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Conharmonic symmetry inheritance and its physical significance in general relativity

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Abstract In this paper, we have studied conharmonic curvature inheritance (Conh CI) symmetry and its physical importance on the spacetime of general relativity. Also, we have obtained some new results which are direct consequence of proper conformal symmetries. The necessary conditions for a Conh CI vector to be a conformal Killing vector (CKV) have been obtained and results for establishing its relationship between the other known symmetries of spacetimes in general and Einstein spacetimes have been derived. Conh CI with conharmonic motion has been discussed and it is found that a proper Conh CI can produce a new physically relevant solution for perfect fluid spacetime. For Conh CI with conformal Killing vector in perfect fluid spacetime, the equations of state have also been obtained.

1 Introduction

Under some transformations, the symmetry of a system is a physical or mathematical characteristic that remains unchanged. In recent years, the study of symmetries of the spacetime has gained more attention (cf., [1]). Some geometrical or physical quantities are preserved by these transforms, including metric, curvature, Ricci, stress-energy and Einstein tensors of spacetime. Existing solutions of Einstein and Einstein-Maxwell equations and their classification were explored earlier through spacetime symmetries. There are several applications of symmetries of spacetime available in their relation with laws of conservation, such as the law of conservation of momentum, energy etc. [2]. Most primary spacetime symmetries include conformal Killing, homothetic, Killing vector, matter, Ricci and curvature collineations. Different types of symmetries have been studied by Stephani [3], Duggal [4], Hall [5] and Ahsan et. al. ([6] - [17]).

Let (V_4, g) be a spacetime with V_4 a four-dimensional connected smooth Hausdorff manifold and a smooth Lorentzian metric g with signature $(-,+,+,+)$. Symbols $\mathcal{R}$ and $\mathcal{C}$ are used for $(1,3)$ type Riemannian curvature tensor and conformal curvature tensor respectively. The components (in local coordinates) are written as R^h_{ijk} for Riemann curvature tensor, C^h_{ijk} for conformal curvature tensor, R_{ij} for Ricci tensor, g_{ij} for metric tensor and R for scalar curvature respectively, where $R_{ij} = R^h_{ihj}$ and $R = R_{ij}g^{ij}$.

In 1992, the notion of curvature inheritance (CI) was introduced by K.L. Duggal in [18]. Curvature inheritance is the generalization of curvature collineation (CC) [19]. A spacetime (V_4, g) admits curvature inheritance if the relation

$$\pounds_\xi R^h_{ijk} = 2\alpha R^h_{ijk}, \tag{1.1}$$

is satisfied, where $\pounds_\xi$ represents the Lie differentiation operator along a vector ξ in (1.1). The vector ξ is called the curvature inheritance vector (CIV) and $\alpha = \alpha(x^i)$ is an inheriting factor. In particular, if $\alpha = 0$ then CI reduces to CC. If $\alpha \neq 0$ then ξ defines a proper CI. Duggal [18] showed that the proper CI has a direct connection with the physically applicable proper conformal Killing vector. Based on definition (1.1), CI does retain the physical importance of Komar's identity [20].

Generally, inheritance symmetry of spacetime is defined by a geometrical or physical object mathematically as

$$\pounds_\xi \mathcal{A} = 2\alpha\mathcal{A}, \tag{1.2}$$

where α is representing the scalar function on V_4, and $\mathcal{A}$ is a geometrical/physical quantity like, g_{ij}, R^h_{ijk}, and R_{ij} etc. By fixing a quantity, we can define a specific inheritance symmetry. As on substitution of g_{ij} in place of $\mathcal{A}$, equation (1.2) defines conformal motion (Conf M) [21]. Similarly it defines for R^h_{ijk}, curvature inheritance (CI) [18], and for R_{ij}, Ricci inheritance (RI) [22]. Here it should be mentioned that for inheritance symmetry along CKV ($\mathcal{A}$=g_{ij}), the function $\alpha(x^i)$ is called as conformal factor and for rest of the inheritance symmetries it is called as inheriting factor. Further, if $\alpha = 0$ in equation (1.2) then all the inheritance types will be reduced to collineations. The most primary symmetry on (V_4, g) is motion (M) or isometry, which is obtained by setting $\mathcal{A} = g_{ij}$ and $\alpha = 0$ in equation (1.2). Then equation (1.2) is called the Killing equation and the vector ξ is called Killing vector.

In [23], Abdussattar and B. Dwivedi introduced conharmonic symmetries, and they have established the relationship between the conharmonic Killing vector (Conh KV) and inheriting symmetries. They have also studied anisotropic fluid spacetimes admitting Conh KV. A spacetime (V_4, g) admits a conharmonic curvature collineation (Conh CC) along a vector ξ it $\pounds_\xi Z^h_{ijk} = 0$, where Z^h_{ijk} is represents the conharmonic curvature tensor of spacetime and the vector ξ in this case is called Conh KV.

Inheritance symmetry vectors are beneficial in study the Einstein's field equations (EFEs) and their exact solutions. EFEs are given by

$$R_{ij} - \frac{1}{2}Rg_{ij} = \kappa T_{ij}, \tag{1.3}$$

where g_{ij}, R_{ij}, and T_{ij} signify the components of metric, Ricci tensor, and energy momentum tensor respectively and scalar curvature is denoted by R and κ is the gravitational constant. In this paper, we consider equation (1.3) for κ = 1. Symmetry vector also determines conservation law ([20], [24]).

However, a recent study [23] shows only the limited use of Conh KV in general relativity, but the continuity of the work prevails. This is due to the restriction $\Box\psi = g^{ij}\psi_{;ij} = 0$ and $\psi_{;ij} \neq 0$, which gives the equation of state in the perfect fluid spacetime cases i.e., one can say that, perfect fluid models are not included. On the other way, Conh KV is significant due to their various applications in cosmology and astrophysics [23]. This suggests that there is a need to modify the concept of Conh KV, compatible with a proper CKV and other symmetries.

In 2021, [1] M. Ali et. al., introduce a new type of symmetry called conharmonic curvature inheritance (Conh CI) along a vector field ξ such that:

$$\pounds_\xi Z^h_{ijk} = 2\alpha Z^h_{ijk}, \tag{1.4}$$

where Z^h_{ijk} represents the conharmonic curvature tensor and $\alpha = \alpha(x^i)$ is an inheriting factor on V_4. In particular, Conh CI implies to Conh CC when $\alpha = 0$ and for $\alpha \neq 0$, equation (1.4) defines a proper Conh CI along a vector ξ. In the article [1], the authors studied conharmonic curvature inheritance symmetry of spacetime of general relativity when inheriting factor (α) and conformal factor (ψ) are same (α = ψ). In this paper, we have studied conharmonic curvature inheritance symmetry when inheriting factor and conformal factor are different ($\alpha \neq \psi$). Equation (1.4) can also be explicitly written as the following :

$$Z^h_{ijk;l}\xi^l - Z^l_{ijk}\xi^h_{;l} + Z^h_{ljk}\xi^l_{;i} + Z^h_{ilk}\xi^l_{;j} + Z^h_{ijl}\xi^l_{;k} = 2\alpha Z^h_{ijk}. \tag{1.5}$$

Here (;) stands for the covariant derivative with respect to (x^i) co-ordinates in the spacetime. The importance of the study of Conh CI is a very helpful tool in finding the invariance properties of some geometrical objects like the Einstein tensor. Einstein's tensor plays a significant role in the theory of general relativity, as it connects with the matter contents of spacetime. Also, Einstein field equations and their solutions depend on these tensors. Such studies also contribute to explore the physical fields in a certain region of spacetime, and their reflections represent the symmetries of the metric. Now using the symmetry aspects of perfect fluid spacetime in the general theory of relativity where the energy-momentum tensor is defined as

$$T_{ij} = (\mu + p)u_i u_j + pg_{ij}, \tag{1.6}$$

where p, μ and u_i are isotropic pressure, energy density and four velocity vector respectively.

