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SUBMANIFOLDS OF COSYMPLECTIC STATISTICAL-SPACE-FORMS

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Abstract In this paper, invariant and anti-invariant submanifolds of cosymplectic statistical-space-forms are considered. Among others a generalized Wintgen inequality on Legendrian submanifolds of such space forms is obtained.

1 Introduction

The statistical structure is a pair of Riemannian metric and a torsion free linear connection satisfying Codazzi equation. It was firstly introduced by Amari [1]. This structure have been studied also in [1, 2, 15, 16, 18, 25, 26, 27, 29] etc. Recently Furuhata et al. [16] initiated the idea of Sasakian statistical-space-form. Later Malek and Akbari [21] studied statistical submanifolds in cosymplectic statistical space form.

Pseudo parallel and Ricci generalized pseudo parallel submanifolds were studied by Deszcz et al. [12]. Later Asperti et al. [4, 5] studied such submanifold in space forms. Wintgen [30] established the inequality $K \leq \|H\|^2 - |K^{\perp}|$ between Gauss curvature K, the squared mean curvature $\|H\|^2$ and normal curvature $K^{\perp}$ of any surface in E^4 and also showed that equality holds if the ellipse of curvature of M^2 in E^4 is a circle. Later in 1999, De Smet et al. [11] have constructed the conjecture on Wintgen inequality for any submanifold in real space form

$$\rho \leq \|H\|^2 - \rho^{\perp} + c, \tag{1.1}$$

where ρ is normalized scalar curvature and $\rho^{\perp}$ is normalized normal scalar curvature. They also proved this conjecture of submanifold with codimension 2 of any dimensional real space form. Thereafter Choi and Lu [10] proved this inequality on any 3-dimensional submanifold of real space form, where the codimension is arbitrary. In 2008, Ge and Tang [17] and in 2011 Lu [19] independently proved Wintgen inequality on submanifold with arbitrary dimension and codimension of real space form. Many authors studied (generalized) Wintgen inequality of certain submanifold of different space form [3, 13, 14, 20, 22, 23, 24, 28].

Recently several authors studied generalized Wintgen inequality on submanifolds of different Statistical manifold [6, 7]. Chen [9] made a detail survey about recent results of Wintgen inequality.

This paper deals with the study of some submanifolds of cosymplectic statistical-space-forms.

2 Preliminaries

Let $\bar{M}^{2n+1}(\phi, \xi, \eta, \bar{g})$ be an almost contact metric manifold and X, Y and $Z \in \Gamma(T\bar{M})$. Then we know that [8]

$$\phi^2(X) = -X + \eta(X)\xi,\ \phi\xi = 0, \tag{2.1}$$

$$\eta(\xi) = 1,\ \bar{g}(X,\xi) = \eta(X),\ \eta(\phi X) = 0, \tag{2.2}$$

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$\bar{g}(\phi X, Y) = -\bar{g}(X, \phi Y). \tag{2.4}$$

The above $\bar{M}^{2n+1}(\phi,\xi,\eta,\bar{g})$ is called cosymplectic manifold if

$$(\bar{\nabla}_X \phi)Y = 0, \tag{2.5}$$

$$\bar{\nabla}_X \xi = 0, \tag{2.6}$$

where $\bar{\nabla}$ is the Riemannian connection on $\bar{M}$. A cosymplectic manifold including constant ϕ-sectional curvature c is called cosymplectic-space-form and its curvature tensor satisfies

$$\begin{aligned} \bar{R}(X,Y)Z &= \frac{c}{4}\Big[\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} + \{\bar{g}(X,\phi Z)\phi Y \\ &- \bar{g}(Y,\phi Z)\phi X + 2\bar{g}(X,\phi Y)\phi Z + \eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + \bar{g}(X,Z)\eta(Y)\xi - \bar{g}(Y,Z)\eta(X)\xi\}\Big]. \end{aligned} \tag{2.7}$$

If $\hat{\bar{\nabla}}$ is a torsion free affine connection on $\bar{M}^{2n+1}(\phi,\xi,\eta,\bar{g})$ such that

$$(\hat{\bar{\nabla}}_X \bar{g})(Y,Z) = (\hat{\bar{\nabla}}_Y \bar{g})(X,Z), \tag{2.8}$$

then $(\hat{\bar{\nabla}},\bar{g})$ is called a statistical structure of $\bar{M}^{2n+1}(\phi,\xi,\eta,\bar{g})$ [1].
Define $\overset{*}{\bar{\nabla}}$ as

$$X\bar{g}(Y,Z) = \bar{g}(\hat{\bar{\nabla}}_X Y, Z) + \bar{g}(Y, \overset{*}{\bar{\nabla}}_X Z) \tag{2.9}$$

for any $X,Y,Z \in \Gamma(T\bar{M})$. Here the connections $\hat{\bar{\nabla}}$ and $\overset{*}{\bar{\nabla}}$ are called dual connection and $(\overset{*}{\bar{\nabla}})^* = \hat{\bar{\nabla}}$. Then $(\overset{*}{\bar{\nabla}},\bar{g})$ is also a statistical structure. The connections $\hat{\bar{\nabla}}$ and $\overset{*}{\bar{\nabla}}$ are related by $\bar{\nabla}_X Y = \frac{1}{2}(\hat{\bar{\nabla}}_X Y + \overset{*}{\bar{\nabla}}_X Y)$.

Remark 2.1. For a statistical structure $(\hat{\bar{\nabla}},\bar{g})$, we set

$$\mathcal{K}_X Y = \hat{\bar{\nabla}}_X Y - \bar{\nabla}_X Y \tag{2.10}$$

for any $X,Y \in \Gamma(TM)$. Then $\mathcal{K} \in \Gamma^{(1,2)}(TM)$ satisfies

$$\mathcal{K}_X Y = \mathcal{K}_Y X,\ g(\mathcal{K}_X Y, Z) = \bar{g}(X, \mathcal{K}_Y Z). \tag{2.11}$$

Conversely, for a Riemannian metric $\bar{g}$ if a given $\mathcal{K} \in \Gamma^{(1,2)}(TM)$ satisfies (2.11), a pair $(\hat{\bar{\nabla}} := \bar{\nabla} + \mathcal{K}, \bar{g})$ becomes a statistical structure.

Definition 2.2. $\bar{M}^{2n+1}(\hat{\bar{\nabla}},\phi,\xi,\eta,\bar{g})$ is called cosymplectic statistical manifold if $(\hat{\bar{\nabla}},\bar{g})$ is statistical structure, $(\phi,\xi,\eta,\bar{g})$ is cosymplectic structure and satisfies [21]

$$\mathcal{K}_X \phi Y + \phi \mathcal{K}_X Y = 0, \tag{2.12}$$

where $\mathcal{K}_X Y = \mathcal{K}(X,Y) = \hat{\bar{\nabla}}_X Y - \bar{\nabla}_X Y$.

