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ON THE HIGHER ORDER GENUS OF THE NILPOTENT GRAPH OF A RING

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Abstract. For a commutative ring R, let $Z_N(R)^* = \{x \in R : xy \in N(R)$ for some $y \in R^*\}$, which is the vertex set of the *nilpotent graph* $\Gamma_N(R)$ and any two vertices x and y are adjacent if and only if $xy \in N(R)$, where $N(R)$ is the set of all nilpotent elements of R. In this article we discuss the higher order(two and three) genus of $\Gamma_N(R)$.

1 Introduction

Algebraic graph theory is an provocative area in which one can innovate graphs from algebraic structures. Relating graph theory with algebraic structures, gives assist in resolving many algebraic and graph theoretic problems. In recent times, a plenty of works have been done in analysing the properties of the graphs constructed from algebraic structures. Motivated by the studies of P.W. Chen in algebraic graph theory, in 2010, Ai. Hua Li and Qi- Sheng Li [1] introduced a kind of graph structure on a ring, called *nilpotent graph* and the graph can be determined as follows: The vertex set of the nilpotent graph $\Gamma_N(R)$ is $Z_N(R)^* = \{x \in R : xy \in N(R)$ for some $y \in R^*\}$ and two vertices are adjacent if and only if their product should be in $N(R)$, where $N(R)$ is the set of all nilpotent elements of R. Also they have investigated some of its basic properties. Consequently, Nikmehr and Khojasteh [7] studied about $\Gamma_N(R)$, when R is non- commutative. Added with that, the topological properties of $\Gamma_N(R)$ was scruntinized by R. Kala and S. Kavitha, see [6]. Influenced by the above researches, we want to bring to light, the higher order genus of $\Gamma_N(R)$.
All over this paper, R denotes a commutative ring with unity and $R^\times$ denotes the set of all unit elements of R. Also $\mathfrak{m}_1^*$ denotes the collection of all non-zero elements of m_1 and $\mathbb{F}_i$ notates a filed with cardinality i. Since the pendant vertices do not disturb the genus embedding, we disgard them in all the embeddings of $\Gamma_N(R)$. For genus and its related problems one can refer [3], for ring theory, graph theory definitions one can see [2] and [4] respectively.

2 Preliminaries

Now, we recollect some of the theorems that are required for the succeeding sections.

Lemma 2.1. ([3, Theorem 4.4.5]) $g(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$ if $n \geq 3$.

Lemma 2.2. ([3, Theorem 4.4.7]) $g(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$ if $m, n \geq 2$ and $n \neq 7$.

Theorem 2.3. ([3, Theorem 4.4.2]) The genus of a connected graph is the sum of the genera of its blocks.

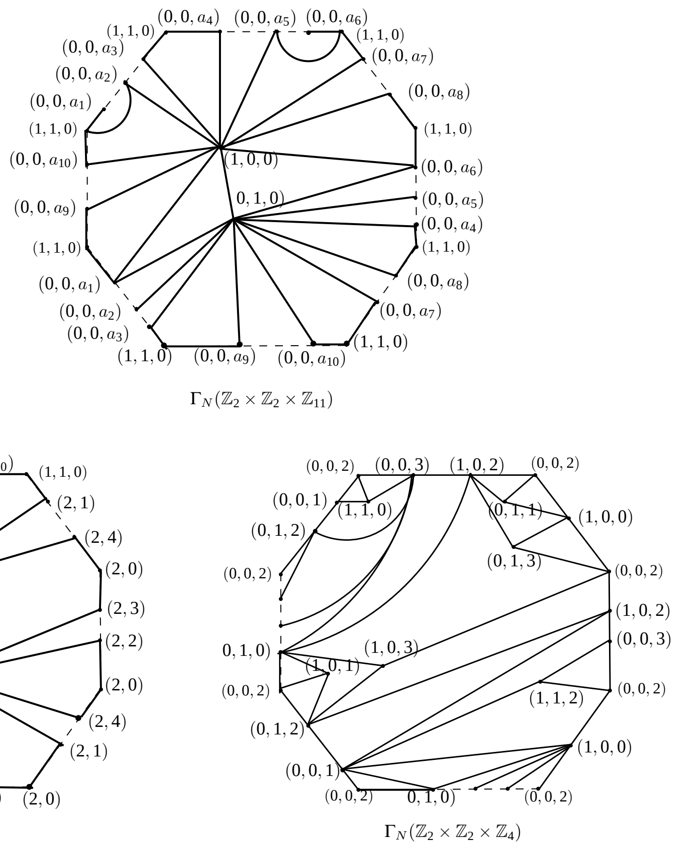

Figure 1. An embedding of graphs in S_2

Theorem 2.4. ([3, Proposition 4.4.4]) For a connected graph G with $n \geq 3$ vertices and q edges, $g(G) \geq \frac{q}{6} - \frac{n}{2} + 1$. Moreover, if G contains no cycle of length 3, then, $g(G) \geq \frac{q}{4} - \frac{n}{2} + 1$

Theorem 2.5. ([3, Euler's Formula]) Let G be a graph with n vertices, q edges which is $2-$ cell embedded in a surface S, then $n - q + f = 2 - 2g$, where f is the number of faces in S.