The layout of the paper is the following. In section 2, we give some basic information about conharmonic curvature inheritance and related results, which is required for this paper. Section 3 is categorized into three subsections. In the first subsection, we study Conh CI with conformal motion and investigate the necessary condition for a Conh CIV to be a CKV. In particular, some results are obtained in the Einstein spacetime . In the next subsection, we derive the relationship of Conh CI with conharmonic motion in general spacetime and Einstein spacetime. In the last subsection of Section 3, we construct a block diagram that represents the relationship of Conh CI with other well-known symmetries of spacetime. Finally, we give a physical application in perfect fluid spacetime, which is discussed in Section 4. A concise summary and valuable discussion regarding physical significance are presented in the last section.

2 Conharmonic Curvature Inheritance

Conharmonic curvature tensor on (V_4, g) is denoted by Z^h_{ijk} and defined as follows ([25]-[26]):

$$Z^h_{ijk} = R^h_{ijk} + \frac{1}{2}(\delta^h_j R_{ik} - \delta^h_k R_{ij} + g_{ik}R^h_j - g_{ij}R^h_k), \tag{2.1}$$

where δ^h_j stands for Kronecker delta and $R^h_k = g^{ih}R_{ik}$.

Definition 2.1. Let (V_4, g) be a spacetime with Lorentzian metric g. A vector field ξ that preserves a spacetime conharmonic curvature tensor constant up to some scalar function all along the way is called a Conh CI vector field, while scalar function in this case is called inheriting factor. Mathematically, a Conh CIV is defined as follows [1]:

$$\pounds_\xi Z^h_{ijk} = 2\alpha Z^h_{ijk}. \tag{2.2}$$

If inheriting factor $\alpha = 0$, Conh CI reduces to Conh CC. If $\alpha \neq 0$, it is said that spacetime admits a proper Conh CI.

Theorem 2.2. *In a Ricci flat spacetime, Conh CI symmetry reduces to CI symmetry.*

Proof. In Ricci flat spacetime (empty spacetime), Ricci tensor vanishes identically ($R_{ij} = 0$) then equation (2.1) implies that conharmonic curvature tensor reduces to Riemannian curvature tensor. Then

$$\pounds_\xi R^h_{ijk} = \pounds_\xi Z^h_{ijk} = 2\alpha Z^h_{ijk} = 2\alpha R^h_{ijk}.$$

This completes the proof. □

Contracting equation (2.1), then

$$Z_{ij} = -\frac{1}{2}g_{ij}R, \tag{2.3}$$

where $Z_{ij} = Z^h_{ihj}$ known as contracted conharmonic curvature tensor [27]. Now contracting equation (2.2) over indices h and j, we obtain

$$\pounds_\xi Z_{ij} = 2\alpha Z_{ij}. \tag{2.4}$$

Definition 2.3. A vector field ξ^i satisfying (2.4) is called contracted Conh CIV.

Thus, in general, every Conh CIV is a contracted Conh CIV, but converse may not be true. Suppose that

$$\hbar_{ij} = \pounds_\xi g_{ij} = \xi_{i;j} + \xi_{j;i}. \tag{2.5}$$

Mathematically the conformal curvature tensor of spacetime is defined as follows (cf., [26])

$$C^h_{ijk} = R^h_{ijk} + \frac{1}{2}(\delta^h_j R_{ik} - \delta^h_k R_{ij} + R^h_j g_{ik} - R^h_k g_{ij}) + \frac{R}{6}(g_{ij}\delta^h_k - g_{ik}\delta^h_j) \tag{2.6}$$

and also, it may be expressed in terms of Z^h_{ijk} and Z_{ij} as follows

$$C^h_{ijk} = Z^h_{ijk} + \frac{1}{3}(Z_{ik}\delta^h_j - Z_{ij}\delta^h_k). \tag{2.7}$$

Similarly Weyl projective tensor for spacetime defined as (cf., [26])

$$W^h_{ijk} = R^h_{ijk} + \frac{1}{3}[\delta^h_j R_{ik} - \delta^h_k R_{ij}] \tag{2.8}$$

and also, it may be expressed in terms of Z^h_{ijk} and Z_{ij} as

$$W^h_{ijk} = Z^h_{ijk} + \frac{1}{12}(Z_{ik}\delta^h_j - Z_{ij}\delta^h_k) + \frac{1}{4}(g^h_k Z_{ij} - g^h_j Z_{ik}). \tag{2.9}$$

In general, for a Conh CIV ξ, use of equations (2.2) and (2.5) leads to the following identities.

Theorem 2.4. *If a spacetime admits a Conh CI then the following identities hold:*

$$(a) \quad \pounds_\xi C^h_{ijk} = 2\alpha C^h_{ijk}, \quad (b) \quad \pounds_\xi W^h_{ijk} = 2\alpha W^h_{ijk} + E^h_{ijk}, \tag{2.10}$$

where $E^h_{ijk} = \frac{1}{4}(\hbar^h_k Z_{ij} - \hbar^h_j Z_{ik})$.

Proof. Operating the Lie differentiation on equation (2.7) and then using (2.2) and (2.4), we obtain

$$\pounds_\xi C^h_{ijk} = 2\alpha C^h_{ijk}, \tag{2.11}$$

which completes the proof of (2.10)(a). Similarly, we can prove (2.10) (b) by taking the Lie derivative of (2.9) and then using equations (2.2), (2.4), and (2.5). □

Remark 2.5. Theorem 2.4 indicates the significance of using Conh CI over CI. It motivates the Conh CI symmetry of spacetime since it implies the conformal curvature inheritance symmetry. While the CI does not inherits symmetry for conformal curvature tensor.

It is known that the spacetime (V_4, g) is conformally flat [28] iff

$$C^h_{ijk} = 0, \tag{2.12}$$

projectively flat [28] iff

$$W^h_{ijk} = 0, \tag{2.13}$$

and conharmonically flat [27] iff

$$Z^h_{ijk} = 0. \tag{2.14}$$

The tensors Z^h_{ijk} and C^h_{ijk} become identical if the scalar curvature of spacetime is zero. Then, conformal curvature tensor satisfies the symmetry inheritance property ((2.10)(a)) provided the spacetime admits the Conh CI along a vector field ξ.