Definition 2.3. [21] Suppose $\bar{M}^{2n+1}(\hat{\bar{\nabla}}, \phi, \xi, \eta, \bar{g})$ is a cosymplectic statistical manifold and $c \in \mathbb{R}$. Define $\bar{S}(X,Y)Z = \frac{1}{2}\{\hat{\bar{R}}(X,Y)Z + \overset{*}{\bar{R}}(X,Y)Z\}$ where $\hat{\bar{R}}$, $\overset{*}{\bar{R}}$ are respectively the curvature tensor with respect to $\hat{\bar{\nabla}}, \overset{*}{\bar{\nabla}}$. If

$$\begin{aligned}\bar{S}(X,Y)Z &= \frac{c}{4}\Big[\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} + \{\bar{g}(X,\phi Z)\phi Y \\ &- \bar{g}(Y,\phi Z)\phi X + 2\bar{g}(X,\phi Y)\phi Z + \eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + \bar{g}(X,Z)\eta(Y)\xi - \bar{g}(Y,Z)\eta(X)\xi\}\Big],\end{aligned} \tag{2.13}$$

then $\bar{M}^{2n+1}(\hat{\bar{\nabla}}, \phi, \xi, \eta, \bar{g})$ is called cosymplectic statistical manifold with constant ϕ-sectional curvature c. We call this as cosymplectic statistical-space-form and denote such space-form by $\bar{M}^{2n+1}_{stat}(c)$.

Lemma 2.4. *In $\bar{M}^{2n+1}_{stat}(c)$, the following relations hold:*

$$\hat{\bar{\nabla}}_X \phi Y - \phi \overset{*}{\bar{\nabla}}_X Y = 0, \tag{2.14}$$

$$\hat{\bar{\nabla}}_X \xi = \bar{g}(\hat{\bar{\nabla}}_X \xi, \xi)\xi \tag{2.15}$$

$\forall\ X, Y$ *tangent to* $\bar{M}^{2n+1}_{stat}(c)$.

Proof. In same way of [16] we prove the above. □

Similarly for $\overset{*}{\bar{\nabla}}$ we get the following:

$$\overset{*}{\bar{\nabla}}_X \phi Y - \phi \hat{\bar{\nabla}}_X Y = 0, \tag{2.16}$$

$$\overset{*}{\bar{\nabla}}_X \xi = \bar{g}(\overset{*}{\bar{\nabla}}_X \xi, \xi)\xi \tag{2.17}$$

$\forall\ X, Y$ tangent to $\bar{M}^{2n+1}_{stat}(c)$.
Consider M^m is a submanifold of $\bar{M}^{2n+1}_{stat}(c)$. Suppose ∇, $\hat{\nabla}$ and $\overset{*}{\nabla}$ are respectively the induced connections of $\bar{\nabla}$, $\hat{\bar{\nabla}}$ and $\overset{*}{\bar{\nabla}}$ and g is the induced metric on M. Then we can write

$$\begin{cases}\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \\ \hat{\bar{\nabla}}_X Y = \hat{\nabla}_X Y + \hat{h}(X,Y), \\ \overset{*}{\bar{\nabla}}_X Y = \overset{*}{\nabla}_X Y + \overset{*}{h}(X,Y).\end{cases} \tag{2.18}$$

$$\begin{cases}\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \\ \hat{\bar{\nabla}}_X V = -\hat{A}_V X + \hat{\nabla}^{\perp}_X V, \\ \overset{*}{\bar{\nabla}}_X V = -\overset{*}{A}_V X + \overset{*}{\nabla}{}^{\perp}_X V,\end{cases} \tag{2.19}$$

where $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. Here h (resp. $\hat{h}$ and $\overset{*}{h}$) are the 2nd fundamental forms and A (resp. $\hat{A}$ and $\overset{*}{A}$) are shape operators with respect to ∇ (resp. $\hat{\nabla}$ and $\overset{*}{\nabla}$). They are related by

$$\begin{cases}g(h(X,Y),V) = g(A_V X, Y), \\ g(\hat{h}(X,Y),V) = g(\overset{*}{A}_V X, Y), \\ g(\overset{*}{h}(X,Y),V) = g(\hat{A}_V X, Y).\end{cases} \tag{2.20}$$

We can also write

$$\nabla_X Y = \frac{1}{2}\{\hat{\nabla}_X Y + \overset{*}{\nabla}_X Y\}, \tag{2.21}$$

$$h(X,Y) = \frac{1}{2}\{\hat{h}(X,Y) + \overset{*}{h}(X,Y)\}. \tag{2.22}$$

If $\hat{h} = 0$ and $\overset{*}{h} = 0$, then M^m is said to be totally geodesic with respect to $\hat{\bar{\nabla}}$ and $\overset{*}{\bar{\nabla}}$ respectively. A submanifold M^m of $\bar{M}^{2n+1}_{stat}(c)$ is named as totally umbilical with respect to $\hat{\bar{\nabla}}, \overset{*}{\bar{\nabla}}$ if the following holds:

$$g(X,Y)\hat{H} = \hat{h}(X,Y), \tag{2.23}$$

$$g(X,Y)\overset{*}{H} = \overset{*}{h}(X,Y), \tag{2.24}$$

respectively, where $\hat{H}$ and $\overset{*}{H}$ are defined by

$$\hat{H} = \frac{1}{m}\sum_{i=1}^{m}\hat{h}(E_i,E_i) \tag{2.25}$$

and

$$\overset{*}{H} = \frac{1}{m}\sum_{i=1}^{m}\overset{*}{h}(E_i,E_i), \tag{2.26}$$

where $\{E_1, E_2, \cdots, E_m\}$ be an orthonormal basis on $\Gamma(TM)$.
M^m is known as an invariant submanifold of $\bar{M}^{2n+1}_{stat}(c)$ if $\phi X \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(TM)$. And M is named as an anti-invariant submanifold of $\bar{M}^{2n+1}_{stat}(c)$ if $\phi X \in \Gamma(T^\perp M))$ for any $X \in \Gamma(TM)$. If $m = n$ and $\xi \in \Gamma(T^\perp M)$ then anti-invariant submanifold is considered as Legendrian submanifold.

