Palestine Journal of Mathematics
Vol. 11(4)(2022) , 373–378

ON NEGATIVITY OF TOEPLITZ–HESSENBERG DETERMINANTS WHOSE ELEMENTS CONTAIN LARGE SCHRÖDER NUMBERS

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Communicated by P. K. Harikrishnan

MSC 2010 Classifications: Primary 11B83; Secondary 11C20, 15A15, 15B05, 26A51.

Keywords and phrases: Negativity, logarithmic convexity, Hessenberg determinant, Toeplitz determinant, large Schröder number, central Delannoy number, Wronski's formula, Kaluza's theorem.

The author appreciates anonymous reviewers for their careful corrections to and valuable comments on the original version of this paper.

Abstract In the paper, by virtue of Wronski's formula and Kaluza's theorem for the power series and its reciprocal, and with the aid of the logarithmic convexity of a sequence constituted by the products of the factorials and the large Schröder numbers, the author presents negativity of a class of Toeplitz–Hessenberg determinants whose elements contain the products of the factorials and the large Schröder numbers.

1 Introduction

It is common knowledge that a sequence a_n for $n \ge 0$ is said to be logarithmically convex if the inequality $a_n^2 \ge a_{n-1}a_{n+1}$ holds for all $n \ge 1$. See [6, p. 369, 3.9.59].

An upper (or a lower, respectively) Hessenberg matrix is an $n \times n$ matrix $H_n = (h_{ij})_{1 \le i,j \le n}$, where $h_{ij} = 0$ for all pairs (i,j) such that $j+1 < i$ (or $i+1 < j$, respectively). See [16, Chapter 10].

A Toeplitz matrix is a matrix of the form

$$\begin{pmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{-n} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & \cdots & t_{-n+1} & t_{-n+2} \\ t_2 & t_1 & t_0 & \cdots & t_{-n+2} & t_{-n+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{n-2} & t_{n-3} & t_{n-4} & \cdots & t_0 & t_{-1} \\ t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_1 & t_0 \end{pmatrix},$$

that is, an $n \times n$ matrix $T_n = (t_{k,j})_{0 \le k,j \le n-1}$, where $t_{k,j} = t_{k-j}$. See [3].

For our convenience, we call the determinants $|H_n|$ and $|T_n|$ the Hessenberg determinant and the Toeplitz determinant, respectively. If an $n \times n$ matrix M_n is both a Hessenberg matrix and a Toeplitz matrix, we call its determinant $|M_n|$ the Toeplitz–Hessenberg determinant.

In combinatorial number theory, there exist two kinds of the Schröder numbers, the large Schröder numbers S_r and the little Schröder numbers s_r. They are named after the German mathematician Ernst Schröder. The large Schröder number S_r can be combinatorially interpreted as the number of paths from the southwest corner $(0,0)$ of an $r \times r$ grid to the northeast corner (r,r), using only single steps north, northeast, or east, that do not rise above the southwest-northeast diagonal. The positive integer sequence of the large Schröder numbers S_r can be analytically generated [1, Theorem 8.5.7] by

$$\frac{1-t-\sqrt{t^2-6t+1}}{2t} = \sum_{r=0}^{\infty} S_r t^r = 1 + 2t + 6t^2 + 22t^3 + 90t^4 + 394t^5 + \cdots. \tag{1.1}$$

Between the large Schröder numbers S_r and the little Schröder numbers s_r, there is the relation

$$S_r = 2s_{r+1}, \quad r \in \mathbb{N}.$$

See [1, Corollary 8.5.8], [9, p. 123], [10, p. 142], [11, p. 718], [12, p. 2], and [13, pp. 24–25].

Let $r \ge 1$ be a positive integer and let $|\alpha_{ij}|_r$ denote a determinant of order r with elements α_{ij}. If α_i for $1 \le i \le r$ are non-negative integers, then the large Schröder numbers S_r satisfy

$$\left|(-1)^{\alpha_i+\alpha_j}(\alpha_i+\alpha_j)!S_{\alpha_i+\alpha_j}\right|_r \ge 0 \quad \text{and} \quad \left|(\alpha_i+\alpha_j)!S_{\alpha_i+\alpha_j}\right|_r \ge 0. \tag{1.2}$$

See [9, Theorem 2.7] and [11, Theorem 1.3].