Theorem 2.6. *A conformally flat spacetime admits Conh CI if and only if there exits a conformal Killing vector ξ, provided the conformal factor and the inheriting factor are the same.*

Proof. Let us assume that spacetime is conformally flat then equation (2.6) reduces to

$$Z^h_{ijk} = \frac{R}{6}(g_{ik}\delta^h_j - g_{ij}\delta^h_k). \tag{2.15}$$

We take Lie derivative of (2.15), then from (2.2), we get

$$\pounds_\xi g_{ij} = 2\alpha g_{ij}. \tag{2.16}$$

Thus, we can say that ξ is a conformal Killing vector field.
Conversely, operating Lie derivative on both sides of (2.15), we obtain

$$\pounds_\xi Z^h_{ijk} = \frac{R}{6}(\delta^h_j \pounds_\xi g_{ik} - \delta^h_k \pounds_\xi g_{ij}). \tag{2.17}$$

If ξ is a conformal Killing vector field then we have

$$\pounds_\xi Z^h_{ijk} = 2\alpha Z^h_{ijk}. \tag{2.18}$$

This completes the proof. □

Theorem 2.7. *If a spacetime (V_4, g) admits a Conh CIV ξ^i, then*

$$(\xi^m R^i_m)_{;i} = R\alpha. \tag{2.19}$$

Proof. A Conh CIV satisfies the explicitly written equation (1.5). Contracting the equation (1.4) then making use of $Z_{ij} = -2R_{ij}$ and the contracted Bianchi identity, we get the equation (2.19). □

Now we consider equation (2.5), which shows variation in metric tensor g_{ij} with respect to a Conh CIV ξ. The following identity holds in general

$$\hbar_{l(h} Z^l_{i)jk} = 0. \tag{2.20}$$

Generally, the solution of above identity (2.20) is expressed as

$$\pounds_\xi g_{ij} = 2\beta g_{ij} + M_{ij}, \tag{2.21}$$

where β is scalar function on (V_4, g) and M_{ij} is a second order symmetric tensor. Equation (2.21) can be classified into the following types:

Type- I. If $M_{ij} = 0$, then (2.21) reduces to Conf M or conformal Killing equation.

Type-II. If $M_{ij} \neq 0$ and $M_{ij} \neq g_{ij}$, then ξ is a non-conformal vector.

It may be noted that the metric inheritance or non-inheritance, depends on whether (V_4, g) belongs to type-I or type-II. In the context of inheritance symmetry, a pervasive study has been done on the symmetries defined by metric with the help of type-I or type-II ([4] - [22]).

At this stage, we quote that as the components of Riemann curvature tensor arise on a spacetime (V_4, g), admitting a connection that may not be a metric connection. Hence we have a reason that why do we compare the curvature or conharmonic curvature symmetries to metric symmetries. The answer can be given two ways for geometry and physics.

From the physics point of view, the answer can be made due to the study of congruence of timelike geodesics and their deviation $\nabla^2_t v^i = R^i_{jkl} u^j u^k u^l$. Here, u^i and v^i are the vectors, tangential to timelike geodesic satisfying $u^i v_i = 0$ and ‘t’ is the proper time (cf., [29], [5]).

And, from the geometry point of view, the answer will be obtained in forming a Lie algebra of finite dimension, generated by the set of conformal Killing vectors even if these are at least of C^3. On contrary, it is not necessary that structure of Lie algebra with smooth C^∞ curvature inheritance symmetry vector of dimension is finite. If smooth differentiablity condition is not considered, then we may lose the structure of Lie algebra for the curvature inheritance vector (cf., [29], [5]).

For type-II, due to the presence of a different tensor M_{ij} in (2.21) we find that the group structure cannot be guaranteed, and many examples justify the comparison. The reader may also find other physical/geometrical reasons for comparison. In this paper our focus is on study related to only type-I, while type- II will be our subsequent work.

3 Conharmonic Curvature Inheritance with Conformal Motion

In this section, we discuss conharmonic curvature inheritance symmetry with conformal motions, which is defined in ([21]) as follows:

$$\hbar_{ij} = \pounds_\xi g_{ij} = 2\psi g_{ij}, \tag{3.1}$$

where $\psi = \psi(x^i)$ is a conformal factor on (V_4, g), ξ^i is called conformal Killing vector and (3.1) is called conformal Killing equation. If $\psi = 0$, then, ξ is called a KV and (3.1) is called Killing equation. If $\psi_{;i} = 0$, then, ξ is called a HV and (3.1) is said to be homothetic motion (HM). If $\psi_{;ij} = 0$ and $\psi_{;i} \neq 0$, then, ξ is said to be a special conformal Killing vector (SCKV) and (3.1) is said to be special conformal motion (S Conf M). Further, it is found that the following equations are satisfied by a CKV (cf., [18], [30]):

$$\begin{aligned} &(a) \quad \pounds_\xi R_{ij} = -\Box\psi g_{ij} - 2\psi_{;ij}, \\ &(b) \quad \pounds_\xi R = -6\Box\psi - 2\psi R, \\ &(c) \quad \pounds_\xi C^h_{ijk} = 0, \end{aligned} \tag{3.2}$$

where Laplacian-Beltrami operator $\Box$ of ψ is the defined as, $\Box\psi = \psi_{;ij}g^{ij}$.

Theorem 3.1. *A Conh CIV ξ^i, which is a CKV and also satisfies Komar's identity, then the vector field ξ^i necessarily holds the following condition:*

$$[[\sqrt{g}(\xi^{i;j} - \xi^{j;i})]_{;j}]_{;i} = 0. \tag{3.3}$$

Proof. It is known that the conharmonic curvature tensor satisfies the identity [1],

$$Z^i_{klm} g_{ij} + Z^i_{jlm} g_{ik} = 0. \tag{3.4}$$

Operating the Lie derivative of (3.4) and using equations (2.2) and (3.1), we obtain

$$Z^i_{klm} \hbar_{ij} + Z^i_{jlm} \hbar_{ik} = 0. \tag{3.5}$$

Now using expression (2.1) for Z^h_{ijk} and (3.1) in (3.5), we get

$$R^i_{klm} \hbar_{ij} + R^i_{jlm} \hbar_{ik} = 0. \tag{3.6}$$

By virtue of the Ricci identity, (3.6) reduces to following form

$$\hbar_{ij;kl} - \hbar_{ij;lk} = 0. \tag{3.7}$$

Thus, a necessary condition for a vector ξ^i which is defined Conh CI as well as CKV are represented by equation (3.7). Multiplying by $\sqrt{g}g^{il}g^{jk}$ in equation (3.7), we obtain (3.3). This leads the proof. □

In the general theory of relativity, Komar's identity [20] directly exhibits a key role for establishing a conservation law. Along any vector field ξ, (3.3) holds, so the Conh CI's are the symmetries of (V_4, g) that are admitted through the group of coordinate transformations for general curvilinear system. Thus, we proceed to the following theorem:

Theorem 3.2. *Let a set $S = \{\Omega_\wedge : \wedge = 1, 2, 3, 4, 5, ..., m\}$ of all those geometric objects which satisfy the conharmonic inheritance symmetry on (V_4, g), and the set $S' = \{\xi_1, \xi_2, ..., \xi_n\}$ of vector fields, under which all $\Omega_\wedge$ inherit conharmonic symmetry, then the set S' forms a Lie algebra of finite dimension.*

Proof of the Theorem 3.2 is obvious by virtue of the Theorem 3.1.