For any $X \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, we have

$$\phi X = PX + FX, \tag{2.27}$$

$$\phi V = tV + sV, \tag{2.28}$$

where $PX, tV \in \Gamma(TM)$ and $FX, sV \in \Gamma(T^\perp M)$.
From (2.18) and (2.19) we have

$$\hat{\bar{R}}(X,Y)Z = \hat{R}(X,Y)Z - \hat{A}_{\hat{h}(Y,Z)}X + \hat{A}_{\hat{h}(X,Z)}Y, \tag{2.29}$$

$$\overset{*}{\bar{R}}(X,Y)Z = \overset{*}{R}(X,Y)Z - \overset{*}{A}_{\overset{*}{h}(Y,Z)}X + \overset{*}{A}_{\overset{*}{h}(X,Z)}Y, \tag{2.30}$$

$$\bar{R}(X,Y)Z = R(X,Y)Z - A_{h(Y,Z)}X + A_{h(X,Z)}Y, \tag{2.31}$$

$$g\left(\hat{R}^\perp(X,Y)V,W\right) = g(\hat{\bar{R}}(X,Y)V,W) + g([\hat{A}_V, \overset{*}{A}_W]X,Y), \tag{2.32}$$

$$g\left(\overset{*}{R}{}^\perp(X,Y)V,W\right) = g(\overset{*}{\bar{R}}(X,Y)V,W) + g([\overset{*}{A}_V, \hat{A}_W]X,Y), \tag{2.33}$$

$$g\left(R^\perp(X,Y)V,W\right) = g(\bar{R}(X,Y)V,W) + g([A_V, \hat{A}_W]X,Y), \tag{2.34}$$

where $R, \hat{R}, \overset{*}{R}$ is the curvature tensor and $R^\perp, \hat{R}^\perp, \overset{*}{R}{}^\perp$ normal curvature tensor on M^m of $\bar{M}^{2n+1}_{stat}(c)$ with respect to ∇, $\hat{\nabla}$ and $\overset{*}{\nabla}$.
From Definition 2.2, (2.29) and (2.30) we get

$$\begin{aligned} 2\bar{S}(X,Y)Z &= 2S(X,Y)Z - \hat{A}_{\hat{h}(Y,Z)}X + \hat{A}_{\hat{h}(X,Z)}Y \\ &- \overset{*}{A}_{\overset{*}{h}(Y,Z)}X + \overset{*}{A}_{\overset{*}{h}(X,Z)}Y, \end{aligned} \tag{2.35}$$

where S is the curvature tensor of M.
Similarly from Definition 2.2, (2.32) and (2.33) we get

$$2g\left(S^{\perp}(X,Y)V,W\right) = 2g(\bar{S}(X,Y)V,W) + g([\hat{A}_V,\overset{*}{A}_W]X,Y) + g([\overset{*}{A}_V,\hat{A}_W]X,Y). \tag{2.36}$$

3 Invariant submanifolds of $\bar{M}^{2n+1}_{stat}(c)$

Lemma 3.1. *Suppose M is an invariant submanifold of $\bar{M}^{2n+1}_{stat}(c)$. Then following holds:*

$$\hat{\nabla}_X\phi Y - \phi\overset{*}{\nabla}_X Y - t\overset{*}{h}(X,Y) = 0, \tag{3.1}$$

$$\hat{h}(X,\phi Y) = s\overset{*}{h}(X,Y), \tag{3.2}$$

$$\hat{\nabla}_X tV - \hat{A}_{sV}X - t\overset{*}{\nabla}^{\perp}{}_X V + \phi\overset{*}{A}_V X = 0, \tag{3.3}$$

$$\hat{h}(X,tV) + \hat{\nabla}^{\perp}_X sV - s\overset{*}{\nabla}^{\perp}{}_X V = 0 \tag{3.4}$$

for $X,Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

Proof. It follows from (2.14) and (2.18) that

$$0 = \hat{\nabla}_X\phi Y + \hat{h}(X,\phi Y) - \phi\overset{*}{\nabla}_X Y - \phi\overset{*}{h}(X,Y). \tag{3.5}$$

Using (2.28) in (3.5) we get

$$\hat{\nabla}_X\phi Y - \phi\overset{*}{\nabla}_X Y - t\overset{*}{h}(X,Y) - s\overset{*}{h}(X,Y) + \hat{h}(X,\phi Y) = 0 \tag{3.6}$$

Comparing tangential and normal parts from both sides of (3.6), (3.1) and (3.2) follows.
Similarly from (2.14), (2.28) and (2.19) we have

$$\hat{\nabla}_X tV + \hat{h}(X,tV) + \hat{\nabla}^{\perp}_X sV \tag{3.7}$$
$$= \hat{A}_{sV}X + t\overset{*}{\nabla}^{\perp}{}_X V + s\overset{*}{\nabla}^{\perp}{}_X V - \phi\overset{*}{A}_V X,$$

from which (3.3) and (3.4) are followed. □

Lemma 3.2. *Consider M is an invariant submanifold of $\bar{M}^{2n+1}_{stat}(c)$. Then following holds:*

$$\overset{*}{\nabla}_X\phi Y - \phi\hat{\nabla}_X Y - t\hat{h}(X,Y) = 0, \tag{3.8}$$

$$\overset{*}{h}(X,\phi Y) = s\hat{h}(X,Y), \tag{3.9}$$

$$\overset{*}{\nabla}_X tV - \overset{*}{A}_{sV}X - t\hat{\nabla}^{\perp}_X V + \phi\hat{A}_V X = 0, \tag{3.10}$$

$$\overset{*}{h}(X,tV) + \overset{*}{\nabla}^{\perp}{}_X sV - s\hat{\nabla}^{\perp}_X V = 0, \tag{3.11}$$

for $X,Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

Proof. In similar way of proof as above, Lemma 3.2 can be proved. □

Lemma 3.3. *Suppose M is an invariant submanifold of $\bar{M}^{2n+1}_{stat}(c)$. Then*

$$\hat{\nabla}_X\xi = \eta(\hat{\nabla}_X\xi)\xi, \tag{3.12}$$

$$\hat{h}(X,\xi) = 0, \tag{3.13}$$

$$\overset{*}{\nabla}_X\xi = \eta(\overset{*}{\nabla}_X\xi)\xi, \tag{3.14}$$

$$\overset{*}{h}(X,\xi) = 0 \tag{3.15}$$

Proof. From (2.15), (2.18) we get (3.12) and (3.13). Similarly from (2.17) and (2.18) we get (3.14) and (3.15). □

Definition 3.4. [4, 5, 12] A submanifold M^m of $\bar{M}^{2n+1}_{stat}(c)$ with respect to $\hat{\bar{\nabla}}$ is said to be pseudo parallel if $\hat{h}$ satisfies

$$\begin{aligned} (\bar{S}(X,Y)\cdot\hat{h})(Z,U) &= S^{\perp}(X,Y)\hat{h}(Z,U)-\hat{h}(S(X,Y)Z,U)-\hat{h}(Z,S(X,Y)U) \\ &= L_{\hat{h}}Q(g,\hat{h})(Z,U;X,Y), \end{aligned} \tag{3.16}$$

where $L_{\hat{h}}$ is some function on $\mathcal{W}=\{x\in M:\ (\hat{h}-\hat{H}g)_x\neq 0\}$, $\forall\ X,Y,Z,U\in\Gamma(TM)$. Certainly, if $L_{\hat{h}}=0$ then M is known as semiparallel of $\bar{M}^{2n+1}_{stat}(c)$ with respect to $\hat{\bar{\nabla}}$.

