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$f-$CLEAN RINGS RELATIVE TO RIGHT IDEALS

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Abstract. An element a of associative ring R with identity is called full if $sat = 1$ for some $s, t \in R$. A ring R is called $f-$clean ring if every element of R is the sum of a full and an idempotent. In this paper, we introduce the concept of $f-$clean ring relative to right ideal. We study various properties of this ring. We give some relations between $f-$clean rings and $f-$clean rings of 2×2 matrices over R, relative to some right ideal $P \neq R$. New characterization obtained include necessary and sufficient conditions of a ring R to be $f-$clean in terms $P-$local and $P-$clean rings. Also, We proved that every ring is $f-$ clean relative to any maximal right ideal of it.

1 Introduction

In their fundamental work [2], Andrunakievich V. A and Ryabukhin Yu. M introduced the notion of quasi-regularity and pimitivity relative to right ideals of a ring to be shown that representations of right ideals in the form of intersections of maximal modular right ideals lead to a generalization of the main structure theorems related to the notions of quasi-regularity and primitivity. Later in [1] the concept of rings relative to right ideal which was extended to regular ring relative to right ideal in as generalization of (Von Neumann) regular ring (also known as $P-$regular rings). In [5], H. Hakmi continued the study of $P-$regular and $P-$potent rings and in [6] and [7], he studied the local and $r-$clean ring relative to right ideal ($P-$local rings). In our paper we continue the studying of rings relative to right ideal from a new point of view that, we study the concept of $f-$clean rings relative to right ideal as generalization of $f-$clean ring, [8]. $f-$clean rings were introduced by B. Li and L. Feng in [8], as extended of clean ring, [10]. An element a of a ring R is said to be clean if $a = u + e$, where $e \in R$ is an idempotent and u is a unit in R. If every element of a ring R is clean, then R is called a clean ring. Clean rings were introduced by W. K. Nicholson in his fundamental paper [9]. The clean rings were further extended to $r-$clean rings and the $r-$clean rings were introduced by Ashrafi and Nasibi [3] and they defined that an element x of a ring R is $r-$clean if $x = a + e$, where $a \in R$ is a regular element and $e \in R$ is an idempotent. A ring R is said to be $r-$clean if each of its element is $r-$clean. Also, in [7] the concept of $r-$clean rings was extended to $r-$clean ring relative to right ideals. An element a of a ring R is said to be full if $sat = 1$ for some elements $s, t \in R$, [8]. An element x of a ring R is called $f-$clean if $x = a + e$, where $a \in R$ is full and $e \in R$ is idempotent. If every element of a ring R is $f-$clean, then R is called $f-$clean ring, [8]. In our paper we study the concept of $f-$clean ring relative to some proper right ideal.

Throughout in this paper rings R are associative with identity. In section 2, we study the fundamental properties of $P-$idempotents and $P-$clean elements. In section 3, we study $P-$full elements. Where we proved that an element a of a ring R is full relative to right ideal P of a ring R if and only if $R = sR + P$ for some $s \in R$ such that $sP \subseteq P$. Furthermore, for $a, b \in R$ such that $a - b \in P$, then a is $P-$full if and only if b is $P-$full. In section 4, we study the concept $f-$clean ring relative to right ideal. Were we proved that every ring R is an $f-$clean relative to every maximal right ideal of R. In addition to that, we obtain that an element x of a ring R is $f-$clean relative to P if and only if $1 - x$ is $f-$clean relative to P. Also, we proved that if in a ring R the set of all $P-$idempotents is $\{0, 1, p, 1 + p\}$ for every $p \in P$, then a ring R is $f-$clean relative to P if and only if for every $x \in R$ either x or $1-x$ is a $P-$full. Also, in this section we study the connection between the $f-$clean elements in a ring R and $f-$clean elements relative to P (relative to Q) in the ring of 2×2 matrices over R. We prove that an element a of a ring R is $f-$clean if and only if there exists $x, y \in R$ such that the element $\begin{bmatrix} x & y \\ 0 & a \end{bmatrix}$ is $f-$clean relative to some proper right ideal P of $M_2(R)$.

2 $P-$idempotent and $P-$clean elements

Let R be a ring and $P \neq R$ be a right ideal of R.
- We say that an element a of a ring R has a right inverse relative to right ideal P, or has a right $P-$inverse for short, if $R = aR + P$. It is clear that if $P = 0$, then an element $a \in R$ has a right $P-$inverse if and only if a has a right inverse in R.
- Recall that an element e of a ring R is an idempotent relative to right ideal P, or $P-$idempotent for short, if $e^2 - e \in P$ and $eP \subseteq P$, [2]. Note that in previous definition it is easily verified that $0, 1 \in R$ are $P-$idempotents. Also, if $P = 0$, then an element $e \in R$ is $P-$idempotent if and only if e is idempotent.

Lemma 2.1. Let R be a ring and $P \neq R$ be a right ideal of R. For every $P-$ idempotent $e \in R$ the following hold:
(1) Elements $e^2 \in R$ and $1 - e \in R$ are $P-$idempotents.
(2) For every positive integer k, $e^k \in R$ is $P-$idempotent.
(3) For every $p \in P$, $e + p \in R$ is $P-$idempotent in R.