3 Genus two nilpotent graphs

In this section we classify all commutative rings that are embeddable in S_2.

Remark 3.1. Suppose $R = F_1 \times F_2$, where each F_i is a field, then $\Gamma_N(R) \cong K_{|F_1^*|,|F_2^*|}$

Remark 3.2. ([6, Theorem 4.3]) There will be no local ring R (but not a field) for which $g(\Gamma_N(R)) = 2$

Theorem 3.3. Suppose R is a product of m local rings(but not a field), then $g(\Gamma_N(R)) \geq 4$ for $m \geq 2$

Proof. The proof is obvious by taking $R = \mathbb{Z}_4 \times \mathbb{Z}_4$ (which is the least possibility). Since $g(\Gamma_N(\mathbb{Z}_4 \times \mathbb{Z}_4))$ contains two blocks, consists of a $K_{4,7}$ and a $K_{3,4}$, by Theorem 2.3, $g(\Gamma_N(R)) \geq 4$, for $m \geq 2$. □

Theorem 3.4. Suppose $R = \prod_{i=1}^{n} R_i \times \prod_{j=1}^{m} F_j$, where each R_i is local with maximal ideal $\mathfrak{m}_i$ and F_j is a field and $n, m \geq 1$. Then $g(\Gamma_N(R)) = 2$ if and only if R is any one of the rings: $\mathbb{Z}_4 \times \mathbb{Z}_5$, $\frac{\mathbb{Z}_2(x)}{(x^2)} \times \mathbb{Z}_5, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \frac{\mathbb{Z}_2(x)}{(x^2)}$

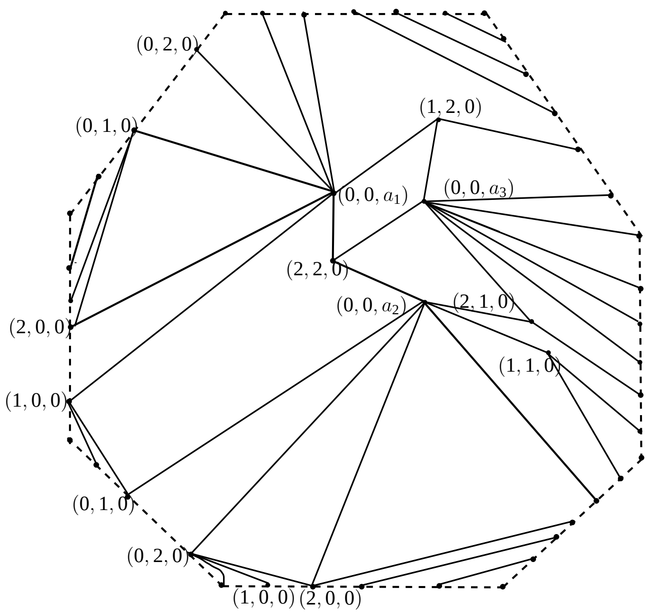

Figure 2. An embedding of $\Gamma_N(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{F}_4)$ in S_2
In the above figure, we can easily insert the six vertices, that are adjacent with $(1, 0, 0), (2, 0, 0)$ and the six vertices, that are adjacent with $(0, 1, 0), (0, 2, 0)$.