Theorem 3.5. *Suppose M is an invariant submanifold of $\bar{M}^{2n+1}_{stat}(c)$. Then M is totally geodesic with respect to $\hat{\bar{\nabla}}$ if and only if it is pseudo parallel with respect to $\hat{\bar{\nabla}}$ provided $L_{\hat{h}}\neq 0$*

Proof. We know that

$$\begin{aligned} Q(g,\hat{h})(Z,U;X,Y) &= g(Y,Z)\hat{h}(X,U)-g(X,Z)\hat{h}(Y,U) \\ &+ g(Y,U)\hat{h}(X,Z)-g(X,U)\hat{h}(Y,Z). \end{aligned} \tag{3.17}$$

From (3.16) and (3.17) we have

$$\begin{aligned} & S^{\perp}(X,Y)\hat{h}(Z,U)-\hat{h}(S(X,Y)Z,U)-\hat{h}(Z,S(X,Y)U) \\ & = L_{\hat{h}}\Big\{g(Y,Z)\hat{h}(X,U)-g(X,Z)\hat{h}(Y,U)+g(Y,U)\hat{h}(X,Z)-g(X,U)\hat{h}(Y,Z)\Big\}. \end{aligned} \tag{3.18}$$

Substituting $X=U=\xi$ in (3.18) and using (3.13), (3.15) and (2.35), we get

$$L_{\hat{h}}\hat{h}(Y,Z)=0. \tag{3.19}$$

By virtue of (3.19) we get the result. The converse is trivial. □

Corollary 3.6. *Suppose M^m is a submanifold of $\bar{M}^{2n+1}_{stat}(c)$. Then M is semiparallel with respect to $\hat{\bar{\nabla}}$ if and only if it is totally geodesic with respect to $\hat{\bar{\nabla}}$.*

Definition 3.7. [4, 5, 12] A submanifold M^m of $\bar{M}^{2n+1}_{stat}(c)$ with respect to $\hat{\bar{\nabla}}$ is said to be Ricci generalized pseudo parallel if $\hat{h}$ satisfies

$$\begin{aligned} (\bar{S}(X,Y)\cdot\hat{h})(Z,U) &= S^{\perp}(X,Y)\hat{h}(Z,U)-\hat{h}(S(X,Y)Z,U)-\hat{h}(Z,S(X,Y)U) \\ &= L_{Ric}Q(Ric,\hat{h})(Z,U;X,Y), \end{aligned} \tag{3.20}$$

where $L_{\hat{h}}$ is some function on $\mathcal{W}=\{x\in M:\ (\hat{h}-\hat{H}g)_x\neq 0\}$, $\forall\ X,Y,Z,U\in\Gamma(TM)$.

Theorem 3.8. *Suppose M^m is an invariant submanifold of $\bar{M}^{2n+1}_{stat}(c)$. Then M is totally geodesic with respect to $\hat{\bar{\nabla}}$ if and only if it is Ricci generalized pseudo parallel with respect to $\hat{\bar{\nabla}}$ provided $(m-1)L_{Ric}\neq 0$.*

Proof. From (3.17) and (3.20) we get

$$\begin{aligned} & S^{\perp}(X,Y)\hat{h}(Z,U)-\hat{h}(S(X,Y)Z,U)-\hat{h}(Z,S(X,Y)U) \\ & = L_{Ric}\{\hat{h}(Y,Z)Ric(X,U)-\hat{h}(X,Z)Ric(Y,U)+\hat{h}(Y,U)Ric(X,Z) \\ & -\hat{h}(X,U)Ric(Y,Z)\}. \end{aligned} \tag{3.21}$$

Substituting $X=U=\xi$ in (3.21) and making use of (3.13), (3.15) and (2.35) we get

$$0=(m-1)L_{Ric}\hat{h}(Y,Z). \tag{3.22}$$

By virtue of (3.22) we get the theorem. □

4 Anti-invariant submanifolds of $\bar{M}^{2n+1}_{stat}(c)$

Lemma 4.1. *Suppose M^m is an anti-invariant submanifold of $\bar{M}^{2n+1}_{stat}(c)$ with $\xi \in \Gamma(T^\perp M)$. Then the following holds:*

$$\hat{\nabla}^\perp_X \phi Y - \phi \overset{*}{\nabla}_X Y - s\overset{*}{h}(X,Y) = 0, \tag{4.1}$$

$$\hat{A}_{\phi Y} X + t\overset{*}{h}(X,Y) = 0, \tag{4.2}$$

$$\hat{\nabla}_X tV - \hat{A}_{sV} X - t\overset{*}{\nabla}{}^\perp_X V = 0, \tag{4.3}$$

$$\hat{h}(X,tV) + \hat{\nabla}^\perp_X sV - s\overset{*}{\nabla}{}^\perp_X V + \phi \overset{*}{A}_V X = 0, \tag{4.4}$$

where $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$.