Proof. (1) Obvious. (2) Since e is $P-$idempotent, $e^2 - e \in P$ and $eP \subseteq P$, so $e^2 = e + p_0$ for some $p_0 \in P$. Proof by induction on k. For $k = 1, 2$ the assertion holds by assumption and by (1). Suppose that e^{k-1} is $P-$idempotent, then

$$(e^{k-1})^2 - e^{k-1} \in P \quad and \quad e^{k-1}P \subseteq P$$

So $(e^{k-1})^2 = e^{k-1} + p_1$ for some $p_1 \in P$. Thus

$$(e^k)^2 = (e^{k-1})^2 e^2 = (e^{k-1} + p_1)(e + p_0)$$
$$= e^k + e^{k-1}p_0 + p_1 e + p_1 p_0$$

Therefore $(e^k)^2 - e^k = p$, where $p = e^{k-1}p_0 + p_1 e + p_1 p_0 \in P$. This shows that

$$(e^k)^2 - e^k \in P \quad and \quad e^k P = ee^{k-1}P \subseteq eP \subseteq P$$

(3) Since $e \in R$ is $P-$idempotent, $e^2 - e \in P$ and $eP \subseteq P$, so $e^2 = e + p_0$ for some $p_0 \in P$. Let $p \in P$ and suppose that $f = e + p$, then

$$f^2 = (e+p)(e+p) = e^2 + ep + pe + p^2$$
$$= e + p_0 + ep + pe + p^2$$
$$= (e+p) + (-p + p_0 + ep + pe + p^2)$$
$$= f + p_2$$

Where $p_2 = -p + p_0 + ep + pe + p^2 \in P$, thus $f^2 - f \in P$. On the other hand, for every $t \in P$, $ft = (e+p)t = et + pt \in eP + P \subseteq P$, so $fP \subseteq P$. Thus $f = e + p$ is $P-$idempotent. □

Let R be a ring and $M_2(R)$ be the ring of all 2×2 matrices over a ring R. It is clear that the sets:

$$P = \{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in R\} \quad and \quad Q = \{\begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} : a, b \in R\}$$

are right ideals in $M_2(R)$ such that $P \neq M_2(R)$, $Q \neq M_2(R)$. The connection between the idempotent elements in R and $P-$idempotents ($Q-$idempotents) in $M_2(R)$ we provide in the following:

Proposition 2.2. Let R be a ring. Then the following hold:

(1) If $e \in R$ is an idempotent, then for every $x, y \in R$ the element $\alpha = \begin{bmatrix} x & y \\ 0 & e \end{bmatrix}$ is $P-$idempotent in $M_2(R)$.

(2) An element $e \in R$ is idempotent in R if and only if the element $\alpha = \begin{bmatrix} x & y \\ 0 & e \end{bmatrix}$ is $P-$idempotent in $M_2(R)$, for some $x, y \in R$.

(3) If $e \in R$ is an idempotent, then for every $x, y \in R$ the element $\alpha = \begin{bmatrix} e & 0 \\ x & y \end{bmatrix}$ is $Q-$idempotent in $M_2(R)$.

(4) An element $e \in R$ is idempotent in R if and only if the element $\alpha = \begin{bmatrix} e & 0 \\ x & y \end{bmatrix}$ is $Q-$idempotent in $M_2(R)$, for some $x, y \in R$.

Proof. (1) Suppose that $e \in R$ is idempotent. Let $x, y \in R$, then

$$\alpha^2 - \alpha = \begin{bmatrix} x^2 & xy + ye \\ 0 & e \end{bmatrix} - \begin{bmatrix} x & y \\ 0 & e \end{bmatrix} = \begin{bmatrix} x^2 - x & xy + ye - y \\ 0 & 0 \end{bmatrix} \in P$$

For every $p = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \in P$, $\alpha p = \begin{bmatrix} xa & yb \\ 0 & 0 \end{bmatrix} \in P$, thus $\alpha P \subseteq P$. This shows that α is $P-$idempotent.
(2) If $e \in R$ is idempotent in R, then by (1) the element α is $P-$idempotent. Conversely, suppose that α is $P-$idempotent for some $x, y \in R$. Since $\alpha^2 - \alpha \in P$,

$$\begin{bmatrix} x^2 & xy + ye \\ 0 & e^2 \end{bmatrix} - \begin{bmatrix} x & y \\ 0 & e \end{bmatrix} = \begin{bmatrix} x^2 - x & xy + ye - y \\ 0 & e^2 - e \end{bmatrix} \in P$$

Thus $e^2 = e$. Similarly, we can prove (3) and (4). □

Let R be a ring and $P \neq R$ be a right ideal of R. Recall that an element x of a ring R is clean relative to right ideal P, or $P-$clean for short, if $x = a + e$, where $a \in R$ has a right $P-$inverse and $e \in R$ is an $P-$idempotent, [6]. It is clear that if $P = 0$, then an element $x \in R$ is clean if and only if x is $P-$clean. A ring R is called $P-$clean if every element of R is $P-$clean. Furthermore, we have the following:

Lemma 2.3. Let R be a ring and $P \neq R$ be a right ideal of R. Then the following hold:
(1) Elements $1, -1, 0$ are $P-$cleans.
(2) Every element of R has an inverse is $P-$clean.
(3) Every element of R has a right inverse is $P-$clean.
(4) Every element of R has a right $P-$inverse is $P-$clean.

Proof. (1) Obvious, because $1 = 1 + 0$, $0 = (-1) + 1$, $-1 = -1 + 0$, where $1, 0$ are idempotents, so $1, 0$ are $P-$idempotents and $1, -1$ such that $R = 1R + P$, $R = (-1)R + P$, i.e., $1, -1$ has a right $P-$inverse. (2) If $a \in R$ has an inverse, then $a = a + 0$ and 0 is $P-$idempotent. (3) Obvious. (4) It is clear. □

Also, we have the following:

Lemma 2.4. Let R be a ring, $P \neq R$ be a right ideal of R. Then for every $P-$idempotent $e \in R$ the following hold:
(1) e is a $P-$clean element.
(2) For every positive integer k, e^k is a $P-$clean element.

