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NUMERICAL ERROR BOUND FOR NON-HOMOGENEOUS STABLE AND UNSTABLE LINEAR DYNAMICAL SYSTEMS

Adnan Daraghmeh

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Abstract In this article we use the Balanced Truncation (BT) method for constructing a reduced order model for both stable and unstable finite dimensional linear time-invariant (LTI) dynamical systems with non-homogeneous initial conditions. The L_2-error bound for the stable system has been obtained by interpolating the non-zero initial conditions as an extra input by choosing the Dirac's delta function $\delta_0 \in L_2$. This approach has been successfully extended to develop a framework for model reduction over a finite interval $[0, T]$ for unstable (LTI) systems. The advantages and flexibility of this approach are demonstrated with variety of numerical examples.

1 Introduction

Control design is one of the central themes in system theory. It examines the system settings through feedback, so that the closed loop system behaves as expected with a minimum cost. There are many physical, chemical and biological phenomenon, which are modeled in terms of partial differential equations. The state space formulation for such model requires infinite dimensionality design control. Thus, for purpose of computation and implementation, this is not practical. Therefore, it is important to find a low order controller for infinite dimensional system. Model reduction is a major issue for control, optimization and simulation of large-scale system that can be used to obtain low order controller [1]. A number of methods have been proposed to reduce order of infinite dimensional linear time invariant [FDLTI] systems such as balanced truncation (BT) [1],[2], Hankel norm approximation and singular perturbation approximation (SPA) [3], [4], [5], [6]. All these methods preserve certain properties of the original system such as stability, passivity and gives an error bound that easily commutable [7], [8], [9]. Although balanced truncation and singular perturbation approximation methods give the same upper bound of the error reduction , but the characteristics of both methods are contrary to each other [10],[11],[12]. It has been shown that the reduced systems by balanced truncation has a smaller error at high frequencies, and tend to be greater at lower frequencies [13]. Moreover, the reduced systems through SPA method behave otherwise, i.e., the error goes to zero at low frequencies and tend to enlarge at high frequencies [14],[15]. Balanced model reduction of linear control systems has been under investigation for quite a long time due to the ubiquity of large-scale linear systems in wide range of applications in science and engineering [11],[13],[16]. The general idea of balanced model reduction is to restrict the system to the subspace of easily controllable and observable states which can be determined by the Hankel singular values associated with the system [17],[18]. All these methods give a stable reduced system and guaranteed upper bound of the approximation error provided that the initial conditions are homogeneous, i.e., that $x_0 = 0$. Balanced model reduction of linear systems with non-homogeneous initial conditions has received very little attention, with [19],[20],[21],[22] being the only exceptions known to authors. In these papers, the authors extend balanced truncation to the case of non-homogeneous initial conditions (in case of [19],[20],[21],[22] using an L_2 regularization of the non-smooth input due to the initial data). Most of these model reduction methodologies have been developed

originally for asymptotically stable dynamical systems, that is, systems having all their poles in the left half-plane. On the other hand, there exist prominent applications where model reduction of unstable systems becomes a vital tool. Examples include the worst-case identification of unstable plants [23], and obtaining low-order approximations to Hamiltonian systems common in physics such as large collections of coupled oscillators [24]. Controllers are designed to derive a plant into desirable and robust performance settings [26]. However, many controller design techniques, such as LQG and $\mathcal{H}_\infty$ techniques, lead to controllers that have the same order as the plant to be controlled [25],[26]. High-order controllers are problematic for real-time applications due to the potential for degraded numerical accuracy, and computational difficulties. Alternatively, one could replace the original high-order controller with a low order but high-fidelity approximation.
Since controllers are usually unstable systems [27], the controller reduction problem leads directly to a model reduction problem involving unstable systems [26],[27],[28],[29],[30]. Most of these ways are based on balanced truncation. However, a different framework have been pursued in [31] which uses rational Krylov methods and an interpolatory framework for model reduction. The linear time-invariant (LTI) system is of the form