Theorem 3.3. *Let a spacetime (V_4, g) admits a conharmonic curvature inheritance vector ξ, which is also a conformal Killing vector. Then*

$$\begin{aligned} &(a) \quad \Box\psi = -\tfrac{\alpha}{3} R, \\ &(b) \quad \psi_{;ij} = \tfrac{\alpha}{6} Z_{ij}, \\ &(c) \quad \alpha = 4\psi + \xi^i \partial_i (\log \sqrt{R}). \end{aligned} \tag{3.8}$$

Proof. Taking the Lie derivative of (2.3) then using equations (3.1) and (3.2)(b), we obtain

$$\pounds_\xi Z_{ij} = 3\Box\psi g_{ij}. \tag{3.9}$$

Now comparing the equation (3.9) with (2.4), and then using (2.3), we get

$$\Box\psi = -\frac{\alpha}{3}R. \tag{3.10}$$

This leads to the proof of the first part. Proof of the second part follows from (3.10), $\Box\psi = \psi_{;ij}g^{ij}$, on multiplication with g_{ij}, then we get

$$\psi_{;ij} = -\frac{\alpha R}{12}g_{ij}. \tag{3.11}$$

From (2.3), equation (3.11) reduces to (3.8) (b). Now we prove the last part. For a CKV, ψ must satisfy the condition, $\psi = \frac{1}{4}\xi^i_{;i}$, with this condition equation (2.19) leads to

$$\alpha R = (4\psi R + \xi^m R^i_{m;i}). \tag{3.12}$$

Now we apply contracted Bianchi identity (cf., [21]), $R^i_{m;i} = \frac{1}{2}\partial_m R$, in (3.12), and we get

$$\alpha = 4\psi + \xi^i\partial_i(\log\sqrt{R}). \tag{3.13}$$

The proof is completed. □

Corollary 3.4. *From Theorem 3.3, it can be deduced that if ξ is a Killing or homothetic or special CKV, then there exists no Conh CIV other than Conh CCV.*

The proof follows from (3.8) (b) with use of condition $\psi_{;ij} = 0$ for Killing, homothetic or special conformal Killing vector.

Corollary 3.5. *If an Einstein spacetime admits inheriting symmetry (1.2) then inheriting factor and conformal factor are scalar multiple each other.*

Proof. As we know, scalar curvature of an Einstein spacetime is constant $\partial_i R = \frac{\partial R}{\partial x^i} = 0$, then equation (3.8)(c) reduces to $\alpha = 4\psi$. Thus, inheriting factor and conformal factor are scalar multiple to each other. □

Corollary 3.6. *From Theorem 3.3 and Corollary 3.5, it is found that operator $\Box$ follows an eigen function equation, $\Box\alpha = \lambda\,\alpha$, where $\lambda = (-\frac{R}{3})$.*

Theorem 3.7. *If an Einstein spacetime admits conharmonic curvature inheritance symmetry, then the Ricci tensor admits the symmetry inheritance property.*

Proof. Let (V_4, g) be an Einstein spacetime with (R$\neq$ 0) which is defined as follows:

$$R_{ij} = \frac{R}{4}g_{ij}, \qquad (\text{R= const}). \tag{3.14}$$

Now (3.14) can be expressed in terms of Z_{ij} as follows:

$$R_{ij} = -\frac{Z_{ij}}{2}. \tag{3.15}$$

Now taking Lie derivative of (3.15) and using (2.4), we get

$$\pounds_\xi R_{ij} = -\alpha Z_{ij}, \tag{3.16}$$

which on using (3.15), equation (3.16) reduces to

$$\pounds_\xi R_{ij} = 2\alpha R_{ij}. \tag{3.17}$$

Thus, Ricci tensor admits the symmetry inheritance property. □

Since Duggal [18] has proved a result for CIV as "Every proper CIV in an Einstein spacetime is a proper CKV with $\alpha = \psi$." From this statement and Theorem 3.7, we have the following.

Corollary 3.8. *In a space of constant curvature, every Conh CIV is a proper CKV with $\psi = \alpha$.*

Theorem 3.9. *A spacetime (V_4, g) admits a proper Conh CI with a proper CKV is necessarily a conformally flat space.*

Proof. For a proper Conh CI, (V_4, g) holds the equation (2.10) (a). Since (V_4, g) also admits a CKV, then it satisfies the equation (3.2) (c). Now combining both equations, we get

$$C^h_{ijk} = 0, \qquad (\alpha \neq 0). \tag{3.18}$$

Thus, from (2.12), (V_4, g) is a conformally flat space. □

Conformally flat spacetime is the important one in the category of flat spaces for both fields viz., differential geometry and general theory of relativity. Thus, the study of spacetime symmetries (curvature collineation) in conformally flat spacetimes has a great importance that has attracted many recent interests. G S Hall [31] and his collaborators have worked on curvature collineation in conformally flat spacetime. Furthermore, [32] G. S. Hall, J. R. Pulham, and A. D. Hossack have discussed the Lie algebra structures of CKV for conformally flat plane waves. Enrique Alvarez and Raquel Santos-Garcia [33] present a new class of conformal field theory, when the background gravitational field considered to be as conformally flat. Conformally flat spacetimes enjoy conformal properties, quite similar to the that of flat spacetimes. Now here we propose the problem of finding all conformally flat spaces which admit a Conh CI vector, including those manifolds, which also admit conformal Killing vector.

Example 3.10. Let (V_4, g) be an Einstein spacetime admits a Conh CI vector ξ. It follows from Corollary 3.8, and use of equation (3.8)(b) leads to

$$\alpha_{;ij} = (-\frac{\alpha R}{12})g_{ij}, \tag{3.19}$$

where α and R are scalar function of spacetime (V_4, g). We consider the single scalar function ϕ instead of $(-\frac{\alpha R}{12})$ in (3.19), then we obtain

$$\alpha_{;ij} = \phi g_{ij}. \tag{3.20}$$

Petrov [2] refers a finding of Sinyukov [34], which explains that, if a spacetime (V_4, g) is admitting a vector field $\phi_{;i}$ satisfying (3.20) for $\phi \neq 0$, then an existing co-ordinate system has metric of the form:

$$ds^2 = g_{11}dx^1dx^1 + (\frac{1}{g_{11}})\Gamma_{ab}(x^2, x^3, x^4)dx^a dx^b, \tag{3.21}$$

where a, b $\neq$ 1 and $g_{11} = [2\int \phi(x^1)dx^1 + C]^{-1}$ and arbitrary function $\phi = \phi(x^1)$.

This example of (V_4, g) with metric (3.21) is well suited with Theorem 3.3 and Corollary 3.8.