Proof. (1) Let $e \in R$ be a $P-$idempotent element, then $e^2 - e \in P$, $eP \subseteq P$, so $e^2 = e + p_0$ for some $p_0 \in P$. So

$$(2e - 1)(2e - 1) = 4e^2 - 2e - 2e + 1 = 4(e + p_0) - 4e + 1 = 1 + 4p_0$$

and so

$$1 = (2e-1)(2e-1) - 4p_0 \in (2e-1)R + P$$

thus $R = (2e-1)R+P$. This shows that $2e-1$ has a right $P-$inverse and $e = (2e-1)+(1-e)$, where $1-e \in R$ is $P-$idempotent. Thus e is $P-$clean.
(2) Since e is $P-$idempotent, by Lemma 2.1 e^k is $P-$idempotent for every positive integer k, so by (1) e^k is a $P-$clean element. □

Lemma 2.5. Let R be a ring, $P \neq R$ be a right ideal. Then the following hold:
(1) If $e \in R$ is $P-$idempotent, then for every $p \in P$, $e+p \in R$ is $P-$ idempotent.
(2) If $a \in R$ has a right $P-$inverse, then for every $p \in P$, $a+p \in R$ has a right $P-$inverse.

Proof. (1) Let $e \in R$ be a $P-$idempotent element, then $e^2 - e \in P$ and $eP \subseteq P$, so $e^2 = e + p_0$ for some $p_0 \in P$. Let $p \in P$ and suppose that $f = e+p$, then

$$f^2 = (e+p)(e+p) = e^2 + ep + pe + p^2$$

$$= e + p_0 + ep + pe + p^2$$

$$= (e+p) + (-p + ep + pe + p^2)$$

$$= f + p_2$$

where $p_2 = -p + ep + pe + p^2 \in P$, thus $f^2 - f \in P$. On the other hand, for every $t \in P$, $ft = (e+p)t = et + pt \in eP + P \subseteq P$, so $fP \subseteq P$. Thus $f = e+p$ is $P-$idempotent.
(2) Let $a \in R$ has a right $P-$inverse, then $R = aR + P$. Let $p \in P$, suppose that $b = a+p$, then $a = b - p$ and

$$R = aR + P = (b-p)R + P \subseteq bR + (-p)R + P \subseteq bR + P \subseteq R$$

this shows that $R = bR + P$, i.e., $b = a+p$ has a right $P-$inverse. □

Lemma 2.6. Let R be a ring and $P \neq R$ be a right ideal. Then for every elements $x, y \in R$ such that $x - y \in P$, y is $P-$clean if and only if x is $P-$ clean.

Proof. Let $x, y \in R$ such that $x - y \in P$, then $x = y + p_0$ for some $p_0 \in P$. Suppose that y is $P-$clean, then $y = a + e$, where $a \in R$ has a right $P-$ inverse and $e \in R$ is $P-$idempotent. Then $x = y + p_0 = a + (e + p_0)$. Since e is $P-$idempotent and $p_0 \in P$, then by Lemma 2.5 $e + p_0$ is $P-$idempotent, this shows that x is $P-$clean. Similarly, we can prove converse. □

3 $P-$full elements

In this section we study the concept of full elements relative to right ideal as generalization of full elements, we start with the following:

Definition 3.1. Let R be a ring and $P \neq R$ be a right ideal of R. We say that an element a of a ring R is a full element relative to right ideal P, or $P-$full for short, if there exists $s, t \in R$ such that $1 - sat \in P$ and $sP \subseteq P$.

Note that in previous definition, it is easy to see that for $P = 0$, an element a of a ring R is full relative to P if and only if a is a full element. Also, it is easy to see that in any ring R, $1, -1$ are full elements relative to any right ideal $P \neq R$ of R. Furthermore, we have the following:

Lemma 3.2. Let R be a ring and $P \neq R$ be a right ideal of R. Then the following statements hold:
(1) An element a of R is $P-$full if and only if $-a$ is a $P-$full element.
(2) An element a of R is $P-$full if and only if there exists $s \in R$ such that $R = saR + P$ and $sP \subseteq P$.
(3) Every element $a \in R$ has a right $P-$inverse is $P-$full.

Proof. (1) Suppose that an element $a \in R$ is $P-$full, then there exists $s, t \in R$ such that $1 - sat \in P$ and $sP \subseteq P$. Since $-a, -t \in R$, $1 - s(-a)(-t) = 1 - sat \in P$, $sP \subseteq P$ which implies that $-a$ is a $P-$full element. Conversely, if $-a$ is $P-$full, then there exists $s, t \in R$ such that $1 - s(-a)t \in P$ and $sP \subseteq P$. Since $-t \in R$ and $1 - sa(-t) = 1 - s(-a)t \in P$ and $sP \subseteq P$, a is $P-$full.
(2) Suppose that a is $P-$full, then there exists $s, t \in R$ such that $1 - sat \in P$ and $sP \subseteq P$, so

$$1 \in satR + P \subseteq saR + P$$

Thus $R = saR + P$. Conversely, suppose that $R = saR + P$ and $sP \subseteq P$ for some $s \in R$, then $1 = sat + p_0$ for some $t \in R$, $p_0 \in P$, so $1 - sat = p_0 \in P$ and $sP \subseteq P$, i.e., a is $P-$full.
(3) If $a \in R$ has a right $P-$inverse, then by definition $R = aR + P$, so by (2) a is $P-$full. □

Lemma 3.3. Let R be a ring and $P \neq R$ be a right ideal of R. If $a, b \in R$ such that $a - b \in P$, then a is $P-$full if and only if b is $P-$full.

Proof. Let $a, b \in R$ such that $a - b \in P$, then $a = b + p_0$ for some $p_0 \in P$.
($\Rightarrow$) Suppose that a is $P-$full, then there exists $s, t \in R$ such that $1 - sat \in P$ and $sP \subseteq P$. So

$$1 - sbt = 1 - s(a - p_0)t = (1 - sat) + sp_0t \in P + sPR \subseteq P + sP \subseteq P$$

and $sP \subseteq P$. Thus b is $P-$full. Similarly, we can prove converse. □

Let R be a ring and $M_2(R)$ be the ring of all 2×2 matrices over a ring R. It is clear that the sets:

$$P = \{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in R\} \quad and \quad Q = \{\begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} : a, b \in R\}$$

are right ideals in $M_2(R)$ such that $P \neq M_2(R)$ and $Q \neq M_2(R)$. The connection between the full elements in R and full elements relative to P (relative to Q) in $M_2(R)$ we provide in the following:

Proposition 3.4. For every element $a \in R$ the following statements hold:
(1) If $a \in R$ is a full element in R, then the element $\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$ is $P-$full in $M_2(R)$.
(2) If the element $\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$ is $P-$full in $M_2(R)$, then a is full in R.
(3) If $a \in R$ is a full element in R, then the element $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$ is $Q-$full in $M_2(R)$.
(4) If the element $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$ is $Q-$full in $M_2(R)$, then a is full in R.