Proof. We observe that $\Gamma_N(\frac{\mathbb{Z}_2(x)}{(x^2)} \times \mathbb{Z}_5) \cong \Gamma_N(\mathbb{Z}_4 \times \mathbb{Z}_5)$. Also it has $K_{3,8}$ as a induced subgraph and by Theorem 2.2, $g(\Gamma_N(R)) \geq 2$. Now Figure.1 explicitly shows that $g(\Gamma_N(R)) = 2$. When $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$, $\Gamma_N(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4)$ has two blocks of $K_{3,3}$ and by Theorem 2.3, $g(\Gamma_N(R)) \geq 2$. Now Figure.1 reveals that $g(\Gamma_N(R)) = 2$.
For the other way, if $n + m \geq 4$, then $|R| \geq 32$ and $|R_1| \geq 4$ and $|\mathfrak{m}_1^*| \geq 1$. Therefore by Theorem 2.5, $g(\Gamma_N(R)) \geq 3$. Hence $n + m \leq 3$.
Case 1: $n + m = 2$
Suppose $|\mathfrak{m}_1^*| = 2$, then $R_1 = \mathbb{Z}_9$ or $\frac{\mathbb{Z}_3(x)}{(x^2)}$. Let $a_1, a_2 \in \mathfrak{m}_1^*$ with $a_1 \neq a_2$, $u_i \in R_1^*, 1 \leq i \leq 6$ with $u_i \neq u_j$. Then the subgraph induced by $A_1 = \{w_1, w_2, \dots w_{11}\,]$ contains $K_{5,6}$ as a subgraph, where $w_1 = (a_1, 0), w_2 = (a_2, 0), w_k = (u_i, 0), 3 \leq k \leq 8, 1 \leq i \leq 6, w_9 = (a_1, 1), w_{10} = (a_2, 1), w_{11} = (0, 1)$. Now using Theorem 2.2, $g(\Gamma_N(R)) \geq 3$. Hence $|\mathfrak{m}_1^*| = 1$.
Now we claim that $|F_1| \leq 5$. For, take $a \in \mathfrak{m}_1^*, b_1, b_2 \in R_1^\times$. Then the subgraph generated by $A_2 = \{x_1, \dots x_{15}\}$ contains $K_{3,12}$, where $x_1 = (a, 0), x_2 = (b_1, 0), x_3 = (b_2, 0), x_j = (0, c_k), 4 \leq j \leq 9, x_t = (a, c_k), 10 \leq t \leq 15, 1 \leq k \leq 6$, $c_k \in F_1$. Using Theorem 2.2, we obtain that $g(\Gamma_N(R)) \geq 3$. Hence $|F_1| = 5$. Hence $R = \mathbb{Z}_4 \times \mathbb{Z}_5$ or $\frac{\mathbb{Z}_2(x)}{(x^2)} \times \mathbb{Z}_5$.
Case 2: $n + m = 3$
Suppose $n = 2$ and $m = 1$, Theorem 3.3 yields that, $g(\Gamma_N(R)) \geq 3$. Hence we arrived at the decision that $R = R_1 \times F_1 \times F_2$. Now by case.1, $R_1 = \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2(x)}{(x^2)}$.
Subcase 1: Suppose $|F_i| \geq 3$ for $i = 1, 2$. Let $y_1 = (1, 0, 0), y_2 = (2, 0, 0), y_3 = (3, 0, 0), y_4 = (0, u_1, 0), y_5 = (0, u_2, 0), y_6 = (0, 0, v_1), y_7 = (0, 0, v_2), y_8 = (0, u_1, v_1), y_9 = (0, u_2, v_1), y_{10} = (0, u_1, v_2), y_{11} = (0, u_2, v_2), y_{12} = (2, 0, v_1), y_{13} = (2, 0, v_2), y_{14} = (2, u_1, 0), y_{15} = (2, u_2, 0)$. Then the subgraph $\langle A_3 \rangle$, where $A_3 = \{y_1, \dots y_{15}\}$ possess a $K_{3,12}$. Hence by Theorem 2.2, $g(\Gamma_N(R)) \geq 3$.
Subcase 2: $|F_2| \geq 3$. Then clearly the collection of vertices $\{(0, 0, u_1), (2, 0, u_1), (0, 0, u_2), (2, 0, u_2), (1, 0, 0), (2, 0, 0), (3, 0, 0), (0, v_1, 0), (1, v_1, 0), ()2, v_1, 0), (3, v_1, 0)$ where $v_i \in F_1, u_i \in F_2\}$ will form a $K_{4,7}$ which has genus greater than 2. Hence $|F_i| = 2$, for $i = 1, 2$. Hence $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \frac{\mathbb{Z}_2(x)}{(x^2)}$. ∎

