f)$ reduces to the well known space ℓ_p and is given by

$$\ell_p = \{u = (u_i) \in \Lambda : \sum_{i=1}^{\infty}(|u_i|^p < \infty\}.$$

Also, for $\mathfrak{f}(u) = u$, then $L(f)$ reduces to the space ℓ_1 and is given by

$$\ell_1 = \{u = (u_i) \in \Lambda : \sum_{i=1}^{\infty}(|u_i| < \infty\}.$$

Several authors including Ganie [5], Maddox [12], Ozturk and Bilgin [15], and some others studied some sequence spaces defined by a modulus function in a detailed way.

Let $\mho$ be a sequence space. Then the sequence space $\mho(f)$ is defined by

$$\mho(f) = \{u = (u_j) \in \Lambda : \sum_{j=1}^{\infty} \mathfrak{f}(|u_j|) \in \mho\}.$$

The author in [11] gave an extension of $\mho(f)$ by considering a sequence of modulus functions $\mathcal{F} = (f_j)$ and defined the space

$$\mho(\mathcal{F}) = \{u = (u_j) \in \Lambda : \sum_{j=1}^{\infty} \mathfrak{f}_j(|u_j|) \in \mho\}.$$

Let $\theta = (t_j)$ be increasing integer sequence. Then it will be called lacunary sequence if $t_0 = 0$ and $t_j = t_j - t_{j-1} \to \infty$. By θ we will denote the intervals of the form $I_j = (t_{j-1}, t_j]$ and with q_j we will denote the ratio $\frac{t_j}{t_{j-1}}$ [4].

For $1 \leq p \leq \infty$, the author in [18] has defined the Cesàro sequence space ces_p is defined as

$$ces_p = \left\{ v = (v_k) : \sum_{i=1}^{\infty} \left(\frac{1}{i} \sum_{j=1}^{i} |v_j| \right)^p < \infty \right\},$$

and proved that it is a Banach space with the norm

$$||v|| = \left[\sum_{i=1}^{\infty} \left(\frac{1}{i} \sum_{j=1}^{i} |v_j| \right)^p \right]^{\frac{1}{p}}.$$

It was further studied by several authors viz., Et [2], Nuray [14] and many others. The author in [13] has introduced the Cesàro sequence spaces $\mathcal{X}_p$ and $\mathcal{X}_\infty$ of non-absolute type and has shown that $\text{ces}_p \subset \mathcal{X}_p$ is strict for $1 \leq p \leq \infty$.

2 Main results

In this section of text, we introduce the space $\mathfrak{C}_{(p)}(\mathcal{F}, \theta)$, and show it is a Fréchet space.

Following Başarir [1], Et. [2], Freedman [4], Ganie [6]-[9], Jagers [10], Nuray [14], Ruckle [16], Savaş [17], we introduce the following spaces:

$$\mathfrak{C}_{(p)}[\mathcal{F}, \theta] = \left\{ v = (\varsigma_k) : \sum_{i=1}^{\infty} \left(\mathcal{F} \left[\frac{1}{h_i} \sum_{k \in I_i} |\varsigma_k| \right] \right)^{p_i} < \infty \right\},$$

where $p = (p_i)$ is a bounded sequence of positive real numbers.

We now begin with the following theorem.

Theorem 2.1. *The space $\mathfrak{C}_{(p)}(\mathcal{F}, \theta)$ is linear spaces over* **C**.

Proof. Let $\varsigma, \tau \in \mathfrak{C}_{(p)}(\mathcal{F}, \theta)$, then

$$\sum_{i=1}^{\infty} \left(\mathcal{F} \left[\frac{1}{h_i} \sum_{k \in I_i} |\varsigma_k \right] \right)^{p_i} < \infty \quad \text{and} \quad \sum_{i=1}^{\infty} \left(\mathcal{F} \left[\frac{1}{h_i} \sum_{k \in I_i} \tau_k \right] \right)^{p_i} < \infty.$$

Now, $a, b \in \mathbf{C}$, we have

$$\sum_{i=1}^{\infty} \left(\mathcal{F} \left[\frac{1}{h_i} \sum_{k \in I_i} |a\varsigma_k + b\tau_k| \right] \right)^{p_i}$$
$$\leq \max(1, 2^{\mathcal{H}-1}) \left[\sum_{i=1}^{\infty} \left(\mathcal{F} \left[\frac{1}{h_i} \sum_{k \in I_i} |\varsigma_k| \right] \right)^{p_i} + \sum_{i=1}^{\infty} \left(\mathcal{F} \left[\frac{1}{h_i} \sum_{k \in I_i} |\tau_k| \right] \right)^{p_i} \right] < \infty.$$

Consequently, $a\varsigma_k + b\tau_k \in \mathfrak{C}_{(p)}(\mathcal{F}, \theta)$ and hence is linear spaces over **C**. □

We first define the following definition:

Definition 2.2. If a linear space V is complete, then it is known as Fréchet space.