Proof. From (2.18) and (2.19) we have

$$\hat{\bar{\nabla}}_X \phi Y - \phi \overset{*}{\bar{\nabla}}_X Y = \hat{\nabla}^\perp_X \phi Y - \hat{A}_{\phi Y} X - \phi \overset{*}{\nabla}_X Y - \phi \overset{*}{h}(X,Y). \tag{4.5}$$

Using (2.14) and (2.28) in (4.5) we have

$$\hat{\nabla}^\perp_X \phi Y - \hat{A}_{\phi Y} X - \phi \overset{*}{\nabla}_X Y - t\overset{*}{h}(X,Y) - s\overset{*}{h}(X,Y) = 0. \tag{4.6}$$

Equating tangential and normal part of (4.6) we have (4.1) and (4.2).
Similarly from (2.18), (2.19) and (2.28) we have

$$\begin{aligned} \hat{\bar{\nabla}}_X \phi V - \phi \overset{*}{\bar{\nabla}}_X V &= \hat{\nabla}_X tV + \hat{h}(X,tV) + \hat{\nabla}^\perp_X sV \\ &- \hat{A}_{sV} X - t\overset{*}{\nabla}{}^\perp_X V - s\overset{*}{\nabla}{}^\perp_X V + \phi \overset{*}{A}_V X. \end{aligned} \tag{4.7}$$

Using (2.14) in (4.7), we get

$$\begin{aligned} 0 &= \hat{\nabla}_X tV + \hat{h}(X,tV) + \hat{\nabla}^\perp_X sV \\ &- \hat{A}_{sV} X - t\overset{*}{\nabla}{}^\perp{}_X V - s\overset{*}{\nabla}{}^\perp{}_X V + \phi \overset{*}{A}_V X. \end{aligned} \tag{4.8}$$

Equating tangential and normal components of (4.8), we get (4.3) and (4.4). □

Lemma 4.2. *Suppose M^m is an anti-invariant submanifold of $\bar{M}^{2n+1}_{stat}(c)$ with $\xi \in \Gamma(T^\perp M)$. Then the following holds:*

$$\overset{*}{\nabla}{}^\perp_X \phi Y - \phi \hat{\nabla}_X Y - s\hat{h}(X,Y) = 0, \tag{4.9}$$

$$\overset{*}{A}_{\phi Y} X + t\hat{h}(X,Y) = 0, \tag{4.10}$$

$$\overset{*}{\nabla}_X tV - \overset{*}{A}_{sV} X - t\hat{\nabla}^\perp_X V = 0, \tag{4.11}$$

$$\overset{*}{h}(X,tV) + \overset{*}{\nabla}{}^\perp_X sV - s\hat{\nabla}^\perp_X V + \phi \hat{A}_V X = 0, \tag{4.12}$$

where $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$.

Proof. In same way of proof of Lemma 4.1, Lemma 4.2 can be proved. □

Lemma 4.3. *Suppose M^m is an anti-invariant submanifold of $\bar{M}^{2n+1}_{stat}(c)$ with $\xi \in \Gamma(T^\perp M)$. Then the following holds:*

$$\hat{\nabla}^\perp_X \xi = \eta(\hat{\nabla}^\perp_X \xi)\xi, \tag{4.13}$$

$$A_\xi X = 0, \tag{4.14}$$

$$\overset{*}{\nabla}{}^\perp{}_X \xi = \eta(\overset{*}{\nabla}{}^\perp_X \xi)\xi, \tag{4.15}$$

$$\overset{*}{A}_\xi X = 0. \tag{4.16}$$

Proof. From (2.15), (2.19) we get (4.13) and (4.14). Similarly from (2.17) and (2.19) we get (4.15) and (4.16). □

Theorem 4.4. *Suppose M^m is an anti-invariant submanifold of $\bar{M}^{2n+1}_{stat}(c)$ with $\xi \in \Gamma(T^\perp M)$. If $\hat{\nabla}^\perp_X sV = s\overset{*}{\nabla}{}^\perp_X V$. Then the following holds:*

$$\hat{A}_{\phi X}Y = \hat{A}_{\phi Y}X, \tag{4.17}$$

$$\overset{*}{A}_{\phi X}Y = \overset{*}{A}_{\phi Y}X, \tag{4.18}$$

$$\hat{h}(X,Y) = \phi\overset{*}{A}_{\phi Y}X, \quad \overset{*}{h}(X,Y) = \phi\hat{A}_{\phi Y}X. \tag{4.19}$$

Proof. From (4.2) we get (4.17). Similarly from (4.10) we get (4.18). By virtue of (4.4) and (4.12) we get (4.19). □

Theorem 4.5. *Suppose M^m is an anti-invariant submanifold of $\bar{M}^{2n+1}_{stat}(c)$ with $\xi \in \Gamma(T^\perp M)$. If $\hat{\nabla}^\perp_X sV = s\overset{*}{\nabla}{}^\perp_X V$ and $\phi\overset{*}{h} \in \Gamma(TM)$. Then the following holds:*

$$S^\perp(X,Y)\phi Z = \phi S(X,Y)Z \tag{4.20}$$

Proof. By taking $\phi\overset{*}{h} \in \Gamma(TM)$ and using (4.1) we get $\hat{\nabla}^\perp_X \phi Y = \phi\overset{*}{\nabla}_X Y$. Using this we get (4.20). □

Theorem 4.6. *Suppose M^m is an anti-invariant submanifold of $\bar{M}^{2n+1}_{stat}(c)$ with $\xi \in \Gamma(T^\perp M)$. If $\hat{\nabla}^\perp_X sV = s\overset{*}{\nabla}{}^\perp_X V$ and $\hat{A}_{\phi X}\overset{*}{A}_{\phi Y} = \overset{*}{A}_{\phi Y}\hat{A}_{\phi X}$ $\forall\, X, Y \in \Gamma(TM)$, then $\bar{S}(X,Y)Z = S(X,Y)Z$.*

Proof. Making use of (4.17), (4.18) and (4.19) we get

$$\hat{A}_{\hat{h}(Y,Z)}X = \hat{A}_{\phi\overset{*}{A}_{\phi Y}Z}X = \hat{A}_{\phi X}\overset{*}{A}_{\phi Y}Z. \tag{4.21}$$

Similarly we get

$$\overset{*}{A}_{\overset{*}{h}(Y,Z)}X = \overset{*}{A}_{\phi X}\hat{A}_{\phi Y}Z. \tag{4.22}$$

By virtue of (4.21) and (4.22) in (2.35) we get

$$\begin{aligned} 2\bar{S}(X,Y)Z - 2S(X,Y)Z &= \hat{A}_{\phi Y}\overset{*}{A}_{\phi X}Z - \hat{A}_{\phi X}\overset{*}{A}_{\phi Y}Z \\ &+ \overset{*}{A}_{\phi Y}\hat{A}_{\phi X}Z - \overset{*}{A}_{\phi X}\hat{A}_{\phi Y}Z. \end{aligned} \tag{4.23}$$

Using $\hat{A}_{\phi X}\overset{*}{A}_{\phi Y} = \overset{*}{A}_{\phi Y}\hat{A}_{\phi X}$ in (4.23) we get the theorem. □

Theorem 4.7. *Let M^m be a minimal anti-invariant submanifold of $\bar{M}^{2n+1}_{stat}(c)$ with $\xi \in \Gamma(T^\perp M)$. Suppose*
(i) $\hat{\nabla}^\perp_X sV = s\overset{*}{\nabla}{}^\perp_X V$, $\hat{A}_{\phi X}\overset{*}{A}_{\phi Y} = \overset{*}{A}_{\phi Y}\hat{A}_{\phi X}$, $\phi\overset{*}{h} \in \Gamma(TM)$ holds.
Then M^m is pseudo parallel with respect to $\hat{\bar{\nabla}}$, if M^m is totally geodesic with respect to $\hat{\bar{\nabla}}$ provided $mL_{\hat{h}} \neq -(1+m)\frac{c}{4}$.