Proof. (1) Suppose that $a \in R$ is a full element, then there exists $s, t \in R$ such that $sat = 1$, so

$$\begin{bmatrix} 0 & 0 \\ 0 & s \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix} \in M_2(R)$$

Such that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & s \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & sat \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in P$$

and $\begin{bmatrix} 0 & 0 \\ 0 & s \end{bmatrix} P \subseteq P$. So $\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$ is a $P-$full element in $M_2(R)$.

(2) Suppose that $\sigma = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$ is a $P-$full element in $M_2(R)$, then there exists $u = \begin{bmatrix} n & m \\ z & s \end{bmatrix}, v =$

$\begin{bmatrix} \alpha & \beta \\ \gamma & t \end{bmatrix} \in M_2(R)$ such that

$$1 - u\sigma v = 1 - \begin{bmatrix} n & m \\ z & s \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} \cdot \begin{bmatrix} \alpha & \beta \\ \gamma & t \end{bmatrix} \in P$$

and $uP = \begin{bmatrix} n & m \\ z & s \end{bmatrix} P \subseteq P$. Since $uP \subseteq P$, $z = 0$ and

$$1 - \begin{bmatrix} n & m \\ 0 & s \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} \cdot \begin{bmatrix} \alpha & \beta \\ \gamma & t \end{bmatrix} = 1 - \begin{bmatrix} \lambda & \mu \\ sa\gamma & sat \end{bmatrix} \in P$$

Where $\lambda, \mu \in R$. This shows that $1 - sat = 0$. i.e., $sat = 1$, thus a is full in R.
(3) Similarly, as in (1).
(4) Similarly, as in (2). □

From Proposition 3.4 we can drive the following:

Corollary 3.5. For every element a of a ring R the following statements hold:
(1) The element $a \in R$ is a full element in R if and only if the element $\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$ is $P-$full in $M_2(R)$.
(2) The element $a \in R$ is a full element in R if and only if the element $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$ is $Q-$full in $M_2(R)$.

Theorem 3.6. For every element $a \in R$ the following statements hold:
(1) If $a \in R$ is a full element in R, then for every $x, y \in R$, the element $\begin{bmatrix} x & y \\ 0 & a \end{bmatrix}$ is $P-$full in $M_2(R)$.
(2) If for some $x, y \in R$, the element $\begin{bmatrix} x & y \\ 0 & a \end{bmatrix}$ is $P-$full in $M_2(R)$, then a is full in R.
(3) If $a \in R$ is a full element in R, then for every $x, y \in R$, the element $\begin{bmatrix} a & 0 \\ x & y \end{bmatrix}$ is $Q-$full in $M_2(R)$.
(4) If for some $x, y \in R$, the element $\begin{bmatrix} a & 0 \\ x & y \end{bmatrix}$ is $Q-$full in $M_2(R)$, then a is full in R.

Proof. (1) Suppose that $a \in R$ is a full element, then there exists $s, t \in R$ such that $sat = 1$, so

$$\begin{bmatrix} 0 & 0 \\ 0 & s \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix} \in M_2(R)$$

Such that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & s \end{bmatrix} \cdot \begin{bmatrix} x & y \\ 0 & a \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & sat \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in P$$

and $\begin{bmatrix} 0 & 0 \\ 0 & s \end{bmatrix} P \subseteq P$. So $\begin{bmatrix} x & y \\ 0 & a \end{bmatrix}$ is a $P-$full element in $M_2(R)$.

(2) Suppose that $\sigma = \begin{bmatrix} x & y \\ 0 & a \end{bmatrix}$ is a $P-$full element in $M_2(R)$ for some $x, y \in R$. Then there

exists $u = \begin{bmatrix} n & m \\ z & s \end{bmatrix}, v = \begin{bmatrix} \alpha & \beta \\ \gamma & t \end{bmatrix} \in M_2(R)$ such that

$$1 - u\sigma v = 1 - \begin{bmatrix} n & m \\ z & s \end{bmatrix} \cdot \begin{bmatrix} x & y \\ 0 & a \end{bmatrix} \cdot \begin{bmatrix} \alpha & \beta \\ \gamma & t \end{bmatrix} \in P$$

and $uP = \begin{bmatrix} n & m \\ z & s \end{bmatrix} P \subseteq P$. Since $uP \subseteq P$, $z = 0$ and

$$1 - \begin{bmatrix} n & m \\ 0 & s \end{bmatrix} \cdot \begin{bmatrix} x & y \\ 0 & a \end{bmatrix} \cdot \begin{bmatrix} \alpha & \beta \\ \gamma & t \end{bmatrix} = 1 - \begin{bmatrix} \lambda & \mu \\ sa\gamma & sat \end{bmatrix} \in P$$

Where $\lambda, \mu \in R$. This shows that $1 - sat = 0$. i.e., $sat = 1$, thus a is full in R.
(3) Similarly, as in (1).
(4) Similarly, as in (2). □

From Theorem 3.6 we can obtain the following:

Corollary 3.7. For every element $a \in R$ the following statements hold:
(1) An element $a \in R$ is a full element in R if and only if there exists $x, y \in R$ such that $\begin{bmatrix} x & y \\ 0 & a \end{bmatrix}$ is $P-$full in $M_2(R)$.
(2) An element $a \in R$ is a full element in R if and only if there exists $x, y \in R$ such that $\begin{bmatrix} a & 0 \\ x & y \end{bmatrix}$ is $Q-$full in $M_2(R)$.