Theorem 2.3. *For* $1 \leq p_i < \infty$*, the space* $\mathfrak{C}_{(p)}\left(\mathcal{F},\theta\right)$ *is a Fréchet space paranormed by*

$$g(\varsigma) = \left[\sum_{i=1}^{\infty}\left(\mathcal{F}\left[\frac{1}{h_i}\sum_{k\in I_i}|\varsigma_k|\right]\right)^{p_i}\right]^{\frac{1}{\mathcal{G}}}, \tag{2.1}$$

where $\mathcal{H} = \sup p_i < \infty$ *and* $\mathcal{G} = \max(1,\mathcal{H})$.

Proof. To prove the theorem, we must show the completeness property of $\mathfrak{C}_{(p)}\left(\mathcal{F},\theta\right)$. So, let $\varsigma^j = (\varsigma_i^j)_i$ be any Cauchy sequence in $\mathfrak{C}_{(p)}\left(\mathcal{F},\theta\right)$ for each $j \in \mathbb{N}$. Therefore, for each positive integer n_0, we have

$$g\left(\varsigma^i - \varsigma^j\right) < \frac{\varepsilon}{r\varsigma_0}\ \forall\, i,j \geq n_0.$$

Hence, by (2.1), we have

$$\left[\sum_{n=1}^{\infty}\left(\mathcal{F}\left[\frac{\frac{1}{h_n}\sum_{k\in I_n}|\varsigma_k^{(i)}-\varsigma_k^{(j)}|}{g\left(\varsigma^{(i)}-\varsigma^{(j)}\right)}\right]\right)^{p_n}\right]^{\frac{1}{\mathcal{G}}}$$

is positive and small and hence so is

$$\sum_{n=1}^{\infty}\left(\mathcal{F}\left[\frac{\frac{1}{h_n}\sum_{k\in I_n}|\varsigma_k^{(i)}-\varsigma_k^{(j)}|}{g\left(\varsigma^{(i)}-\varsigma^{(j)}\right)}\right]\right)^{p_n}.$$

But, $1 \leq p_n < \infty$, hence for each $n \geq 1$, we have

$$\mathcal{F}\left[\frac{\frac{1}{h_n}\sum_{k\in I_n}|\varsigma_k^{(i)}-\varsigma_k^{(j)}|}{g\left(\varsigma^{(i)}-\varsigma^{(j)}\right)}\right]$$

is small enough but positive. We choose $r > 0$ such that $\frac{\varsigma_0}{2}rp\left(\frac{\varsigma_0}{2}\right) \geq 1$, where p is the kernel associated $\mathcal{F}$. We thus have

$$\mathcal{F}\left[\frac{\frac{1}{h_n}\sum_{k\in I_n}|\varsigma_k^{(i)}-\varsigma_k^{(j)}|}{g\left(\varsigma^{(i)}-\varsigma^{(j)}\right)}\right] \leq \frac{\varsigma_0}{2}rp\left(\frac{\varsigma_0}{2}\right).$$

for each $n \in \mathbf{N}$. Using the integral representation of modulus function $\mathcal{F}$, we see for all $i,j \geq n_0$ that

$$\frac{1}{h_n}\sum_{k\in I_n}|\varsigma_k^{(i)}-\varsigma_k^{(j)}| \leq \frac{r\varsigma_0}{2}g\left(\varsigma^{(i)}-\varsigma^{(j)}\right) < \frac{\varepsilon}{2}.$$

Hence for each fixed k, $\varsigma_k^{(i)}$ is a Cauchy sequence in $\mathbf{C}$ and $\mathbf{C}$ being complete, it converges, say, $\varsigma_k^{(j)} \to \varsigma_k$ for $j \to \infty$.

Thus, for a given $\varepsilon > 0$, choose a natural number $m_0 > 1$ such that $g\left(\varsigma^{(i)}-\varsigma^{(j)}\right) < \varepsilon$ for all $i,j \geq m_0$. But

$$\left[\sum_{n=1}^{r}\left(\mathcal{F}\left[\frac{1}{h_n}\sum_{k\in I_n}|\varsigma_k^{(i)}-\varsigma_k^{(j)}|\right]\right)^{p_n}\right]^{\frac{1}{\mathcal{G}}},$$

being finite and as small as we please for $\forall i,j \geq m_0$, therefore, using continuity of $\mathcal{F}$ and letting $j \to \infty$, we conclude that

$$\left[\sum_{n=1}^{r}\left(\mathcal{F}\left[\frac{1}{h_n}\sum_{k\in I_n}|\varsigma_k^{(i)}-\varsigma_k|\right]\right)^{p_n}\right]^{\frac{1}{\mathcal{G}}},$$

is small and positive $\forall i \geq m_0$. Now, letting $r \to \infty$, we have

$$g\left(\varsigma^{(i)} - \varsigma\right) \leq \varepsilon \;\forall i \geq m_0,$$

for ε small and positive. This, shows that $(\varsigma^{(i)})$ converges to ς in the paranorm of $\mathfrak{C}_{(p)}\left(\mathcal{F}, \theta\right)$. But, $(\varsigma^{(i)}) \in \mathfrak{C}_{(p)}\left(\mathcal{F}, \theta\right)$ and $\mathcal{F}$ is continuous, it follows that $\varsigma \in \mathfrak{C}_{(p)}\left(\mathcal{F}, \theta\right)$. □