In combinatorial number theory, central Delannoy numbers $D(r)$ for $r \ge 0$ are the number of "king walks" from the $(0,0)$ corner of an $r \times r$ square to the upper right corner (r,r). The sequence of central Delannoy numbers $D(r)$ can be analytically generated by

$$\frac{1}{\sqrt{t^2-6t+1}} = \sum_{r=0}^{\infty} D(r)t^r = 1 + 3t + 13t^2 + 63t^3 + \cdots. \tag{1.3}$$

For $r \in \mathbb{N}$, the large Schröder numbers S_r and central Delannoy numbers $D(r)$ have relations

$$S_r = \frac{(-1)^r}{2}\begin{vmatrix} D(1) & 1 & 0 & \cdots & 0 & 0 & 0 \\ D(2) & D(1) & 1 & \cdots & 0 & 0 & 0 \\ D(3) & D(2) & D(1) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ D(r-1) & D(r-2) & D(r-3) & \cdots & D(1) & 1 & 0 \\ D(r) & D(r-1) & D(r-2) & \cdots & D(2) & D(1) & 1 \\ D(r+1) & D(r) & D(r-1) & \cdots & D(3) & D(2) & D(1) \end{vmatrix} \tag{1.4}$$

and

$$D(r) = 2^r \begin{vmatrix} \frac{3}{2} & -\frac{1}{2} & 0 & 0 & \cdots & 0 & 0 & 0 \\ S_1 & \frac{3}{2} & -\frac{1}{2} & 0 & \cdots & 0 & 0 & 0 \\ S_2 & S_1 & \frac{3}{2} & -\frac{1}{2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ S_{r-3} & S_{r-4} & S_{r-5} & S_{r-6} & \cdots & \frac{3}{2} & -\frac{1}{2} & 0 \\ S_{r-2} & S_{r-3} & S_{r-4} & S_{r-5} & \cdots & S_1 & \frac{3}{2} & -\frac{1}{2} \\ S_{r-1} & S_{r-2} & S_{r-3} & S_{r-4} & \cdots & S_2 & S_1 & \frac{3}{2} \end{vmatrix}. \tag{1.5}$$

See [9, Theorem 2.9] and [11, Theorem 1.5]. The relations (1.4) and (1.5) were derived in [11, Theorem 1.5] by using Wronski's formula (2.1) in Lemma 2.1 below and by connecting the factors $\sqrt{t^2-6t+1}$ in (1.1) and (1.3) respectively.

In [14, Theorem 1.1], central Delannoy numbers $D(r)$ for $r \in \mathbb{N}$ were proved to satisfy

$$\begin{vmatrix} D(1) & 1 & 0 & \cdots & 0 & 0 & 0 \\ D(2) & D(1) & 1 & \cdots & 0 & 0 & 0 \\ D(3) & D(2) & D(1) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ D(r-2) & D(r-3) & D(r-4) & \cdots & D(1) & 1 & 0 \\ D(r-1) & D(r-2) & D(r-3) & \cdots & D(2) & D(1) & 1 \\ D(r) & D(r-1) & D(r-2) & \cdots & D(3) & D(2) & D(1) \end{vmatrix}$$

$$= -\frac{1}{6^r}\sum_{\ell=1}^{r}(-1)^\ell 6^{2\ell}\frac{(2\ell-3)!!}{(2\ell)!!}\binom{\ell}{r-\ell}. \tag{1.6}$$

Comparing (1.4) with (1.6) and simplifying reveal

$$S_r = \frac{(-1)^{r+1}}{2\times 6^{r+1}}\sum_{\ell=1}^{r+1}(-1)^\ell 6^{2\ell}\frac{(2\ell-3)!!}{(2\ell)!!}\binom{\ell}{r-\ell+1}, \quad r\in\mathbb{N}. \tag{1.7}$$

The explicit formula (1.7) or its variants for the large Schröder numbers S_r can also be found in [9, Theorem 2.1], [10, Theorem 1], and [12, Theorem 1].