Theorem 3.11. *If a spacetime (V_4, g) admits projective collineation and $E^h_{ijk} = 0$, then the spacetime does not admit Conh CIV other than a Conh CCV.*

Proof. Let (V_4, g) admits a projective collineation (PC) vector ξ such that

$$\pounds_\xi W^h_{ijk} = 0. \tag{3.22}$$

If ξ is also a Conh CIV, it is evident from (2.10) (b) that, either α=0 $\Rightarrow \xi$ is Conh CCV or W^h_{ijk}=0. Thus, (V_4, g) has constant curvature and so V_4 is Einstein spacetime, then use of corollary 3.8 leads to the statement that every PCV is also CKV. Yano [21] has proved a result as "If ξ is both CKV and PCV, then it is homothetic vector."Applying this, we obtain that $\alpha = 0$ is only possibility, hence it proves the Theorem. □

Theorem 3.12. *If ξ is a CKV and CIV in a (V_4, g), then the symmetry of the conharmonic curvature tensor is an inheriting symmetry.*

Proof. Taking the Lie differentiation of the equation (2.1) and use of the inheritance symmetry properties of Riemannian curvature tensor (cf., [18]), i.e. $\pounds_\xi R^h_{ijk} = 2\alpha R^h_{ijk}$, $\pounds_\xi R_{ij} = 2\alpha R_{ij}$ and $\pounds_\xi R^i_j = 2\alpha R^i_j - R^i_m \hbar^m_j$ and equation (3.1), we obtain

$$\pounds_\xi Z^h_{ijk} = 2\alpha Z^h_{ijk} + M^h_{ijk}, \tag{3.23}$$

where

$$M^h_{ijk} = \frac{1}{2}[g_{ij}\hbar^h_m R^m_k - g_{ik}\hbar^h_m R^m_j + R^h_j \hbar_{ik} - R^h_k \hbar_{ij}]. \tag{3.24}$$

Again using equation of CKV and (3.1) in (3.24) we get $M^h_{ijk} = 0$, and equation (3.23) thus reduces to

$$\pounds_\xi Z^h_{ijk} = 2\alpha Z^h_{ijk}.$$

Thus, conharmonic curvature tensor satisfies symmetry inheritance property. □

Remark 3.13. Theorem 3.12 also holds good, if conformal factor is equal to inheriting factor in spacetime (V_4, g).

Theorem 3.14. *If ξ is a CKV and RIV in a (V_4, g) under consideration of conharmonic curvature inheritance symmetry, then the symmetry of Riemannian curvature tensor is an inheriting one.*

Proof. Taking Lie differentiation of the equation (2.1) and then using the inheritance symmetry properties of conharmonic curvature tensor $\pounds_\xi Z^h_{ijk} = 2\alpha Z^h_{ijk}$, Ricci tensor $\pounds_\xi R_{ij} = 2\alpha R_{ij}$ and $\pounds_\xi R^i_j = 2\alpha R^i_j - R^i_m \hbar^m_j$ in (3.1). Then, we obtain

$$\pounds_\xi R^h_{ijk} = 2\alpha R^h_{ijk} + M^h_{ijk},$$

where M^h_{ijk} is denoted in equation (3.24). Since $M^h_{ijk} = 0$, for a CKV, we obtain CI along a vector field ξ^i. The proof is competed. □

Remark 3.15. Every CIV is RIV, but the converse need not be true in general. However, when spacetime admits RIV and CKV under consideration of conharmonic curvature inheritance symmetry then converse also holds good. This shows the utility of our Theorem (3.9).

3.1 Conharmonic Motion

Abdussatar and Babita Dwivedi [23] introduced a new type of conformal symmetries known as conharmonic symmetries. The conharmonic motion (Conh M) is given by

$$\Box\psi = g^{ij}\psi_{;ij} = 0, \qquad \alpha_{;ij} \neq 0. \tag{3.25}$$

Similarly a Conharmonic Collineation (Conh C) is admitted conformal Collineation (Conf C) when following equation holds together with condition (3.25)

$$\pounds_\xi \Gamma^i_{jk} = \delta^i_j \psi_{;k} + \delta^i_k \psi_{;j} - g_{jk} g^{il} \psi_{;l}. \tag{3.26}$$

If a vector field ξ satisfies

$$\pounds_\xi Z^h_{ijk} = 0, \tag{3.27}$$

then (V_4, g) admits a conharmonic curvature collineation (Conh CC). Clearly every Conh M is a Conh CC and Conh C but the converse need not be true. From equation (2.7) it is evident that Conh CC is a Conf C. The conharmonic motion satisfies

$$\pounds_\xi R^h_{ijk} = \delta^h_j \psi_{;ik} - \delta^h_k \psi_{;ij} + \psi^{;h}_{;j} g_{ik} - \psi^{;h}_{;k} g_{ij}, \tag{3.28}$$

$$\pounds_\xi R_{ij} = -2\psi_{;ij}, \tag{3.29}$$

$$\pounds_\xi R^j_k = -2\psi^{;j}_{;k} - 2\psi R^j_k, \tag{3.30}$$

$$\pounds_\xi R = -2\psi R. \tag{3.31}$$

Multiplying by g^{ij} in equation (3.29) and in view of (3.25), we get

$$g^{ij}\pounds_\xi R_{ij} = 0. \tag{3.32}$$

Thus, we can say that Conh M reduces to a contracted Ricci collineation, but converse need not be true. We also have the following:

Theorem 3.16. *If a spacetime (V_4, g) admits a Conh CI and proper Conh M, then following holds:*

(a) scalar curvature vanishes,
(b) $(\alpha - 4\psi) = 0$,
(c) $Z_{ij} = 0$,
(d) the conformal curvature tensor follows inheritance symmetry property.

Proof. As every proper Conh M is proper Conf M, and under the hypothesis of Theorem 3.3 and using (3.8) (a), its comparing with (3.25), we find that scalar curvature vanishes i.e., $R = 0$. Proof of second part follows from (3.8) (c) and $R = 0$. For proof of next part, we use equation (2.3) under consideration of first part, leads to $Z_{ij} = 0$. Finally, we use equation (2.7) with $R = 0$ and then take the Lie derivative, which implies that symmetry of the conformal curvature tensor of the spacetime is inherited, i.e., $\pounds_\xi C^h_{ijk} = 2\alpha C^h_{ijk}$. □

Theorem 3.17. *If a spacetime (V_4, g) admits proper Conh CI and Conh M. Then that spacetime is conharmonically flat.*

Proof directly follows from equation (3.27).

Example 3.18. Now, we are taking a cosmological model of plane-symmetric perfect fluid distribution [23] which illustrates the Theorem 3.17. This cosmological model was introduced by Singh and Singh in [35]. This model is also plane-symmetric perfect fluid of class one and not conformally flat. The metric of such a model is as follows,

$$ds^2 = (at+1)^2(-dt^2 + dx^2 + dy^2) + (bt+1)dz^2, \tag{3.33}$$

here a, b ≥ 0, ≤ 0 or =0. We find out a conharmonic Killing vector of metric (3.33) is, $\xi = (\frac{A}{a})\delta^i_0$, which is also a Conh CIV (Conf M + CI $\Rightarrow$ Conh CI) only when a=b with $\alpha = \psi = \frac{A}{at+1}$, here A is constant. However, if a = b the cosmological model represented by metric (3.33), reduces to a conformally flat, and also in case (k=0) it becomes FRW model of universe filled with uneven radiation.