3 Inclusion relations on $\mathfrak{C}_{(p)}\left(\mathcal{F}, \theta\right)$

We now investigate some inclusion relations concerning $\mathfrak{C}_{(p)}\left(\mathcal{F}, \theta\right)$.

We first consider the following definitions:

Definition 3.1. For any set $\mathcal{D}$ of sequences, the space of multipliers of $\mathcal{D}$, denoted by $S(\mathcal{D})$, is given by

$$S(\mathcal{D}) = \{u \in \Lambda : u\varsigma \in \mathcal{D} \; for\; all\; \varsigma \in \mathcal{D}\}.$$

Definition 3.2. We say that g satisfies a Δ_2-condition, or that $g \in \Delta_2$, if there is a constant $K \geq 2$ such that $g(2t) \leq Kg(t)$ for all $t \geq 0$.

Theorem 3.3. *If $p = (p_j)$ and $t = (t_j)$ are bounded sequences of positive real numbers with $0 < p_j \leq t_j < \infty$ for each j, then $\mathfrak{C}_{(p)}\left(\mathcal{F}, \theta\right) \subseteq C_{(t)}\left(\mathcal{F}, \theta\right)$, for any modulus function $\mathcal{F}$.*

Proof. Let $\varsigma \in \mathfrak{C}_{(p)}\left(\mathcal{F}, \theta\right)$, then

$$\sum_{j=1}^{\infty}\left(\mathcal{F}\left[\frac{1}{h_j}\sum_{k\in I_i}|\varsigma_k|\right]\right)^{p_j} < \infty.$$

Thus, for sufficiently large j, say $j \geq j_0$, we have for ε small and positive that

$$\mathcal{F}\left[\frac{1}{h_i}\sum_{k\in I_i}|\varsigma_k|\right] < \varepsilon,$$

some fixed $j_0 \in \mathbf{N}$. But $\mathcal{F}$ being non-decreasing with $p_j \leq q_j$, we ahve

$$\sum_{j\geq j_0}^{\infty}\left(\mathcal{F}\left[\frac{1}{h_j}\sum_{k\in I_i}|\varsigma_k|\right]\right)^{t_j} \leq \sum_{j\geq j_0}^{\infty}\left(\mathcal{F}\left[\frac{1}{h_j}\sum_{k\in I_i}|\varsigma_k|\right]\right)^{p_j} < \infty.$$

Consequently, $\varsigma \in C_{(t)}\left(\mathcal{F}, \theta\right)$. □

Theorem 3.4. *If Δ_2-condition is satisfied by the modulus function $\mathcal{F}$, then*

$$\ell_\infty \subset S(\mathfrak{C}_{(p)}\left(\mathcal{F}, \theta\right).$$

Proof. Let $\nu \in \ell_\infty$ with $\mathcal{T} = \sup_j |\nu_j|$ and $\varsigma \in \mathfrak{C}_{(p)}\left(\mathcal{F}, \theta\right)$, then

$$\sum_{j=1}^{\infty}\left(\mathcal{F}\left[\frac{1}{h_i}\sum_{k\in I_i}|\varsigma_k|\right]\right)^{p_j} < \infty.$$

Since $\mathcal{F}$ satisfies the Δ_2-condition, there exists a constant $\mathcal{B}$ such that

$$\begin{aligned}\sum_{j=1}^{\infty}\left(\mathcal{F}\left[\frac{1}{h_i}\sum_{k\in I_i}|\nu_j\varsigma_k|\right]\right)^{p_j} &\leq \sum_{j=1}^{\infty}\left(\mathcal{F}\left[\frac{1}{h_j}\sum_{k\in I_i}|\nu_j||\varsigma_k|\right]\right)^{p_j}\\ &\leq (\mathcal{B}(1+[\mathcal{T}]))^{\mathcal{H}}\sum_{j=1}^{\infty}\left(\mathcal{F}\left[\frac{1}{h_i}\sum_{k\in I_i}|\varsigma_k|\right]\right)^{p_j}\\ &< \infty,\end{aligned}$$

where $[\mathcal{T}]$ represents the integer part of $\mathcal{T}$. Consequently, $\nu \in \mathfrak{C}_{(p)}\left(\mathcal{F}, \theta\right)$. □

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