Proof. By virtue of Theorem 4.6, (3.16) and (2.35) we get

$$\begin{aligned} S^\perp(X,Y)\hat{h}(Z,U) &= \left[L_h + \frac{c}{4}\right]\Big\{g(Y,Z)\phi\overset{*}{A}_{\phi U}X - g(X,Z)\phi\overset{*}{A}_{\phi U}Y \\ &+ g(Y,U)\phi\overset{*}{A}_{\phi Z}X - g(X,U)\phi\overset{*}{A}_{\phi Z}Y\Big\}. \end{aligned} \tag{4.24}$$

Using (4.20) and (2.35) in (4.24) we get

$$\begin{aligned} &\frac{c}{4}\{g(Y,\overset{*}{A}_{\phi U}Z)\phi X - g(X,\overset{*}{A}_{\phi U}Z)\phi Y\} \\ &= \left[L_h + \frac{c}{4}\right]\Big\{g(Y,Z)\phi\overset{*}{A}_{\phi U}X - g(X,Z)\phi\overset{*}{A}_{\phi U}Y + g(Y,U)\phi\overset{*}{A}_{\phi Z}X - g(X,U)\phi\overset{*}{A}_{\phi Z}Y\Big\}. \end{aligned} \tag{4.25}$$

Equation (4.25) can be written as

$$\frac{c}{4}\{g(Y,\overset{*}{A}_{\phi Z}U)X - g(Z,\overset{*}{A}_{\phi U}X)Y\} \tag{4.26}$$
$$= \left[L_h + \frac{c}{4}\right]\left\{g(Y,Z)\overset{*}{A}_{\phi U}X - \overset{*}{A}_{g(X,Z)\phi U}Y + \overset{*}{A}_{\phi Z}g(Y,U)X - g(X,U)\overset{*}{A}_{\phi Z}Y\right\}.$$

Contracting (4.26) over X, U we get

$$\frac{c}{4}\{\overset{*}{A}_{\phi Z}Y - g(m\phi H, Z)Y\} \tag{4.27}$$
$$= \{L_h + \frac{c}{4}\}\{g(Y,Z)m\phi H - \overset{*}{A}_{\phi Z}Y + \overset{*}{A}_{\phi Z}Y - m\overset{*}{A}_{\phi Z}Y\}.$$

Since M^m is minimal, so from (4.27) we get $\{mL_{\hat{h}} + (m+1)\frac{c}{4}\}\overset{*}{A}_{\phi Z}Y = 0$.
This proves the theorem. □

Definition 4.8. An anti-invariant submanifold of $\bar{M}^{2n+1}_{stat}(c)$ is said to be Einstein if

$$Ric(Y,Z) = ag(Y,Z),$$

where $Y, Z \in \Gamma(TM^m)$ and a is a constant.

Definition 4.9. An anti-invariant submanifold of $\bar{M}^{2n+1}_{stat}(c)$ is said to be η-Einstein if

$$Ric(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z),$$

where $Y, Z \in \Gamma(TM^m)$ and a, b are constants.

Theorem 4.10. *Suppose M is an anti-invariant submanifold of $\bar{M}^{2n+1}_{stat}(c)$ with $\xi \in \Gamma(T^{\perp}M)$. If M is totally umbilical with respect to $\hat{\hat{\nabla}}, \overset{*}{\hat{\nabla}}$, then M is Einstein.*

Proof. Using (2.23) and (2.24) in (2.35) we get

$$S(X,Y,Z,U) = \frac{c}{4}\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\} \tag{4.28}$$
$$+ \; m^2 g(\hat{H},\overset{*}{H})\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\}.$$

Contracting the above equation over X, U, we get

$$Ric(Y,Z) = \left[\frac{c}{4} + m^2 g(\hat{H},\overset{*}{H})\right](m-1)g(Y,Z). \tag{4.29}$$

This proves the theorem. □

Theorem 4.11. *Suppose M^m is an anti-invariant submanifold of $\bar{M}^{2n+1}_{stat}(c)$ with $\xi \in \Gamma(TM)$. If M is totally umbilical with respect to $\hat{\hat{\nabla}}, \overset{*}{\hat{\nabla}}$, then M is η-Einstein.*

Proof. Using (2.23) and (2.24) in (2.35) we get

$$\begin{aligned} S(X,Y,Z,U) = {} & \frac{c}{4}\Big[\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\} \\ & + \{\eta(X)\eta(Z)g(Y,U) - \eta(Y)\eta(Z)g(X,U) \\ & + g(X,Z)\eta(Y)\eta(U) - g(Y,Z)\eta(X)\eta(U)\}, \\ & + m^2 g(\hat{H},\overset{*}{H})\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\}\Big]. \end{aligned} \tag{4.30}$$

Contracting the above equation over X, U, we get

$$\begin{aligned} Ric(Y,Z) = {} & \left\{(m-1)\frac{c}{4} + m^2(m-2)g(\hat{H},\overset{*}{H})\right\}g(Y,Z) \\ & - (m-2)\frac{c}{4}\eta(Y)\eta(Z). \end{aligned} \tag{4.31}$$

This proves the theorem. □

5 Generalized Wintgen inequality on Legendrian submanifolds of $\bar{M}^{2n+1}_{stat}(c)$

The normalized scalar curvature on M^n is given by

$$\hat{\rho} = \frac{2}{n(n-1)} \sum_{1\le i<j\le n} S(E_i, E_j, E_j, E_i). \tag{5.1}$$

By using (2.13) and (2.35) in (5.1) we get

$$\hat{\rho} = \frac{c}{4} + \frac{2}{n(n-1)} \sum_{r=1}^{n} \sum_{1\le i<j\le n} \{\hat{h}^r_{ii}\overset{*}{h}{}^s{}_{jj} + \hat{h}^r_{jj}\overset{*}{h}{}^s{}_{ii} - 2\hat{h}^r_{ij}\overset{*}{h}{}^s{}_{ij}\}. \tag{5.2}$$