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \\ x(t_0) &= x_0, \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^n$ is an n-dimensional, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^l$ are the states, inputs and outputs of (1.1), and $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{l\times n}$ are matrices of appropriate size, $x_0 \in \mathbb{R}^n$ is the non-zero initial condition prescribed at $t_0 = 0$. A common feature of (1.1) is that it is high-dimensional, with n ranging from a few tens to several thousands as in control problems for large flexible space structures, and that it displays a variety of time scales. If the time in the system are well separated, it is possible to eliminate the fast degrees of freedom and to derive low-order reduced models, using averaging and homogenization techniques. In this article we use the Balanced Truncation (BT) method for constructing a reduced order model for both stable and unstable finite dimensional linear time-invariant (LTI) dynamical systems with non-homogeneous initial conditions. The L_2-error bound for the stable system has been obtained by interpolating the non-zero initial conditions as an extra input by choosing the Dirac's delta function $\delta_0 \in L_2$. This approach has been successfully extended to develop a framework for model reduction over a finite interval $[0, T]$ for unstable (LTI) systems. The advantages and flexibility of this approach are demonstrated with variety of numerical examples.

2 Error Bound for Stable In-homogeneous System

Consider the linear system described in equation (1.1) and x_0 is the non-zero initial condition can be partitioned into the form

$$x(t_0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix},$$

Assumption 2.1. Assume the system described by equation (1.1) is Controllable, Observable and Asymptotically Stable.

Now, we choose a non-singular matrix $W \in \mathbb{R}^{n\times n}$ and applying the method of balanced truncation to obtain a reduced order model written in the form:

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u, \\ \bar{y} &= \bar{C}\bar{x}, \end{aligned} \tag{2.1}$$

where

$$\bar{A} = WAW^{-1}, \quad \bar{B} = WB, \qquad \bar{C} = CW^{-1}$$

and the initial condition of the reduced system is

$$\bar{x}(t_0) = W^{-1}x(t_0),$$

Assumption 2.2. Assume the system described by equations (2.1) is Controllable, Observable and Asymptotically Stable.

We introduce the following two Lyapunov equations:

$$AC_o + C_oA^T = -BB^T,$$

and

$$A^TO_b + AO_b = -C^TC,$$

where C_o and O_b are the Controllability and Observability Gramian respectively defined as:

$$C_o = \int_0^\infty e^{At}\, BB^T\, e^{A^Tt} dt,$$

$$O_b = \int_0^\infty e^{A^Tt}\, C^TC\, e^{At} dt,$$

The balancing transformation W satisfies the following equation:

$$W^{-1}C_oW^{-T} = W^TO_bW^T = \Sigma = diag(\sigma_1, \cdots, \sigma_n),$$

where

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0.$$

are the Hankel Singular Values (HSVs).
In the case the initial condition $x(0)$ is zero and for all $u : (t_0, \infty) \longrightarrow \mathbb{R}^m$ and $r < n$, equations (1.1) and (2.1) satisfy the standard error bound [22]

$$\|y - \bar{y}\|_{L_2(t_0,\infty)} \leq 2 \sum_{r=i+1}^{n} \sigma_i \|u\|_{L_2(t_0,\infty)}, \tag{2.2}$$

We introduce a new error bound between the output of the full system (1.1) and its reduced system (2.1). We extend the system in equation (1.1) to get the following system

$$\begin{aligned} \dot{x}_e &= Ax_e + \begin{pmatrix} B & X_0 \end{pmatrix} \begin{pmatrix} u \\ \delta_0 \end{pmatrix}, \\ y_e &= Cx_e, \end{aligned} \tag{2.3}$$

where A, B, C are defined the same as in equation (1.1), x_e is the state vector, y_e is the output vector and $X_0 = x(t_0)$ is the initial condition and δ_0 defined as

$$\delta_0(t) = \lim_{\epsilon \to 0} \delta_\epsilon(t)$$

The Dirac's delta function denoted by $\delta(t)$ is defined as:

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

such that

$$\int_{t_1}^{t_2} \delta(t) dt = 1,$$

where $0 \in [t_1, t_2]$.
The important property of the delta function is the following relation

$$\int f(t)\delta(t - t_0) dt = f(t_0),$$

Assumption 2.3. Assume the system described by equations (2.3) is Controllable, Observable and Asymptotically Stable.