Remark 3.19. Siddiqui and Ahsan [36] have studied relativistic significance of conharmonically flat spacetime. A conharmonically flat spacetime has a space of constant curvature (Einstein spacetime). The importance of such spaces is due to the interest in study of the cosmology. For further details, see [37].

Let the spacetime (V_4, g) be an Einstein spacetime with nonzero scalar curvature $(R \neq 0)$, which is expressed as follows:

$$R_{ij} = \frac{R}{4}g_{ij}, \qquad R = \text{constant}. \tag{3.34}$$

On operating Lie derivative on (3.34), we get

$$\pounds_\xi R_{ij} = \frac{R}{4}\pounds_\xi g_{ij}. \tag{3.35}$$

Let us consider the Einstein spacetime admitting a Conh M along a vector ξ, then using (3.29) and (3.1) in equation (3.35), we obtain

$$-2\psi_{;ij} = \frac{R}{2}(\psi g_{ij}). \tag{3.36}$$

Multiplying both sides of (3.36) by g^{ij}, we get $\Box\psi + R\psi = 0$. For Conh M, $\Box\psi = 0$, thus $R = 0$, which is contradiction. Hence, we state the next theorem as follows

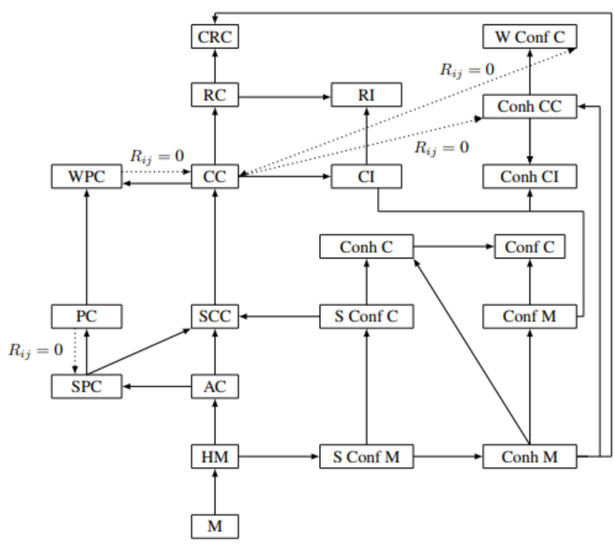
Theorem 3.20. *An Einstein spacetime (V_4, g) with nonzero scalar curvature does not admit Conh M along any vector field ξ.*

3.2 Relation with other spacetime symmetries

In this subsection, we construct a block diagram (Fig.1), that summarizes the relationship of Conh CI with some other well-defined symmetries of the spacetime, few are discussed in this article earlier. The block diagram may be read in the following manner: If a spacetime (V_4, g) admits symmetry and it is denoted in any block of Fig.1., then a subcategory of the symmetries described in adjacent blocks directed by the arrows is leaded from the given block. For example, if an AC is admitted by (V_4, g), then the transformation which defines the AC also fulfills the requirement for being a SCC and SPC. Similarly we have given different symmetries through the diagram.

We mention clearly, that if the Ricci tensor vanishes, then the dashed arrows should be considered. Specially, dashed arrows in the cases of curvature collineation to Weyl conformal collineation and Conh CC are directed to reverse also, i.e. if $R_{ij} = 0$ then curvature collineation is a Weyl curvature collineation, similarly for all CC is a conharmonic curvature collineation and vice-versa also hold good in both cases. This block diagram corrects an error formed in the previous block diagram constructed by Abdussatar and Babita Dwivedi 1998 (Fig.1, [23]), block Conh M to block Conh CC by solid line and also block CC to Conh CC by dashed arrows in both directions. Furthermore, Katzin and his collaborators 1969 and 1972 (Fig.1, [19] and Fig.1 [20]) and Ahsan in 1987 [8] also provided a block diagram for relationship between spacetime symmetries. In addition, three new symmetry blocks(CI,RI and Conh CI) have been introduced with several new connecting lines.

Abbreviation of spacetime symmetries in the block diagram (Fig.1.) is same as block diagram of references the ([8] and [19] - [23]) except CRC, CI, RI and Conh CI and furthermore, details in this context also refer with same references.



(**Fig. 1.** Relationship between conharmonic curvature inheritance and other well known symmetries)

1. CI - Curvature Inheritance $(\pounds_\xi R^h_{ijk} = 2\alpha R^h_{ijk})$.
2. RI - Ricci Inheritance $(\pounds_\xi R_{ij} = 2\alpha R_{ij})$.
3. CRC - Contracted Ricci Collineation $(g^{ij} \pounds_\xi R_{ij} = 0)$.
4. Conh CI - Conharmonic Curvature Inheritance $(\pounds_\xi Z^h_{ijk} = 2\alpha Z^h_{ijk})$.

It may be noted that the block diagram can be improved further, by adding some new symmetries and way of defining the terms. In present diagram we give place to few blocks showing symmetries, which are studied in this paper.

4 Application in fluid spacetime

Let (V_4, g) represents the perfect fluid spacetime for general theory of relativity and satisfying the Einstein field equations without cosmological constant,

$$R_{ij} - \frac{R}{2} g_{ij} = T_{ij}, \tag{4.1}$$

where R denotes for the scalar curvature and T_{ij} represent the stress-energy tensor, which for perfect fluid spacetime is given by

$$T_{ij} = (p + \mu) u_i u_j + p g_{ij}, \tag{4.2}$$

where u^i, p, and μ are representing the fluid flow velocity, isotropic pressure, and energy density, respectively. We are considering that equation (4.2) is self similar then it admits a HM which is defined as follows,

$$\pounds_\xi g_{ij} = 2\psi g_{ij}, \qquad \psi_{,i} = 0. \tag{4.3}$$

Since homothetic motion generates self- similarity on the stress energy tensor. Therefore under consideration of dimensional analysis [4], the physical quantities (μ, p, u^i) admits symmetry inheritance along a vector ξ^i, which is expressed as follows:

$$\pounds_\xi \mu = -2\psi\mu, \tag{4.4}$$

$$\pounds_\xi p = -2\psi p, \tag{4.5}$$

$$\pounds_\xi u^i = \psi u^i. \tag{4.6}$$

If ξ is homothetic vector then necessarily it implies that $\pounds_\xi T_{ij} = 0$, from this we conclude that energy momentum tensor is invariant along the homothetic vector ξ. Invariant characteristic of the HM for stress energy tensor has been very applicable in finding the exact solutions of the EFEs (4.1). Due to the invariant characteristic of symmetry only, HV has been suitable for other vectors corresponding to higher symmetries. Equation (4.3) is also referred to as conformal Killing equation ([18],[4]).
Now here, we are considering the exceptional case for Einstein spacetime to be the perfect fluid spacetime with $\mu + p = 0$, then EFEs (4.1) reduces to

$$R_{ij} - \frac{R}{2} g_{ij} = -\mu g_{ij}, \qquad (R = 4\mu). \tag{4.7}$$

Consider the de- Sitter spacetime metric as following

$$ds^2 = (dx_1^2 + dx_2^2 + dx_3^2)e^{2nx_4} - dx_4^2, \qquad \text{where } (x_1, x_2, x_3, x_4) = (x, y, z, t). \tag{4.8}$$

Metric (4.8) admits a proper conformal Killing vector $\xi^i = (e^{nx_4}, 0, 0, 0,)$, when $\psi = \partial_{x_4}(e^{nx_4}) = ne^{nx_4}$. Now we compute the components of Riemann curvature tensor and Ricci tensor. Use of equation (2.1) (taking Lie derivative), yields that ξ is a Conh CI when $\alpha = \psi$ and $\mu = 3n^2$.