4 $f-$Clean Rings Relative to Right Ideals

In this section we study the concept of $f-$clean rings relative to right ideal as generalization of $f-$clean ring, we start with the following:

Definition 4.1. Let R be a ring and $P \neq R$ be a right ideal of R.
- We say that an element x of a ring R is $f-$clean relative to right ideal P, if $x = a + e$, where $e \in R$ is $P-$idempotent and $a \in R$ is a $P-$full element.
- We say that a ring R is $f-$clean relative to right ideal P, if every element x in R is $f-$clean relative to P.

Note that in previous definition, it is easy to see that for $P = 0$, a ring R is $f-$clean relative to P if and only if R is $f-$clean. Furthermore, we have the following:

Lemma 4.2. Let R be a ring and $P \neq R$ be a right ideal of R. Then the following hold:
(1) Elements $1, -1, 0$ are $f-$clean elements relative to P.
(2) Every $P-$full element in R is $f-$clean relative to P.
(3) Every element of R has a right inverse is $f-$clean relative to P.
(4) Every element of R has an inverse is $f-$clean relative to P.

Proof. (1) Obvious, because $1 = 1 + 0$, $0 = (-1) + 1$, $-1 = -1 + 0$, where $1, 0$ are $P-$idempotents and $1, -1$ are $P-$full elements.
(2) If $a \in R$ is $P-$full, then $a = a + 0$ is $f-$clean relative to P.
(3) If $a \in R$ has a right inverse, then $R = aR \subseteq aR + P \subseteq R$, so $R = aR + P$ and so $1 = at + p_0$ for some $t \in R$, $p_0 \in P$. Thus, $1 - at = p_0 \in P$, so a is $P-$full. Since $a = a + 0$, a is $f-$clean relative to P.
(4) Obvious by (3). □

Lemma 4.3. Let R be a ring and $P \neq R$ be a right ideal of R. Then the following hold:
(1) Every elements has a right $P-$inverse is $f-$clean relative to P.
(2) Every $P-$clean element is $f-$clean relative to P.
(3) If $x \in R$ is an $f-$clean element relative to P, then for every $p \in P$, $x + p$ is $f-$clean relative to P.

Proof. (1) Let $a \in R$ has a right $P-$inverse, then by Lemma 3.2 a is $P-$full and so $a = a + 0$ is $f-$clean relative to P.
(2) Let $x \in R$ be a $P-$clean element, then $x = a + e$, where $a \in R$ has a right $P-$inverse and $e \in R$ is $P-$idempotent, so by Lemma 3.2 $a \in R$ is $P-$full and so x is $f-$clean relative to P.
(3) Suppose that $x \in R$ is $f-$clean relative to P, then $x = a + e$, where $a \in R$ is $P-$full and $e \in R$ is $P-$idempotent, so by Lemma 2.1 for every $p \in P$ $e + p \in R$ is $P-$idempotent. Thus $x + p = (a + e) + p = a + (e + p)$, where $a \in R$ is $P-$full and $e + p \in R$ is $P-$idempotent. Therefore x is $f-$clean relative to P. □

Lemma 4.4. Let R be a ring and $P \neq R$ be a right ideal of R. Then for every $P-$idempotent $e \in R$ the following statements hold:
(1) e is an $f-$clean element relative to P.
(2) For every positive integer k, e^k is an $f-$clean element relative to P.

Proof. (1) Let $e \in R$ be an $P-$idempotent element, then $e^2 - e \in P$ and $eP \subseteq P$, so $e^2 = e + p_0$ for some $p_0 \in P$. So,

$$(2e-1)(2e-1) = 4e^2 - 4e + 1 = 4(e + p_0) - 4e + 1 = 4p_0 + 1$$

$$1 = (2e-1)(2e-1) + 4(-p_0) \in (2e-1)R + P$$

Therefore $R = (2e-1)R + P$, thus $2e - 1 \in R$ has a right $P-$inverse and by Lemma 4.3, $2e-1$ is an $P-$full element. Since $e = (2e-1) + (1-e)$ and $1 - e \in R$ is $P-$idempotent, implies that $e \in R$ is $f-$clean relative to P.
(2) Since e is $P-$idempotent, by Lemma 2.1 e^k is $P-$idempotent for every positive integer k, so by (1) e^k is an $f-$clean element. □

Lemma 4.5. Let R be a ring and $P \neq R$ be a right ideal of R. Suppose that $x, y \in R$ such that $x - y \in P$, then x is $f-$clean relative to P if and only if y is $f-$clean relative to P.

Proof. Let $x, y \in R$ such that $x - y \in P$, then $x = y + p_0$ for some $p_0 \in P$. Suppose that x is $f-$clean relative to P, then $x = a + e$, where $a \in R$ is $P-$full and $e \in R$ is $P-$idempotent. So

$$y = x - p_0 = (a + e) + (-p_0) = a + (e + (-p_0))$$

where $a \in R$ is $P-$full and $e + (-p_0) \in R$ is $P-$idempotent by Lemma 2.1. Thus y is $f-$clean relative to P. Similarly, we can prove converse. □

Proposition 4.6. Let R be a ring, $P \neq R$ be a right ideal of R and $x \in R$. Then x is $f-$clean relative to P if and only if $1 - x$ is $f-$clean relative to P.

Proof. Let $x \in R$ is $f-$clean relative to P, then $x = a + e$, where $a \in R$ is $P-$full and $e \in R$ is $P-$idempotent. So $1 - x = (-a) + (1 - e)$. Since $a \in R$ is $P-$full, by Lemma 3.2 $-a \in R$ is $P-$full and $1 - e \in R$ is $P-$idempotent, thus $1 - x$ is $f-$clean relative to P. Conversely, if $1 - x$ is $f-$clean relative to P, write $1 - x = a + e$, where $a \in R$ is $P-$full and $e \in R$ is $P-$idempotent. So $-x = a + e - 1$ and so $x = (-a) + (1 - e)$. Since a is $P-$full, by Lemma 3.2 $-a \in R$ is $P-$full and $1 - e \in R$ is $P-$idempotent. Therefore x is $f-$clean relative to P. □

Proposition 4.7. Every ring is $f-$clean relative to any maximal right ideal of it.