Again the normalized normal scalar curvature on M^n is given by

$$\begin{aligned} \hat{\rho}^\perp &= \frac{2}{n(n-1)} \left[\sum_{1\le r<s\le n} \sum_{1\le i<j\le n} \{g(S^\perp(E_i,E_j)E_{n+r}, E_{n+s})\}^2 \right]^{\frac{1}{2}} \\ &= \frac{2}{n(n-1)} \Big[\sum_{1\le r<s\le n} \sum_{1\le i<j\le n} \Big\{ \frac{c}{4}\{\delta_{is}\delta_{jr} - \delta_{ir}\delta_{js}\} + g([\hat{A}_{E_{n+r}}, \overset{*}{A}_{E_{n+s}}]E_i, E_j) \\ &+ g([\overset{*}{A}_{E_{n+r}}, \hat{A}_{En+s}]E_i, E_j) \Big\}^2 \Big]^{\frac{1}{2}}. \end{aligned} \tag{5.3}$$

Equation (5.3) can be written as

$$\begin{aligned} \hat{\rho}^\perp &= \frac{2}{n(n-1)} \Big[\sum_{1\le r<s\le n} \sum_{1\le i<j\le n} \Big\{ \frac{c}{4}\{\delta_{is}\delta_{jr} - \delta_{ir}\delta_{js}\} + \sum_{k=1}^{n} \Big(\hat{h}^s_{ik}\overset{*}{h}{}^r{}_{jk} \\ &- \hat{h}^r_{ik}\overset{*}{h}{}^s{}_{jk} + \overset{*}{h}{}^s{}_{ik}\hat{h}^r_{jk} - \hat{h}^r_{ik}\overset{*}{h}{}^s{}_{jk} \Big) \Big\}^2 \Big]^{\frac{1}{2}} \\ &= \frac{2}{n(n-1)} \Big[\sum_{1\le r<s\le n} \sum_{1\le i<j\le n} \Big\{ \frac{c}{4}\{\delta_{is}\delta_{jr} - \delta_{ir}\delta_{js}\} + \sum_{k=1}^{n} \Big(4(h^s_{ik}h^r_{jk} - h^r_{ik}h^s_{jk}) \\ &+ (\hat{h}^s_{ik}\hat{h}^r_{jk} - \hat{h}^r_{ik}\hat{h}^s_{jk}) + (\overset{*}{h}{}^s{}_{ik}\overset{*}{h}{}^r{}_{jk} - \overset{*}{h}{}^r{}_{ik}\overset{*}{h}{}^s{}_{jk}) \Big) \Big\}^2 \Big]^{\frac{1}{2}}. \end{aligned} \tag{5.4}$$

Using Minkowski inequality $\Big[\sum_i (a_i+b_i+c_i+d_i)^2\Big]^{\frac{1}{2}} \le \Big[\sum_i (a_i)^2\Big]^{\frac{1}{2}} + \Big[\sum_i (b_i)^2\Big]^{\frac{1}{2}} + \Big[\sum_i (c_i)^2\Big]^{\frac{1}{2}} + \Big[\sum_i (d_i)^2\Big]^{\frac{1}{2}}$, where $a_i, b_i, c_i, d_i \in \mathbb{R}^+ \cup \{0\}$ in equation (5.4) we get

$$\begin{aligned} \frac{n(n-1)}{2}\hat{\rho}^\perp &\le \Big[\sum_{1\le r<s\le n} \sum_{1\le i<j\le n} \Big\{ \frac{c^2}{16}\{\delta_{is}\delta_{jr} - \delta_{ir}\delta_{js}\}^2 \Big\} \Big]^{\frac{1}{2}} \\ &+ \Big[\sum_{1\le r<s\le n} \sum_{1\le i<j\le n} \Big(4\sum_{k=1}^{n} (h^s_{ik}h^r_{jk} - h^r_{ik}h^s_{jk}) \Big)^2 \Big]^{\frac{1}{2}} \\ &+ \Big[\sum_{1\le r<s\le n} \sum_{1\le i<j\le n} \Big(\sum_{k=1}^{n} (\hat{h}^s_{ik}\hat{h}^r_{jk} - \hat{h}^r_{ik}\hat{h}^s_{jk}) \Big)^2 \Big]^{\frac{1}{2}} \\ &+ \Big[\sum_{1\le r<s\le n} \sum_{1\le i<j\le n} \Big(\sum_{k=1}^{n} (\overset{*}{h}{}^s{}_{ik}\overset{*}{h}{}^r{}_{jk} - \overset{*}{h}{}^r{}_{ik}\overset{*}{h}{}^s{}_{jk}) \Big)^2 \Big]^{\frac{1}{2}}. \end{aligned} \tag{5.5}$$

Also from [6] we get the following:

$$\begin{aligned} & n^2\|H\|^2 - \frac{2n}{n-1} \sum_{r=1}^{n} \sum_{1\le i<j\le n} \{h^r_{ii}h^r_{jj} - (h^r_{ij})^2\} \\ & \ge \frac{2n}{n-1} \Big[\sum_{1\le r<s\le n} \sum_{1\le i<j\le n} \Big(\sum_{k=1}^{n} (h^s_{ik}h^r_{jk} - h^r_{ik}h^s_{jk}) \Big)^2 \Big]^{\frac{1}{2}}, \end{aligned} \tag{5.6}$$

$$n^2\|\hat{H}\|^2 - \frac{2n}{n-1}\sum_{r=1}^{n}\sum_{1\leq i<j\leq n}\{\hat{h}^r_{ii}\hat{h}^r_{jj} - (\hat{h}^r_{ij})^2\} \tag{5.7}$$

$$\geq \frac{2n}{n-1}\Big[\sum_{1\leq r<s\leq n}\sum_{1\leq i<j\leq n}\Big(\sum_{k=1}^{n}(\hat{h}^s_{ik}\hat{h}^r_{jk} - \hat{h}^r_{ik}\hat{h}^s_{jk})\Big)^2\Big]^{\frac{1}{2}},$$

$$n^2\|\overset{*}{H}\|^2 - \frac{2n}{n-1}\sum_{r=1}^{n}\sum_{1\leq i<j\leq n}\{\overset{*}{h}{}^r{}_{ii}\overset{*}{h}{}^r{}_{jj} - (\overset{*}{h}{}^r{}_{ij})^2\} \tag{5.8}$$

$$\geq \frac{2n}{n-1}\Big[\sum_{1\leq r<s\leq n}\sum_{1\leq i<j\leq n}\Big(\sum_{k=1}^{n}(\overset{*}{h}{}^s{}_{ik}\overset{*}{h}{}^r{}_{jk} - \overset{*}{h}{}^r{}_{ik}\overset{*}{h}{}^s{}_{jk})\Big)^2\Big]^{\frac{1}{2}}.$$