The field equation (4.7) for an Einstein spacetime (V_4, g) and Corollary 3.8 indicate that if (V_4, g) admits a Conh CI vector ξ, then its energy momentum tensor, $T_{ij} = -\mu g_{ij}$ is a symmetry inheritance along the vector ξ. Thus, Theorem 3.7 can be extended in fluid spacetime other than perfect fluid, and also we get the following symmetry inheritance equation:

$$\pounds_\xi T_{ij} = 2\alpha T_{ij}. \tag{4.9}$$

Equation (4.9) leads to the following result,

Theorem 4.1. *Let a spacetime (V_4, g), with Einstein field equation (4.1), admits a Conh CIV ξ. Then, ξ is also satisfies a Conf M iff stress energy tensor T_{ij} admits the inheritance symmetry.*

Theorem 4.2. *If* (V_4, g) *be a perfect fluid spacetime admits a Conh CIV* ξ *as well as a CKV. Then* (V_4, g) *satisfying the following equations:*

$$\begin{aligned} &(a) \quad \pounds_\xi T_{ij} = 2\alpha T_{ij}, \\ &(b) \quad \pounds_\xi u_i = \psi u_i, \\ &(c) \quad \pounds_\xi \mu = 2(\alpha - \psi)\mu, \\ &(d) \quad \pounds_\xi p = 2(\alpha - \psi)p. \end{aligned} \tag{4.10}$$

Proof. Firstly we operate $\pounds_\xi$ over equations (4.1) and then use of equation (3.1) with (3.2) (a) and (3.2) (b), leads to

$$\pounds_\xi T_{ij} = 2\Box\psi g_{ij} - 2\psi_{;ij}. \tag{4.11}$$

Now using equation (4.2) in left part of (4.11), and then contracting with $u^i u^j$, h^{ij}, $h^j_k u^i$, and $h^i_k h^j_l - \frac{1}{3}h^{ij}h_{kl}$, we get following equations respectively

$$\pounds_\xi \mu = -2\mu\psi - 2\Box\psi - 2u^i u^j \psi_{;ij}, \tag{4.12}$$

$$\pounds_\xi p = -2p\psi - \frac{4}{3}\Box\psi - \frac{2}{3}u^i u^j \psi_{;ij}, \tag{4.13}$$

$$(p + \mu)v_i = 2[u^j \psi_{;ij} + u_i u^k u^l \psi_{;kl}], \tag{4.14}$$

and
$$\psi_{;ij} = \frac{1}{3}u_i u_j[\Box\psi - 2u^k u^l \psi_{;kl}] + \frac{1}{3}g_{ij}[\Box\psi + u^k u^l \psi_{;kl}] - \psi_{;ik}u^k u_j - \psi_{;jk}u^k u_i. \tag{4.15}$$

For perfect fluid equations (4.1) and (4.2) together, we have

$$R_{ij} - \frac{R}{2}g_{ij} = (\mu + p)u_i u_j + p g_{ij}. \tag{4.16}$$

Contracting it with u^j, using $R = (\mu - 3p)$ and $u^j u_j$ = -1 (u^j supposed to be a time like vector) indicates that u^i is an eigen vector of R_{ij}

$$R_{ij}u^j = -(\frac{3p + \mu}{2})u_i. \tag{4.17}$$

Now using equation (4.17) in the equation (3.11) and by multiplication with $u^i u^j$, we get

$$\psi_{;ij}u^i u^j = -\frac{2}{3}(3p + 2\mu). \tag{4.18}$$

Using the above equation in (4.12) with consideration of equation (3.8)(a), energy density inherits a symmetry

$$\pounds_\xi \mu = 2(\alpha - \psi)\mu. \tag{4.19}$$

Next, use of equation(4.18) in equation (4.13) with under consideration of equation (3.8)(a), isotropic pressure inherits a symmetry

$$\pounds_\xi p = 2(\alpha - p)\mu. \tag{4.20}$$

As we know, for a CKV vector field ξ in fluid spacetime, the following result holds [38]:

$$\pounds_\xi u_i = -\alpha u_i + v_i. \tag{4.21}$$

Again use of equation(4.18) in equation (4.14) and due to virtue of $\psi_{;ij}u^j = (\frac{\psi}{3})[3p + 2\mu]u_i$, we obtain

$$v_i = 0, \qquad (\mu + p) \neq 0. \tag{4.22}$$

From above equation with (4.21) implies to $\pounds_\xi u_i = -\alpha u_i$. Finally, we prove the first part of the Theorem use of equations (4.15), (4.18) and $\psi_{;ij}u^j = (\frac{\psi}{3})[3p + 2\mu]u_i$. Further, use of (4.11) and (3.8)(a), from this implies that energy momentum tensor inherits a symmetry

$$\pounds_\xi T_{ij} = 2\alpha T_{ij}. \tag{4.23}$$

This completes the proof. □

If a conformal Killing vector ξ satisfies (3.1), then following dynamic result holds [38]:

$$(R^{ij}\xi_j)_{;i} = -3\Box\psi. \tag{4.24}$$

Due to Einstein field equation, then equation (4.24) takes another form as following,

$$[(T^{ij} + \frac{1}{2}Rg^{ij})\xi_j]_{;i} = -3\Box\psi, \qquad R = -T = (\mu - 3p). \tag{4.25}$$

Equation (4.25) is equivalent to newly formed dynamical equation (5.1) if (V_4, g) also admits a conharmonic curvature inheritance vector. From equation (3.8) (a) in Theorem 3.3 ($-3\Box\psi = \alpha R$), thus equation (4.25) reduces to

$$[(T^{ij} + \frac{1}{2}Rg^{ij})\xi_j]_{;i} = \alpha R, \qquad R = -T. \tag{4.26}$$

Hence, under consideration with Theorem 4.2, equation (4.26) describes the equations of state for a various form of fluid distribution. Finally, applying the Theorem 4.2 to describe physical meaning of equations of state for two cases. The energy momentum tensor T^{ij} for perfect fluid is defined as equation (4.2), we have

$$[(T^{ij} + \frac{1}{2}Rg^{ij})\xi_j] = [(\mu + p)u^i u^j + pg^{ij} + \frac{1}{2}Rg^{ij}]\xi_j. \tag{4.27}$$