In this paper, we will present negativity of a class of Toeplitz–Hessenberg determinant determinants whose elements contain the products of the factorials $r!$ and the large Schröder numbers S_r. This new result is different from those in (1.2) and (1.5).

2 Lemmas

In this paper, we need the following lemmas.

Lemma 2.1 (Wronski's formula). *If $a_0\ne 0$ and $P(t)=a_0+a_1t+a_2t^2+\cdots$ is a formal series, then the coefficients of the reciprocal series $\frac{1}{P(t)}=b_0+b_1t+b_2t^2+\cdots$ are given by*

$$b_r=\frac{(-1)^r}{a_0^{r+1}}\begin{vmatrix} a_1 & a_0 & 0 & 0 & \cdots & 0 & 0 & 0\\ a_2 & a_1 & a_0 & 0 & \cdots & 0 & 0 & 0\\ a_3 & a_2 & a_1 & a_0 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ a_{r-2} & a_{r-3} & a_{r-4} & a_{r-5} & \cdots & a_1 & a_0 & 0\\ a_{r-1} & a_{r-2} & a_{r-3} & a_{r-4} & \cdots & a_2 & a_1 & a_0\\ a_r & a_{r-1} & a_{r-2} & a_{r-3} & \cdots & a_3 & a_2 & a_1 \end{vmatrix}, \quad r=1,2,\dots. \tag{2.1}$$

Wronski's formula (2.1) in Lemma 2.1 can be found in [2, p. 17, Theorem 1.3], [4, p. 347], [8, Lemma 2.4], [15, Section 2], and the paper [17].

Lemma 2.2 (Kaluza's theorem). *Let $P(t)=a_0+a_1t+a_2t^2+\cdots$ be a formal power series over the field of real numbers such that the sequence a_r for $r=0,1,2,\dots$ is positive and logarithmically convex. If the reciprocal series $\frac{1}{P(t)}=b_0+b_1t+b_2t^2+\cdots$, then $b_r<0$ for $r=1,2,\dots$.*

Kaluza's theorem stated in Lemma 2.2 can be found in [2, p. 13, Problem 6] and the paper [5].

Lemma 2.3 ([9, Corollary 2.1], [10, Theorem 3], [11, Corollary 1.1], and [14, Corollary 1.2]). *The sequence $r!S_r$ for $r\ge 0$ is logarithmically convex.*

3 Negativity of a class of Toeplitz–Hessenberg determinants

In this section, we present negativity of a class of the Hessenberg determinants whose elements contain the products of the factorials $r!$ and the large Schröder numbers S_r. This new result is different from those in (1.2) and (1.5).

Theorem 3.1. *For $m\in\{0\}\cup\mathbb{N}$ and $r\in\mathbb{N}$, if*

$$a_0>\frac{(m!S_m)^2}{(m+1)!S_{m+1}}, \tag{3.1}$$

then

$$(-1)^r \begin{vmatrix} m!S_m & a_0 & 0 & \cdots & 0 & 0 \\ (m+1)!S_{m+1} & m!S_m & a_0 & \cdots & 0 & 0 \\ (m+2)!S_{m+2} & (m+1)!S_{m+1} & m!S_m & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (m+r-3)!S_{m+r-3} & (m+r-4)!S_{m+r-4} & (m+r-4)!S_{m+r-4} & \cdots & a_0 & 0 \\ (m+r-2)!S_{m+r-2} & (m+r-3)!S_{m+r-3} & (m+r-4)!S_{m+r-4} & \cdots & m!S_m & a_0 \\ (m+r-1)!S_{m+r-1} & (m+r-2)!S_{m+r-2} & (m+r-3)!S_{m+r-3} & \cdots & (m+1)!S_{m+1} & m!S_m \end{vmatrix} < 0. \tag{3.2}$$