Now, the extended system in (2.3) can be reduced using balanced truncation method to get the following reduced system:

$$\begin{aligned} \dot{\bar{x}}_e &= \bar{A}\bar{x}_e + \begin{pmatrix} \bar{B} & \bar{X}_0 \end{pmatrix} \begin{pmatrix} u \\ \delta_0 \end{pmatrix}, \\ \bar{y}_e &= \bar{C}\bar{x}_e, \end{aligned} \tag{2.4}$$

where $\bar{A}, \bar{B}, \bar{C}$ the same as in equation (2.1).
The non-zero initial condition for the reduced system (2.4) is given as:

$$\bar{X}_0 = W^{-1}x(t_0),$$

The solution $x_e(t)$ of the extended system in equation (2.3) found to be:

$$\begin{aligned} x_e(t) &= \int_{t_0}^{t} e^{A(t-\tau)} \begin{pmatrix} B & X_0 \end{pmatrix} \begin{pmatrix} u \\ \delta_0 \end{pmatrix} d\tau \\ &= \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau)d\tau + \int_{t_0}^{t} e^{A(t-\tau)} X_0\delta_0(\tau)d\tau \\ &= \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau)d\tau + e^{At}X_0 \\ &= e^{At}X_0 + \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau)d\tau \\ &= x(t), \end{aligned} \tag{2.5}$$

Systems (1.1) and (2.3) have the same state space solution and hence the outputs of the systems must be the same, that is:

$$y_e(t) = y(t) = C\left(e^{At}X_0 + \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau)d\tau \right), \tag{2.6}$$

Also the reduced systems (2.4) and (2.1) have the same solution:

$$\begin{aligned} \bar{x}_e(t) &= \int_{t_0}^{t} e^{\bar{A}(t-\tau)} \begin{pmatrix} \bar{B} & \bar{X}_0 \end{pmatrix} \begin{pmatrix} u \\ \delta_0 \end{pmatrix} d\tau \\ &= \int_{t_0}^{t} e^{\bar{A}(t-\tau)} \bar{B}u(\tau)d\tau + \int_{t_0}^{t} e^{\bar{A}(t-\tau)} \bar{X}_0\delta_0(\tau)d\tau \\ &= \int_{t_0}^{t} e^{\bar{A}(t-\tau)} \bar{B}u(\tau)d\tau + e^{\bar{A}t}\bar{X}_0 \\ &= e^{\bar{A}t}\bar{X}_0 + \int_{t_0}^{t} e^{\bar{A}(t-\tau)} \bar{B}u(\tau)d\tau \\ &= \bar{x}(t), \end{aligned} \tag{2.7}$$

and the outputs of the two reduced systems are equal:

$$\bar{y}_e(t) = \bar{y}(t) = \bar{C}\left(e^{\bar{A}t}\bar{X}_0 + \int_{t_0}^{t} e^{\bar{A}(t-\tau)}\bar{B}u(\tau)d\tau\right), \tag{2.8}$$

Let C_o be the Controllability Gramian of the full system (1.1) and C_{oe} be the Controllability Gramian of the extended system (2.3), then we show that the two matrices are not equal:

$$\begin{aligned} C_{oe} &= \int_{t_0}^{\infty} e^{At}\begin{pmatrix} B & X_0 \end{pmatrix}\begin{pmatrix} B & X_0 \end{pmatrix}^T e^{A^T t}dt \\ &= \int_{t_0}^{\infty} e^{At}\begin{pmatrix} BB^T + X_0X_0^T \end{pmatrix} e^{A^T t}dt \\ &= \int_{t_0}^{\infty} e^{At}BB^Te^{A^T t}dt + \int_{t_0}^{\infty} e^{At}X_0X_0^Te^{A^T t}dt \\ &= C_o + \int_{t_0}^{\infty} e^{At}X_0X_0^Te^{A^T t}dt, \end{aligned} \tag{2.9}$$