Now, for perfect fluid, we deal with two cases:
Case - I. In this case we assume that $u^i \parallel \xi^i$, then right part of equation (4.27) reduces to $-(\frac{3p+\mu}{2})\xi^i$. When this is substituted in equation (4.26), and making use of equations (4.10) (c), (4.10) (d) and $\xi^i_i = 4\psi$, we get

$$\psi(3p + \mu) = -2\alpha\mu. \tag{4.28}$$

(a) For $\alpha = 0$ and $\psi \neq 0$, equation (4.28) leads to $(3p + \mu) = 0$.
(b) For $\alpha = \psi \neq 0$, (4.28) reduces to equation of state, $(p + \mu) = 0$, then $T_{ij} = -\mu g_{ij}$.
(c) For $\alpha \neq 0$ (proper Conh CI), equations (4.28) provides meaningful state equation for different value of ψ and α. Consider the comparatively simpler case for which

$$\alpha = -m^2\psi, \qquad \text{m =constant.} \tag{4.29}$$

Equation (4.29) provide the following equation of state for $\psi \neq 0$,

$$\mu = \frac{3p}{2m^2 - 1}. \tag{4.30}$$

where m^2 depends upon the energy conditions.
Case - II. In this case we assume that $u^i \perp \xi^i$, then right part of equation (4.27) reduces to $(\frac{\mu - p}{2})\xi^i$. When this is substituted in equations (4.26) with use of equations (4.10) (c), (4.10) (d) and $\xi^i_i = 4\psi$, we obtain

$$\psi(\mu - p) = -2\alpha p. \tag{4.31}$$

(a) For $\alpha = 0$ and $\psi \neq 0$, then equation (4.31) implies to $(p - \mu) = 0$.
(b) For $\alpha = \psi \neq 0$, (4.31) reduces to $(p + \mu) = 0$, then $T_{ij} = -\mu g_{ij}$.
(c) For $\alpha \neq 0$ (proper Conh CI), equations (4.31) get equation of state for different value of ψ and α. We consider the simplest type

$$\alpha = -m^2\psi, \qquad \text{m =const.} \tag{4.32}$$

Generally, (4.32) reduces to following equation of state when $\psi \neq 0$

$$\mu = p(1 + 2m^2), \tag{4.33}$$

here m^2 depends upon the energy conditions.

Now we discuss the another application, which is based on the Theorem 3.9. Similar work has been given by G.S.Hall in [32]. Literature in [32] and [4] will be helpful for readers to understand such an application.

Let G_p be a the Grossman manifold of dimension four and of having 2-dimensional sub-spaces of the tangent space $T_p(M)$ at $p \in M$. Mark $\bar{G}_p$, the 4-dimensional open sub-manifold of G_p. The sectional curvature is given by

$$K_p(E) = \frac{E^{ij}E^{kl}R_{ijkl}}{2E^{ij}E^{kl}g_{i[k}g_{l]j}}, \tag{4.34}$$

where $E \in G_p$ and E_{ij} is any non-null simple bi-vector.

Let V_4 admits a smooth vector field ξ, which is nowhere zero associated with f_t a smooth and local diffeomorphism. Further let E and f_{t^*}(E) be generated by U & V which belong to $T_p(M)$. Also, $f_{t*}(U)$ & $f_{t*}(V)$ belong to $T_q(M)$, where f_{t*} is the differential of f_t. The vector field ξ preserves the sectional curvature if

$$K_p(E) = K_q(f_{t^*}(E)). \tag{4.35}$$

Now, consider the metric of generalised plane-wave spacetime (V_4, g) which is conformally flat ([3]). It is defined that such spacetimes admit a nowhere zero, covariantly constant null vector ξ, $\xi_{a;b} = 0$. The metric is given by the following equation

$$ds^2 = dx^2 + dy^2 - 2du\,dv - \frac{1}{2}\phi^2(u)(x^2 + y^2)du^2, \tag{4.36}$$

for which

$$R_{ij} = -\phi(u)l_il_j, \qquad R = 0. \tag{4.37}$$

Theorem 4.3. *[32] Let (V_4, g) be a conformally flat generalised plane-wave spacetime. Then, a global smooth vector field ξ on M is sectional-preserving iff*

$$(a) \quad \pounds_\xi g_{ij} = 2\psi g_{ij}, \qquad (b) \quad \pounds_\xi R_{ij} = 2\psi R_{ij}. \tag{4.38}$$

Theorem 4.3 directly links with Theorem 3.3 in the following manner. Globally, a smooth vector field ξ on a conformally flat generalized plane-wave spacetime is sectional-preserving. In (4.38) (a) and (4.38) (b), ξ is a RIV and a CKV when $\psi = \alpha$ (comparing with equations (4.38), (3.17) and (3.8)). Furthermore, equations (4.38) hold in conformally flat spaces also. From (3.17), we can easily show that (V_4, g) admits conharmonic curvature inheritance iff it also admits Ricci inheritance. Thus, under consideration of hypothesis of Theorem 4.3, M belongs to Type- I $(M_{ij} = 0)$. It follows from equations(3.8), (4.37), and equation (4.38) that

$$\psi_{;ij} = \phi\psi l_il_j, \quad \text{and} \quad \Box\psi = 0. \tag{4.39}$$

5 Discussion and Conclusion

In this paper, we explore the conharmonic curvature inheritance symmetry of spacetime. From (2.10)(a), we conclude that the conformal curvature tensor inherits the symmetry defined by Conh CIV ξ. In contrast, from (2.10)(b), the Weyl projective curvature tensor is non-inheriting due to the presence of tensor E, which is non-zero in general. Hence the natural problem arises :

"What condition(s) is to be applied on (V_4, g), with a proper Conh CI symmetry, such that E vanishes i.e. Weyl projective tensor inherits the symmetry."

It is observed that Theorem 2.2 is similar to a result for CIV [4] and also the generalization of result for a RCV [39] when $\alpha = 0$. From the application point of view above result is quite important as it generates the basis of new solutions for various fluid spacetimes. It is because of the non-vanishing term αR on its right side. Dynamical equation derived from Einstein field equations (1.3) by using (2.19) is,

$$[\xi_j(T^{ij} - \frac{1}{2}Tg^{ij})]_{;i} = \alpha R, \qquad R = -T. \tag{5.1}$$

If α = 0, we observe as a geometrical point of view that, (3.8) (a) and (b) reduces to special CKV for which $\psi_{;ij}$=0 and also doesn't look much helpful in physics. It means that there exists a covariant constant hypersurface orthogonal and geodesic vector $\psi_{;i}$. This type of spacetimes must admit either a repeated null vector or two distinct null vectors of the energy-momentum tensor. Their application is minimal in the theory of general relativity. Suppose α =0, then perfect fluid spacetime and Friedman- Robertson-Walker(FRW) spacetimes do not admit special CKV. On the other hand, proper CKV is more significant because of its spatial application in cosmology. In particular, FRW and perfect fluid spacetimes admit CKV [43]. Thus, Conh CI with proper CKV seems to be a very important, and such a study is highly desirable to further research.

It is notable that the paper in [32], Hall et. al., have studied structure of Lie algebra for the set of vector fields on V_4, preserving the sectional curvature, satisfying Theorem 4.3. A structure Lie algebra with finite-dimensional of Conh CIVs and its existence is an important reason for our comparison with Theorem 3.3. For the better understanding of the readers, we state only main part of the theorem in paper [32] as follows:

" The set of (global) sectional curvature-preserving vector fields on a conformally flat generalised plane-wave spacetime is a finite dimensional sub-algebra of the Lie algebra of CKV's. The associated conformal function ξ satisfies (4.39)*."*

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