Proof. Let R be a ring and M be a maximal right ideal of R. Let $a \in R$, then:
- If $a \in M$, then $1 - a \notin M$ and so $a - 1 \notin M$, thus $R = (a-1)R + M$. This shows that $a - 1 \in R$ has a right $M-$inverse, so by Lemma 3.2 $a - 1$ is a $M-$full element. Since $a = (a-1) + 1$, a is $f-$clean relative to M.
- If $a \notin M$, then $R = aR + M$, this shows that a has a right $M-$inverse, by Lemma 3.2 a is $M-$full. So $a = a + 0$ is $f-$clean. Thus a ring R is $f-$clean relative to M. □

Proposition 4.8. Let R be a ring and $P \neq R$ be a right ideal of R. Then the following conditions are equivalent:
(1) A ring R is $f-$clean relative to right ideal P.
(2) For every $x \in R$, $x = a - e$, where $a \in R$ is $P-$full and $e \in R$ is $P-$ idempotent.

Proof. (1) $\Rightarrow$ (2) Suppose that R is an $f-$clean ring relative to P. Let $x \in R$, then $-x \in R$ and $-x = a + e$, where $a \in R$ is a $P-$full element and $e \in R$ is $P-$idempotent, so $x = (-a) - e$. Since a is $P-$full, then by Lemma 3.2 there exists $s \in R$ such that $R = saR + P$, so $R = (-s)(-a)R + P$, where $-s \in R$. This shows that $-a \in R$ is $P-$full and $e \in R$ is $P-$idempotent.
(2) $\Rightarrow$ (1) Let $x \in R$, then $-x \in R$ and by assumption $-x = a - e$, where $a \in R$ is a $P-$full element and $e \in R$ is $P-$idempotent, so $x = (-a) + e$. Since a is $P-$full, then by Lemma 3.2 there exists $s \in R$ such that $R = saR + P$, so $R = (-s)(-a)R + P$, where $-s \in R$. This shows that $-a \in R$ is $P-$full and $e \in R$ is $P-$idempotent. Thus, x is $f-$clean relative to P and so a ring R is $f-$clean relative to P. □

Theorem 4.9. Let R be a ring and $P \neq R$ be a right ideal of R. If the set of all $P-$idempotents in R is $\{0, 1, p, 1 + p\}$ for every $p \in P$. Then the following conditions are equivalent:
(1) A ring R is $f-$clean relative to P.
(2) For every $x \in R$, either x or $1 - x$ is a $P-$full element.

Proof. (1) $\Rightarrow$ (2) Suppose that R is $f-$clean relative to P. Let $x \in R$, then $x = a + e$ where $a \in R$ is $P-$full and $e \in R$ is $P-$idempotent. So by assumption:
- If $e = 0$, then $x = a$ is $P-$full.
- If $e = p$, then $x = a+p$, since $p \in P$ and a is $P-$full, $R = saR+P$, where $s \in R$ and $sP \subseteq P$, then we have

$$R = saR + P = s(x - p)R + P \subseteq sxR + s(-p)R + P \subseteq$$
$$\subseteq sxR + P \subseteq R$$

Thus $R = sxR + P$ and $sP \subseteq P$, thus by Lemma 3.2 x is a $P-$full element.
- If $e = 1$, then $x = a + 1$, so $1 - x = -a$. Since a is $P-$full then by Lemma 3.2 an element $-a \in R$ is $P-$full, so $1 - x$ is $P-$full.
- If $e = 1 + p$, then $x = a + e = a + 1 + p$, so $1 - x = -(a + p)$. Since a is $P-$full, by Lemma 3.2, $R = saR + P$, where $s \in R$ and $sP \subseteq P$, Thus

$$R = saR + P = s(a + p - p)R + P \subseteq s(a + p)R + s(-p)R + P \subseteq$$
$$\subseteq s(a + p)R + P \subseteq R$$

Thus $R = s(a + p)R + P$ and $sP \subseteq P$, thus by Lemma 3.2 $a + p$ is a $P-$full element and by Lemma 3.2 $-(a + p)$ is $P-$full, therefore $1 - x = -(a + p)$ is a $P-$full element.
(2) $\Rightarrow$ (1) Let $x \in R$, by assumption, either x or $1 - x$ is $P-$full. If x is $P-$full, then $x = x + 0$ is $f-$clean relative to P. Suppose that $1 - x$ is $P-$full and by Lemma 3.2 $x - 1$ is $P-$full and so $x = (x - 1) + 1$ is $f-$clean relative to P. Thus, R is an $f-$clean ring relative to P. □

Definition 4.10. Let R be a ring and $P \neq R$ be a right ideal of R. We say that a ring R is a $P-$local if for every $x \in R$ either x or $1 - x$ has a right $P-$inverse.

Proposition 4.11. Let R be a ring and $P \neq R$ be a right ideal of R. If R is a $P-$local ring, then R is $f-$clean relative to right ideal P and the set of all $P-$idempotents in R only is $\{0, 1, p, 1 + p\}$ for every $p \in P$.