Let O_b be the controllabilty Gramian of the full system (1.1) and O_{be} be the Observability Gramian of the extended system (2.3), then the two matrices are equal:

$$Roe = \int_{t_0}^{\infty} e^{A^T t}C^TCe^{At}dt = R_o, \tag{2.10}$$

Let $\sigma_1 \geq, \sigma_1 \geq \cdots, \geq \sigma_1 \geq 0$ are the Hankel Singular Values, then we write:

$$\Sigma_1 = diag(\sigma_1, \cdots, \sigma_r), \quad \Sigma_2 = diag(\sigma_{r+1}, \cdots, \sigma_n), \quad r < n$$

and

$$\Sigma = \begin{pmatrix} \Sigma_1 & \\ & \Sigma_2 \end{pmatrix},$$

Since $O_{be} = O_b$, then using the balanced transformation W, we obtain:

$$W^TO_bW = W^TO_{be}W = \Sigma = diag(\sigma_1, \cdots, \sigma_n), \tag{2.11}$$

We factorize the Observability Gramian as:

$$R_{oe} = R_o = L^T, \quad L \in \Re^{n\times n},$$

where

$$L^TL = \int_{t_0}^{\infty} e^{A^T t}C^TCe^{At}dt,$$

The following Theorem contains our new error bound between the outputs of the full and reduced systems.

Theorem 2.4. *[32]*
Let $W \in \mathbb{R}^{n\times n}$ be a non-singular transformation matrix and let

$$\sigma_1 \geq, \sigma_1 \geq \cdots, \geq \sigma_1 \geq 0,$$

are the Hankel Singular Values of the extended system (2.5-2.6), then for all

$$u \in L_2(t_0, \infty),$$

the error bound between $y \in L_2(t_0,\infty)$ and $\bar{y} \in L_2(t_0,\infty)$ is:

$$\|y-\bar{y}\|_{L_2(t_0,\infty)} \leq \|LX_0\|_2^2 + \|\sqrt{\Sigma_1}\bar{X}_0\|_2^2 + 2\sum_{r=i+1}^{n} \sigma_i \|u\|_{L_2(t_0,\infty)}, \tag{2.12}$$

Proof. We consider the two cases $x(t_0)=0$ and $u=0$

$$\begin{aligned}
\|y-\bar{y}\|_{L_2(t_0,\infty)} &= \|y_e-\bar{y}_e\|_{L_2(t_0,\infty)} \\
&= \|\int_{t_0}^{t} e^{A(t-\tau)} \begin{pmatrix} B & X_0 \end{pmatrix} \begin{pmatrix} u \\ \delta_0 \end{pmatrix} d\tau - \int_{t_0}^{t} e^{A(t-\tau)} \begin{pmatrix} B & X_0 \end{pmatrix} \begin{pmatrix} u \\ \delta_0 \end{pmatrix} d\tau\| \\
&= \|e^{At}X_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau - e^{\bar{A}t}\bar{X}_0 - \int_{t_0}^{t} e^{\bar{A}(t-\tau)}\bar{B}u(\tau)d\tau\| \\
&\leq \|e^{At}X_0 - e^{\bar{A}t}\bar{X}_0\| + \|\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau - \int_{t_0}^{t} e^{\bar{A}(t-\tau)}\bar{B}u(\tau)d\tau\|,
\end{aligned}$$

for the case $x(t_0)=0$, we have the error bound

$$\|\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau - \int_{t_0}^{t} e^{\bar{A}(t-\tau)}\bar{B}u(\tau)d\tau\| = 2\sum_{r=i+1}^{n} \sigma_i \|u\|_{L_2(t_0,\infty)},$$

for the case $u=0$, we have

$$\|e^{At}X_0 - e^{\bar{A}t}\bar{X}_0\| \leq \|e^{At}X_0\| + \|e^{\bar{A}t}\bar{X}_0\|,$$

since O_b is factorized as

$$L^TL = \int_{t_0}^{\infty} e^{A^Tt}C^TCe^{At}dt,$$

then we want to estimate the first term

$$\|e^{At}X_0\|_{L_2(t_0,\infty)},$$

This can be achieved by

$$\begin{aligned}
\|e^{At}X_0\|_{L_2(t_0,\infty)} &\leq X_0^T \left(\int_{t_0}^{\infty} e^{A^Tt}C^TCe^{At}dt \right) X_0 \\
&\leq \|X_0^TL^TLX_0\| \\
&\leq \|(LX_0)^T(LX_0)\| \\
&\leq \|(LX_0)^T\|\|LX_0\| \\
&\leq \|LX_0\|_{L_2},
\end{aligned}$$