Using (5.6), (5.7) and (5.8) in (5.5) we get

$$\begin{aligned}\hat{\rho}^\perp \;&\leq\; \|\hat{H}\|^2 + \|\overset{*}{H}\|^2 + 4\|H\|^2 + \frac{c}{2\sqrt{2n(n-1)}} \\ &-\; \frac{2}{n(n-1)}\sum_{r=1}^{n}\sum_{1\leq i<j\leq n}\Big[4\{h^r_{ii}h^r_{jj} - (h^r_{ij})^2\} + \{\hat{h}^r_{ii}\hat{h}^r_{jj} - (\hat{h}^r_{ij})^2\} \\ &+\; \{\overset{*}{h}{}^r{}_{ii}\overset{*}{h}{}^r{}_{jj} - (\overset{*}{h}{}^r{}_{ij})^2\}\Big] \\ &=\; \|\hat{H}\|^2 + \|\overset{*}{H}\|^2 + 4\|H\|^2 + \frac{c}{2\sqrt{2n(n-1)}} \\ &-\; \frac{2}{n(n-1)}\sum_{r=1}^{n}\sum_{1\leq i<j\leq n}\Big[8\{h^r_{ii}h^r_{jj} - (h^r_{ij})^2\} + \{\hat{h}^r_{ii}\overset{*}{h}{}^r{}_{jj} + \overset{*}{h}{}^r{}_{ii}\hat{h}^r{}_{jj} - 2\hat{h}^r_{ij}\overset{*}{h}{}^r{}_{ij}\}\Big].\end{aligned} \tag{5.9}$$

Making use of (5.2) in (5.9) we get

$$\begin{aligned}\hat{\rho}^\perp \;&\leq\; \|\hat{H}\|^2 + \|\overset{*}{H}\|^2 + 4\|H\|^2 + \frac{c}{2\sqrt{2n(n-1)}} \\ &-\; \frac{2}{n(n-1)}\sum_{r=1}^{n}\sum_{1\leq i<j\leq n}\Big[8\{h^r_{ii}h^r_{jj} - (h^r_{ij})^2\}\Big] - (\hat{\rho} - \frac{c}{4}).\end{aligned} \tag{5.10}$$

Also from (2.7) and (2.31) we get

$$(\rho - \frac{c}{4}) = \frac{2}{n(n-1)}\sum_{r=1}^{n}\sum_{1\leq i<j\leq n}\{h^r_{ii}h^r_{jj} - (h^r_{ij})^2\}, \tag{5.11}$$

where $\rho = \frac{2}{n(n-1)}\sum_{1\leq i<j\leq n} R(E_i, E_j, E_j, E_i)$ is the normalized scalar of M with respect to ∇. By using of (5.10) and (5.11) we get the following:

Theorem 5.1. *Suppose M^n is a Legendrian submanifold of $\bar{M}^{2n+1}_{stat}(c)$. Then the following holds:*

$$\hat{\rho}^\perp \;\leq\; \|\hat{H}\|^2 + \|\overset{*}{H}\|^2 + 4\|H\|^2 - 8\rho - (\hat{\rho} - \frac{9c}{4}) + \frac{c}{2\sqrt{2n(n-1)}}. \tag{5.12}$$

Example 5.2. Consider the cosymplectic manifold $\bar{M} = \mathcal{C}^3 \times \mathbb{R}$ with the structure tensor $(\phi, \xi, \eta, \bar{g})$ is given by

$$\phi\Big(\sum_{i=1}^{3}(X_i\frac{\partial}{\partial x_i} + Y_i\frac{\partial}{\partial y_i})\Big) = \sum_{i=1}^{3}(Y_i\frac{\partial}{\partial x_i} - X_i\frac{\partial}{\partial y_i}),\; \phi(\frac{\partial}{\partial t}) = 0, \tag{5.13}$$

$$\xi = \frac{\partial}{\partial t},\;\; \eta = dt, \tag{5.14}$$

$$\bar{g} = dt^2 + \sum_{i=1}^{3} dx_i \otimes dx_i + dy_i \otimes dy_i. \tag{5.15}$$

Now we consider $\mathcal{K}(X,Y) = \bar{g}(X,\xi)\bar{g}(Y,\xi)\xi$, then $\mathcal{K}$ satisfies (2.10), (2.11) and (2.12). Hence $(\bar{M}, \hat{\bar{\nabla}} = \bar{\nabla} + \mathcal{K}, \phi, \xi, \eta, \xi)$ is cosymplectic statistical manifold.
Now we consider a submanifold M of $\bar{M}$ as an immersion defined by $\chi(x,y,t) = (0,0,e^{-t}(x+y),0,0,e^{-t}(x-y),t)$.
The vector fields $E_3 = \frac{1}{e^t}\{\frac{\partial}{\partial x_3} + \frac{\partial}{\partial y_3}\}$, $E_4 = \frac{1}{e^t}\{\frac{\partial}{\partial x_3} - \frac{\partial}{\partial y_3}\}$ and $E_7 = \frac{\partial}{\partial t}$ are linearly independent. The tensor field ϕ is given by $\phi E_3 = \frac{1}{e^t}\{\frac{\partial}{\partial x_3} - \frac{\partial}{\partial y_3}\}$ and $\phi E_4 = -\frac{1}{e^t}\{\frac{\partial}{\partial x_3} + \frac{\partial}{\partial y_3}\}$ and $\phi E_7 = 0$. Then it can be checked that $\phi X \in \Gamma(TM)$, $\forall\ X \in \Gamma(TM)$ and therefore M is invariant submanifold of $\bar{M}$.

Example 5.3. Let us consider a submanifold M of cosymplectic statistical manifold $\bar{M} = \mathcal{C}^3 \times \mathbb{R}$ defined in Example 5.2 as an immersion defined by $\chi(x,y,t) = (e^{-t}x, 0,0,0,e^{-t}y,0,t)$.
The vector field $E_1 = \frac{1}{e^t}\frac{\partial}{\partial x_1}$, $E_2 = \frac{1}{e^t}\frac{\partial}{\partial y_2}$ and $E_7 = \frac{\partial}{\partial t}$ are linearly independent.
The tensor field ϕ is given by $\phi E_1 = -\frac{1}{e^t}\frac{\partial}{\partial y_1}$, $\phi E_2 = \frac{1}{e^t}\frac{\partial}{\partial x_2}$ and $\phi E_7 = 0$. Then it can be checked that $\phi X \in \Gamma(T^{\perp}M)\ \forall\ X \in \Gamma(TM)$ and therefore M is anti-invariant submanifold of $\bar{M}$.

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