Proof. Suppose that R is a $P-$local ring. Let $x \in R$, then either x or $1-x$ has a right $P-$inverse.
- If x has a right $P-$inverse, then by Lemma 4.3 x is an $f-$clean relative to P.
- If $1 - x$ has a right $P-$inverse, then by Lemma 4.3 $1 - x$ is an $f-$clean relative to P, thus by Proposition 4.6 x is an $f-$clean element relative to P. Therefore R is $f-$clean relative to P.
Let $e \in R$ be a $P-$idempotent.
- If $e = 0$ or $e = 1$, our proof is completed. Suppose that $e \neq 0$ and $e \neq 1$, then by assumption either e or $1 - e$ has a right $P-$inverse.
- If e has a right $P-$inverse, then $R = eR + P$, so $1 = ex + p_1$ for some $x \in R$ and $p_1 \in P$, thus $e = e^2x + ep_1$. Since e is $P-$idempotent, then $e^2 = e + p_0$ for some $p_0 \in P$. So

$$e = e^2x + ep_1 = (e + p_0)x + ep_1 = ex + p_0x + ep_1$$
$$= 1 - p_1 + p_0x + ep_1 = 1 + p$$

Where $p = -p_1 + p_0x + ep_1 \in P$. Thus $e = 1 + p$.
- If $1 - e$ has a right $P-$inverse, then $R = (1 - e)R + P$, so $1 = (1 - e)y + p_2$ for some $y \in R$, $p_2 \in P$, thus $e = (e - e^2) + ep_2 \in P$. Our proof is completed. □

Lemma 4.12. For every element $u \in R$ the following statements hold:

(1) If $u \in R$ is an $f-$clean element in R, then the element $\begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix}$ is $f-$clean relative to P in $M_2(R)$.

(2) If the element $\begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix}$ is $f-$clean relative to P in $M_2(R)$, then u is $f-$ clean in R.

(3) If $u \in R$ is an $f-$clean element in R, then the element $\begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix}$ is $f-$clean relative to Q in $M_2(R)$.

(4) If the element $\begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix}$ is $f-$clean relative to Q in $M_2(R)$, then u is $f-$ clean in R.

Proof. (1) Suppose that $u \in R$ is an $f-$clean, then $u = a + e$, where $a \in R$ is full and $e \in R$ is idempotent. Since a is full in R, by Proposition 3.4 the element $\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$ is $P-$full in $M_2(R)$. Also, since e is idempotent in R, by Proposition 2.2 $\begin{bmatrix} 0 & 0 \\ 0 & e \end{bmatrix}$ is $P-$idempotent in $M_2(R)$. Thus the element

$$\begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & a+e \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & e \end{bmatrix}$$

is $f-$clean relative to P in $M_2(R)$.

(2) Suppose that $\begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix}$ is $f-$clean relative to P in $M_2(R)$, then

$$\begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix} = \begin{bmatrix} x & y \\ z & a \end{bmatrix} + \begin{bmatrix} \alpha & \beta \\ \gamma & e \end{bmatrix}$$

Where $\begin{bmatrix} x & y \\ z & a \end{bmatrix}$ is $P-$full in $M_2(R)$ and $\begin{bmatrix} \alpha & \beta \\ \gamma & e \end{bmatrix}$ is $P-$idempotent in $M_2(R)$. So $z = -\gamma$ and $u = a + e$. Since $\begin{bmatrix} \alpha & \beta \\ \gamma & e \end{bmatrix}$ is $P-$idempotent, $\begin{bmatrix} \alpha & \beta \\ \gamma & e \end{bmatrix} P \subseteq P$, which implies $\gamma = 0$, so $z = -\gamma = 0$. Thus

$$\begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & a \end{bmatrix} + \begin{bmatrix} \alpha & \beta \\ 0 & e \end{bmatrix}$$

Since $\begin{bmatrix} x & y \\ 0 & a \end{bmatrix}$ is $P-$full in $M_2(R)$, so by Theorem 3.6 the element a is full in R. Also, since $\begin{bmatrix} \alpha & \beta \\ 0 & e \end{bmatrix}$ is $P-$idempotent in $M_2(R)$, so by Proposition 2.2 the element e is idempotent in R. Thus the element $u = a + e$ is $f-$clean in R.

(3) Similarly, as in (1).

(4) Similarly, as in (2). □

From Lemma 4.12 we can drive the following:

Corollary 4.13. For every element $u \in R$ the following statements hold:

(1) The element $u \in R$ is $f - clean$ in R if and only if the element $\begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix}$ is $f-$ clean relative to P in $M_2(R)$.

(2) The element $u \in R$ is $f-$clean in R if and only if the element $\begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix}$ is $f-$clean relative to Q in $M_2(R)$.

Proposition 4.14. For every element $u \in R$ the following statements hold:
(1) If $u \in R$ is an $f-$clean element in R, then for every $x, y \in R$ the element $\begin{bmatrix} x & y \\ 0 & u \end{bmatrix}$ is $f-$clean relative to P in $M_2(R)$.
(2) If for some $x, y \in R$, the element $\begin{bmatrix} x & y \\ 0 & u \end{bmatrix}$ is $f-$clean relative to P in $M_2(R)$, then u is $f-$clean in R.
(3) If $u \in R$ is an $f-$clean element in R, then for every $x, y \in R$ the element $\begin{bmatrix} u & 0 \\ x & y \end{bmatrix}$ is $f-$clean relative to Q in $M_2(R)$.
(4) If for some $x, y \in R$, the element $\begin{bmatrix} u & 0 \\ x & y \end{bmatrix}$ is $f-$clean relative to Q in $M_2(R)$, then u is $f-$clean in R.

Proof. (1) Suppose that $u \in R$ is an $f-$clean, then $u = a + e$, where $a \in R$ is full and $e \in R$ is idempotent. Since a is full in R, by Theorem 3.6, for every $x, y \in R$ the element $\begin{bmatrix} x & y \\ 0 & a \end{bmatrix}$ is $P-$full in $M_2(R)$. Also, since e is idempotent in R, by Proposition 2.2 the element $\begin{bmatrix} 0 & 0 \\ 0 & e \end{bmatrix}$ is $P-$idempotent in $M_2(R)$. Thus the element

$$\begin{bmatrix} x & y \\ 0 & u \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & a+e \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & e \end{bmatrix}$$

is $f-$clean relative to P in $M_2(R)$.
(2) Suppose that for some $x, y \in R$, the element $\begin{bmatrix} x & y \\ 0 & u \end{bmatrix}$ is $f-$clean relative to P in $M_2(R)$, then

$$\begin{bmatrix} x & y \\ 0 & u \end{bmatrix} = \begin{bmatrix} n & m \\ z & a \end{bmatrix} + \begin{bmatrix} \alpha & \beta \\ \gamma & e \end{bmatrix}$$