We do the same for the second part

$$\|e^{\bar{A}t}\bar{X}_0\|,$$

and obtain this bound

$$\|e^{\bar{A}t}\bar{X}_0\|_{L_2(t_0,\infty)} \leq \|\sqrt{\Sigma_1}\bar{X}_0\|_{L_2},$$

Consequently, we get the error bound:

$$\|y-\bar{y}\|_{L_2(t_0,\infty)} \leq \|LX_0\|_{L_2} + \|\sqrt{\Sigma_1}\bar{X}_0\|_{L_2} + 2\sum_{r=i+1}^{n} \sigma_i \|u\|_{L_2(t_0,\infty)}, \tag{2.13}$$

□

3 Error Bound for Un-Stable In-homogeneous System

In this section, we introduce an approach for computing the $L_2[t_0, T]$ induced norm to obtain the the error bound between the outputs of the original and reduced order un-stable systems with non-zero initial condition. To estimate the $L_2[0, T]$-induced norm for the system defined by equations (2.3) and (2.4), we compute the H_∞ norm of a shifted version of the system under consideration. This bound can be used to solve the problem of model reduction for unstable system over finite horizon with non-zero initial condition. Officially, the $L_2[0, T]$-induced norm of a given LTI system is equivalent to $L_2[0, \infty)$-induced norm of a time-variant system with convolution kernel [30].
Assume that the error bound in equation (2.13) is written as:

$$\|y - \bar{y}\|_{L_2(t_0,\infty)} \leq \beta, \tag{3.1}$$

where

$$\beta = \|LX_0\|_{L_2} + \|\sqrt{\Sigma_1}\bar{X}_0\|_{L_2} + 2\sum_{r=i+1}^{n} \sigma_i \|u\|_{L_2},$$

The linear dynamical systems in equations (2.3) and (2.4) is stable and in virtue of [30],[33] we have the following Lemma:

Lemma 3.1. *Consider a strictly proper, finite dimensional LTI stable system, given as:*

$$\left(\begin{array}{c|c} A & [B \quad X_0] \\ \hline C & \end{array}\right),$$

Then the following are equivalent:

(i) The $L_2[0, \infty)$-induced gain is bounded by $\beta > 0$:

$$\|y - \bar{y}\|_{L_2(t_0,\infty)} \leq \beta,$$

.

(ii) The following linear matrix inequality admits a positive definite solution $X > 0$:

$$\begin{pmatrix} A^T X + XA + C^T C & XH \\ H^T X & -\beta^2 I \end{pmatrix} < 0, \tag{3.2}$$

where $H = [B \quad X_0]$

Now, in the case that the linear dynamical system in equations (2.3) and (2.4) is unstable and in virtue of [30],[33] with finite interval $[t_0, T]$, we have the following lemma:

Lemma 3.2. *Consider a strictly proper, finite dimensional not necessarily stable LTI system, given as:*

$$\left(\begin{array}{c|c} A & [B \quad X_0] \\ \hline C & \end{array}\right),$$

Assume the following differential matrix inequality avow a positive definite solution $X(t), \forall t \in [0, T]$:

$$\begin{pmatrix} A^T X + XA + \dot{X} + C^T C & X[B \quad X_0] \\ [B \quad X_0]^T X & -\beta^2 I \end{pmatrix} < 0, \tag{3.3}$$

where $H = [B \quad X_0]$

Let $u \in L_2[t_0, T]$ be an arbitrary input and y be the corresponding output, then the following holds [30]:

$$\int_0^T y^T y dt < \beta^2 \int_0^T u^T u dt,$$

The following corollary contain the L_2-norm for a finite interval $[t_0, T]$:

Corollary 3.3. *If the inequality (3.3) holds, then:*

$$\|y-\bar{y}\|_{L_2[t_0,T]} < \beta, \tag{3.4}$$

Consider the following step window function defined as:

$$W(t) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & otherwise \end{cases},$$

and define the convolution kernel as:

$$(y-\bar{y})(t,\tau) = W(t)(y-\bar{y})(t-\tau), \tag{3.5}$$

To make the computations efficient, we approximate the step window by exponential window of the form e^{-at} where the time constant a satisfy $e^{-at} < 1$ for $t > T$.
If we use this approximation, then the resulting kernel $e^{-at}(y-\bar{y})(t)$ can be associated with the new LTI system whose frequency response is a shifted version of the frequency response of the original system. The constant a can be chosen such that, our new LTI system is stable. The next step is to compute its $L_2[0,\infty)$-induced norm or H_∞ norm.

To approximate the error bound between the original and its reduced system defined in equations (2.3) and (2.4) using $L_2[0,T]$-induced norm, we introduce the following theorem which contains the main result.

Theorem 3.4. *Consider a strictly proper finite dimensional, LTI (not necessarily stable) system G, given as:*

$$\left(\begin{array}{c|c} A & [B \quad X_0] \\ \hline C & \end{array}\right),$$

if there exist a such that:

$$\left(\begin{array}{c|c} A-aI & [B \quad X_0] \\ \hline C & \end{array}\right),$$

is stable with $\|(y-\bar{y})_a\|_\infty < \beta$, then the following holds:

$$\|(y-\bar{y})\|_{L_2[0,T],ind} < \beta e^{aT}, \tag{3.6}$$

Proof. Assume that the bound

$$\|e^{aT}(y-\bar{y})_a\|_\infty < \beta e^{aT},$$

From Lemma (3.1) there exists $X_a \geq 0$ such that:

$$\begin{pmatrix} A_a^T X_a + X_a A_a + e^{aT} C^T C e^{aT} & X_a H \\ H^T X_a & -\beta^2 e^{2aT} I \end{pmatrix} < 0, \tag{3.7}$$

where $A_a = A - aI$ and $H = [B \quad X_0]$
The next step, we define the solution $X(t), \quad \forall t \in [0,T]$ as:

$$X(t) = e^{-2at} X_a,$$

If we multiply equation (3.7) by $e^{-2at} I(n+m, n+m)$, then we have:

$$\begin{pmatrix} A_a^T X_a e^{-2at} + e^{-2at} X_a A_a + e^{2a(T-t)} C^T C & e^{-2at} X_a H \\ H^T X_a e^{-2at} & -\beta^2 e^{2a(T-t)} I \end{pmatrix} < 0$$

$$\begin{pmatrix} A^T X + XA + \dot{X} + e^{2a(T-t)} C^T C & XH \\ H^T X & -\beta^2 e^{2a(T-t)} I \end{pmatrix} = 0,$$

$$\begin{pmatrix} A^T X + XA + \dot{X} + C^T C & XH \\ H^T X & -\beta^2 e^{2aT} I \end{pmatrix} \leq 0$$

where the last inequality relies on $a > 0$ and $t \leq T$.
From Lemma (3.2) and corollary (3.3), we obtain our result and hence the proof is complete. □

Finally, we can write the error bound in theorem (3.4) in the form:

$$\|(y-\bar{y})\|_{L_2[0,T],ind} < e^{aT}\|\sqrt{\Sigma}\|_2^2\|X_0\|_2^2 + 2e^{aT}\sum_{i=r+1}^{n}\sigma_i, \tag{3.8}$$

4 Numerical Examples and Results

To illustrate the effectiveness of our approach we consider the following numerical examples for both stable and unstable systems with non-homogeneous initial conditions.

Example 4.1. Consider the following continuous linear time-invariant stable dynamical model reduction of a build example system [34] with $n = 48$ degrees of freedom and the size of the reduced system is $r = 3$.

The maximum error bound of the dynamical system obtained by applying the result in equation (2.13) is illustrated in Figure (2). The L_2 norm for the difference between the two outputs y and $\bar{y}$ in equation (2.13) is computed for $r = 3$ and is shown in Figure (1). Moreover, Table (1) contains the L_2 norm of difference between the two outputs y and $\bar{y}$ using the result in equation (2.13).