Proof. For fixed $m \in \{0\} \cup \mathbb{N}$, let

$$a_r = (m+r-1)!S_{m+r-1}, \quad r \geq 1. \tag{3.3}$$

Using the sequence a_r for $r = 0, 1, 2, \dots$ and utilizing Wronski's formula (2.1) in Lemma 2.1, we obtain

$$b_r = \frac{(-1)^r}{a_0^{r+1}} \begin{vmatrix} m!S_m & a_0 & 0 & \cdots & 0 & 0 \\ (m+1)!S_{m+1} & m!S_m & a_0 & \cdots & 0 & 0 \\ (m+2)!S_{m+2} & (m+1)!S_{m+1} & m!S_m & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (m+r-3)!S_{m+r-3} & (m+r-4)!S_{m+r-4} & (m+r-4)!S_{m+r-4} & \cdots & a_0 & 0 \\ (m+r-2)!S_{m+r-2} & (m+r-3)!S_{m+r-3} & (m+r-4)!S_{m+r-4} & \cdots & m!S_m & a_0 \\ (m+r-1)!S_{m+r-1} & (m+r-2)!S_{m+r-2} & (m+r-3)!S_{m+r-3} & \cdots & (m+1)!S_{m+1} & m!S_m \end{vmatrix}$$

for $m \in \{0\} \cup \mathbb{N}$ and $r \in \mathbb{N}$. Since the condition (3.1) is equivalent to $a_0a_2 - a_1^2 > 0$, further using Lemma 2.3, we conclude that the sequence a_r defined by (3.1) and (3.3) for $r \geq 0$ is logarithmically convex. Utilizing Lemma 2.2 and employing the logarithmic convexity of the sequence (3.3) arrive at the negativity of the sequence b_r, that is, the negativity in (3.2) is valid for $m \in \{0\} \cup \mathbb{N}$ and $r \in \mathbb{N}$. The proof of Theorem 3.1 is complete. □

Corollary 3.2. *Let $r \in \mathbb{N}$. If $a_0 > \frac{1}{2}$, we have*

$$(-1)^r \begin{vmatrix} S_0 & a_0 & 0 & \cdots & 0 & 0 & 0 \\ 1!S_1 & S_0 & a_0 & \cdots & 0 & 0 & 0 \\ 2!S_2 & 1!S_1 & S_0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ (r-3)!S_{r-3} & (r-4)!S_{r-4} & (r-4)!S_{r-4} & \cdots & S_0 & a_0 & 0 \\ (r-2)!S_{r-2} & (r-3)!S_{r-3} & (r-4)!S_{r-4} & \cdots & 1!S_! & S_0 & a_0 \\ (r-1)!S_{r-1} & (r-2)!S_{r-2} & (r-3)!S_{r-3} & \cdots & 2!S_2 & 1!S_1 & S_0 \end{vmatrix} < 0. \tag{3.4}$$

If $a_0 > \frac{1}{3}$, we have

$$(-1)^r \begin{vmatrix} 1!S_1 & a_0 & 0 & \cdots & 0 & 0 & 0 \\ 2!S_2 & 1!S_1 & a_0 & \cdots & 0 & 0 & 0 \\ 3!S_3 & 2!S_2 & 1!S_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ (r-2)!S_{r-2} & (r-3)!S_{r-3} & (r-3)!S_{r-3} & \cdots & 1!S_1 & a_0 & 0 \\ (r-1)!S_{r-1} & (r-2)!S_{r-2} & (r-3)!S_{r-3} & \cdots & 2!S_2 & 1!S_1 & a_0 \\ r!S_r & (r-1)!S_{r-1} & (r-2)!S_{r-2} & \cdots & 3!S_3 & 2!S_2 & 1!S_1 \end{vmatrix} < 0. \tag{3.5}$$

Proof. When taking $m = 0, 1$ in (3.2), we derive (3.4) and (3.5) readily. □

4 Remarks

Finally we give several remarks.

Remark 4.1. An anonymous reviewer pointed out that, for $p > 0$ and setting

$$a_r = [(m + r - 1)!S_{m+r-1}]^p$$

instead of the sequence defined by (3.3), the result in Theorem 3.1 can be extended and this will give a "two-dimensional version" (m, p) of Theorem 3.1.

Remark 4.2. An anonymous reviewer observed that the numbers taken as a lower bound for a_0 (starting with $m = 1$) increase extremely quickly. For $m = 10$ this requires a_0 larger than 67 milliards.

Remark 4.3. This paper is a companion of the article [7].

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Received: August 21, 2021.
Accepted: December 27, 2021.