Where $\begin{bmatrix} n & m \\ z & a \end{bmatrix}$ is $P-$full in $M_2(R)$ and $\begin{bmatrix} \alpha & \beta \\ \gamma & e \end{bmatrix}$ is $P-$idempotent in $M_2(R)$. So $z = -\gamma$ and $u = a + e$. Since $\begin{bmatrix} \alpha & \beta \\ \gamma & e \end{bmatrix}$ is $P-$idempotent, $\begin{bmatrix} \alpha & \beta \\ \gamma & e \end{bmatrix} P \subseteq P$, which implies $\gamma = 0$, so $z = -\gamma = 0$. Thus

$$\begin{bmatrix} x & y \\ 0 & u \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & a \end{bmatrix} + \begin{bmatrix} \alpha & \beta \\ 0 & e \end{bmatrix}$$

Since $\begin{bmatrix} x & y \\ 0 & a \end{bmatrix}$ is $P-$full in $M_2(R)$, so by Theorem 3.6 the element a is full in R. Also, since $\begin{bmatrix} \alpha & \beta \\ 0 & e \end{bmatrix}$ is $P-$idempotent in $M_2(R)$, so by Proposition 2.2 the element e is idempotent in R. Thus the element $u = a + e$ is $f-$clean in R.
(3) Similarly, as in (1).
(4) Similarly, as in (2). □

From Proposition 4.14 we can obtain the following:

Corollary 4.15. For every element $u \in R$ the following statements hold:
(1) An element $u \in R$ is $f-$clean in R if and only if there exists $x, y \in R$ such that the element $\begin{bmatrix} x & y \\ 0 & u \end{bmatrix}$ is $f-$clean relative to P in $M_2(R)$.

(2) An element $u \in R$ is $f-$clean in R if and only if there exist $x, y \in R$ such that the element $\begin{bmatrix} u & 0 \\ x & y \end{bmatrix}$ is $f-$clean relative to Q in $M_2(R)$.
(3) For every $x, y \in R$, the element $\begin{bmatrix} x & y \\ 0 & u \end{bmatrix}$ is $f-$clean relative to P in $M_2(R)$ if and only if the element $\begin{bmatrix} u & 0 \\ x & y \end{bmatrix}$ is $f-$clean relative to Q in $M_2(R)$.

We again use the notation, let R be a ring and $S = M_2(R)$ be the ring of all 2×2 matrices over a ring R. It is clear that the set

$$S_0 = \{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x, y \in R\}$$

is a subring in S with identity element. Also, the sets

$$P_0 = \{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in R\} \quad and \quad Q_0 = \{ \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} : a \in R\}$$

are right ideals in S_0 and $P_0 \neq S_0$, $Q_0 \neq S_0$. Then we have the following:

Theorem 4.16. For any ring R the following hold:
(1) A ring R is an $f-$clean ring if and only if the ring S_0 is an $f-$clean ring relative to right ideal P_0.
(2) A ring R is an $f-$clean ring if and only if the ring S_0 is an $f-$clean ring relative to right ideal Q_0.
(3) The ring S_0 is an $f-$clean ring relative to right ideal P_0 if and only if the ring S_0 is an $f-$clean ring relative to right ideal Q_0.

Proof. (1)($\Rightarrow$) Suppose that a ring R is $f-$clean. Let $\alpha = \begin{bmatrix} x & 0 \\ 0 & u \end{bmatrix} \in S_0$, where $x, u \in R$. Since R is $f-$clean, then the element $u \in R$ is $f-$clean, so by Corollary 4.15 the element $\alpha = \begin{bmatrix} x & 0 \\ 0 & u \end{bmatrix}$ is $f-$clean relative to P_0-regular element in S_0.
($\Leftarrow$) Let $a \in R$, then for every $x \in R$, $\alpha = \begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix} \in S_0$. Since S_0 is an $f-$clean ring relative to P_0, the element α is $f-$clean relative to P_0, so by Corollary 4.15 a is $f-$clean, thus a ring R is $f-$clean. Similarly, we can prove (2). (3) Follows immediately from (1) and (2). □

References

[1] A. V. Andrunakievich and V. A. Andrunakievich, *Rings that are regular relative to right ideal*, Mat. Zametki. Vol. **49**, No. (3) (1991), pp. 3–11.

[2] V. A. Andrunakievich and Yu. M. Ryabukhin, *Quasiregularity and Pimitivity Relative to Right Ideals of a Ring*, Mat. Sb. Vol. **134**, No. (4) (1987), pp. 451–471.

[3] N. Ashrafi and E. Nasibi, *$r-$clean rings*, Math. Reports. Vol. **15(65)**, No.(2), (2013), pp. 125–132.

[4] K. R. Goodearl, *Von Neumann regular rings*, Pitman. London. (1979).

[5] H. Hamza, *On Some Characterizations of Regular and Potent Rings Relative to Right Ideals*, Novi Sad J. Math. Vol. **48**, No. (1), (2018), pp. 1–7.

[6] H. Hamza, *$P-$Regular and $P-$Local Rings*, Journal of Algebra and Related Topic. Vol. **9**, No. (1), (2021), pp. 1–19.

[7] H. Hamza, *$r-$Clean Rings Relative to Right Ideals*, Journal of Algebraic Systems. Vol. **10**, No. (2), (2023), pp. 203–224.

[8] B. Li and L. Feng, *f-Clean rings and rings having many full elements*, J. Korean Math. Soc. Vol. **47**, No. (2), (2010), pp. 247–261.

[9] W.K. Nicholson, *Lifting idempotents and exchange rings*, Trans. Amer. Math. Soc. Vol. **229**, (1977), pp. 269–278.

[10] H. Zhang and V. Camillo, *On clean rings*, Communications in Algebra. Vol. **44**, (2016), pp. 2475–1481.

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