Table 1. The Maximum Error Bound and The Outputs of The Results in equation (2.13) With $n = 48, r = 3$ for The Build Example.

r	$\lvert y-\bar{y}\rvert_{L_2}$	$2\sum_{i=r+1}^{48}\sigma_i$
1	1.3302e-05	0.0243
2	1.3322e-05	0.0194
3	1.3334e-05	0.0156
4	1.3345e-05	0.0117
5	1.3349e-05	0.0103
6	1.3353e-05	0.0089
7	1.3356e-05	0.0076
8	1.3359e-05	0.0064
9	1.3361e-05	0.0055
10	1.3362e-05	0.0047
11	1.3364e-05	0.0042
15	1.3368e-05	0.0022

Example 4.2. In this example, we study a continuous linear time-invariant unstable dynamical model reduction of a resistor–capacitor circuit (RC-circuit) described in Figure (3). We take the size of the full dynamical system and its reduced system to be $n = 20$ and $r = 2$ respectively.

We use the result in equation (3.8) to compute the approximating values of the L_2 norm for the outputs of the full and its reduced order model. The maximum error bound in equation (3.8) is computed and shown in Figure (4). The $L_2[0,T], ind$ norm for the difference between the outputs y and $\bar{y}$ with $r = 2$ is illustrated in Figure (5). Table (2) contains the $L_2[0,T], ind$ norm of difference between the two outputs y and $\bar{y}$ using the result in equation (3.8).

5 Conclusion

In this work we used the Balanced Truncation (BT) method for constructing a reduced order model for both stable and unstable finite dimensional linear time-invariant (LTI) dynamical systems with non-homogeneous initial conditions. The L_2-error bound for the stable system has

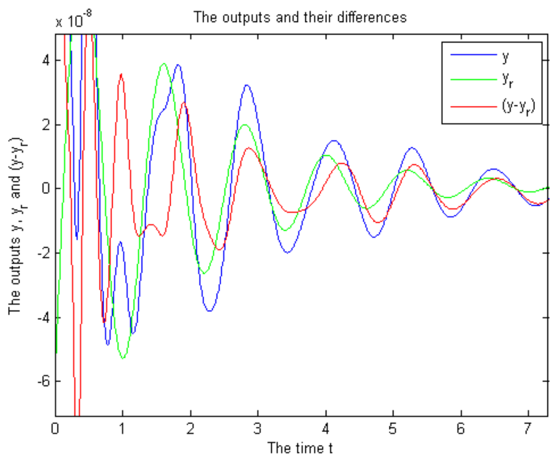

Figure 1. The L_2 Norm of The Difference Between The Outputs of The Full And The Reduced Models with $n = 48, r = 3$ for The Build Example

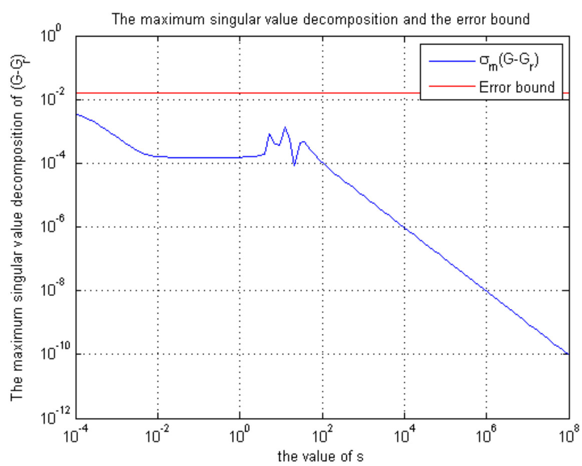

Figure 2. The Maximum Error Bound of The Build Example With $n = 48, r = 3$

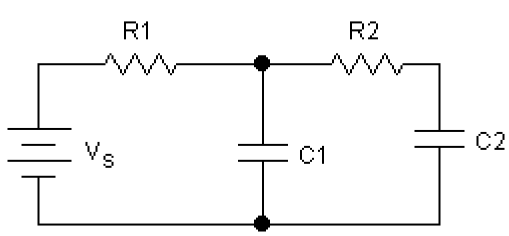

Figure 3. Simple RC-circuit

been obtained by interpolating the non-zero initial conditions as an extra input by choosing the Dirac's delta function $\delta_0 \in L_2$. This approach has been successfully extended to develop a framework for model reduction over a finite interval $[0, T]$ for unstable (LTI) systems. The theoretical results have been validated numerically to show the advantages and flexibility of this approach.

Conflicts of Interest: The authors declare that they have no conflict of interest.

Table 2. The Maximum Error Bound and The Outputs of The Results in (3.4) and (2.4) With $n = 48, r = 3$ for The RC Example.

r	$\|y - \bar{y}\|_{L_2[0,T],ind}$	$2 \sum_{i=r+1}^{20} \sigma_i$
2	0.0264	0.6807
4	0.0041	0.0925
6	0.000194	0.0167
8	7.5842×10^{-7}	0.0089
10	5.2526×10^{-9}	0.0056
12	3.026×10^{-13}	0.0031
14	2.3145×10^{-15}	4.6517×10^{-4}

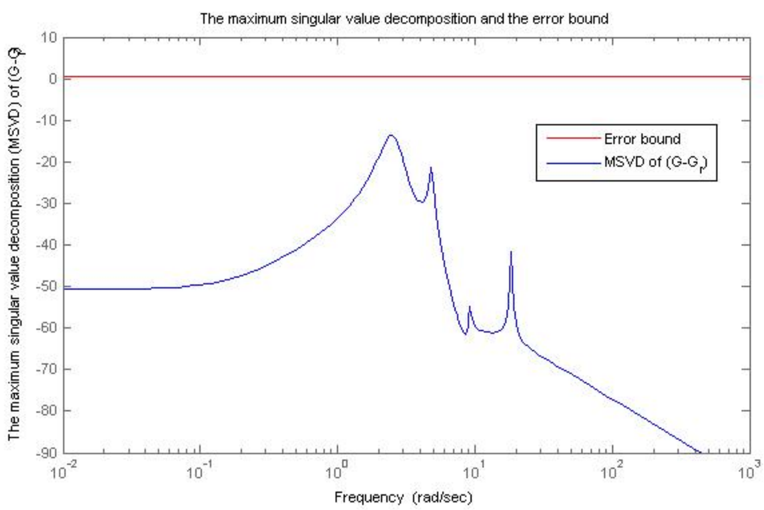

Figure 4. The MSVD and the error bound when $r = 2$

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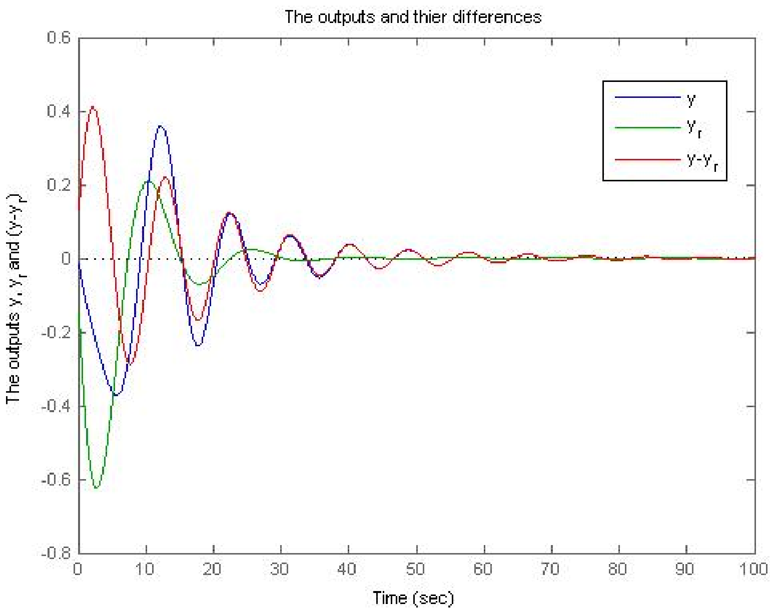

Figure 5. The outputs and their difference when $r = 2$

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Author information

Adnan Daraghmeh, Department of Mathematics, An-Najah National University, Nablus, Palestine.
E-mail: `adn.daraghmeh @ najah.edu`

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