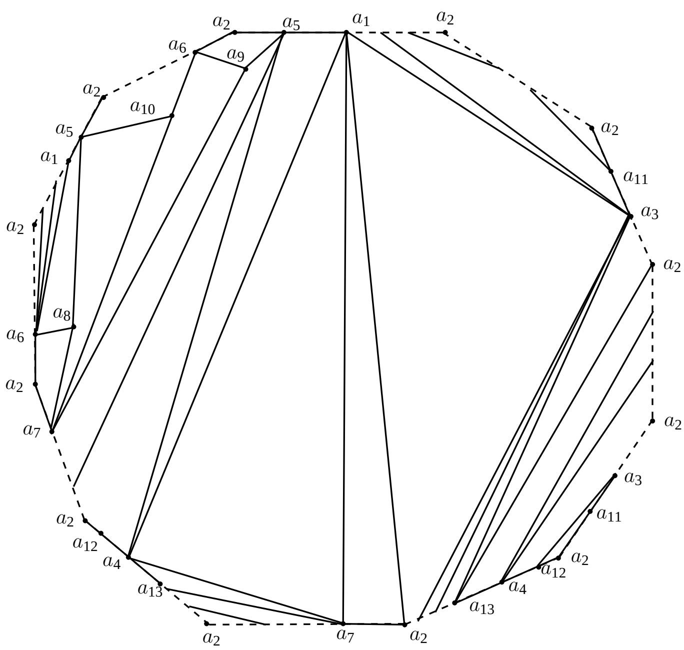

Figure 3. An embedding of $\Gamma_N(\mathbb{Z}_2 \times \mathbb{F}_4 \times \mathbb{F}_4)$ in S_3

In Figure. 3, a_i for $1 \leq i \leq 13$ denote $(1,0,0), (0,x_1,0), (0,x_2,0), (0,x_3,0), (0,0,y_1), (0,0,y_2),$ $(0,0,y_3), (1,x_1,0), (1,x_2,0), (1,x_3,0), (1,0,y_1), (1,0,y_2), (1,0,y_3)$ respectively.

Theorem 3.5. Let$R = \prod_{i=1}^{n} F_i$, where each F_i is a field. Then $g(\Gamma_N(R)) = 2$ if and only if R is any one of the consequent ring:
$\mathbb{Z}_5 \times \mathbb{Z}_7, \mathbb{F}_4 \times \mathbb{F}_8, \mathbb{F}_4 \times \mathbb{F}_9, \mathbb{F}_4 \times \mathbb{F}_{11}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_8, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_9, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_{11}, \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{F}_5, \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{F}_4$

Proof. We proceed with the following cases:
Case 1 : $n \geq 4$
Suppose $|F_1| \geq 3$ and $|F_2| \geq 3$. Then let us take $A_5 = (F_1 \times 0 \times F_3 \times 0 \times \ldots \times 0)^*, A_6 = (0 \times F_2 \times 0 \times F_4 \times 0 \times \ldots \times 0)^*$. Then $|A_5| \geq 5$, $|A_6| \geq 5$ and $\Gamma_N(R)$ must contain a copy of $K_{5,5}$. By Theorem 2.2, we arrived at a contradiction. So we have the only possibility that, R must be $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times I$, where I is a field with $|I| \geq 3$. Suppose $|I| \geq 7$, then also we obtain a contradiction as above. Hence R must be any one of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$. By example 3.6 of [8], we come to know that $g(\Gamma_N(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)) \geq 3$. Also $\Gamma_N(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$ is a subgraph of both $\Gamma_N(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4)$ and $\Gamma_N(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5)$. Hence they will obtain genus greater than two.
Case 2: $n = 3$
Suppose $|F_1| \geq 4$ and $|F_2| \geq 5$, then let $B_1 = (F_1 \times 0 \times F_3)^*, B_2 = (0 \times F_2 \times 0)^*$. Then $|B_1| \geq 7$, $|B_2| \geq 4$ and $\Gamma_N(R)$ must contain a copy of $K_{4,7}$. By Theorem 2.2, we obtain a contradiction. Suppose $|F_3| \geq 3$, then letting $B_3 = (F_1 \times F_2 \times 0)^*, B_4 = (0 \times F_2 \times 0),^*$ we will get $|B_3| \geq 3$ and $|B_4| \geq 12$ and $\Gamma_N(R)$ must contain a copy of $K_{3,12}$. By Theorem 2.2, we obtain a contradiction. Hence R must be $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_8, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_9, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_{11}, \mathbb{Z}_2 \times \mathbb{Z}_3 \times F, |F| \geq 5,$ or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times J, |J| \geq 4$, where F and J are fields. Suppose $|J| \geq 5$, then letting $B_7 = (\mathbb{Z}_3 \times \mathbb{Z}_3 \times 0)^*, B_8 = (0 \times 0 \times J)^*$, we get a copy of $K_{4,8}$, which is a contradiction. Suppose $|F| \geq 7$, then the graph necessarily holds a $K_{6,6}$, which leads to a conflict. Hence $R = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{F}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$.
Case 3: $n = 2$
In this case, by Remark 3.1, R is any one of the preceeding ring: $\mathbb{Z}_5 \times \mathbb{Z}_7, \mathbb{F}_4 \times \mathbb{F}_8, \mathbb{F}_4 \times \mathbb{F}_9$ or $\mathbb{F}_4 \times \mathbb{F}_{11}$.
For the converse portion, we know that $\Gamma_N(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_8)$ and $\Gamma_N(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_9)$ are subgraphs of $\Gamma_N(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_{11})$. Also by Figure 6 of [5] shows that $\Gamma_N(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5) = 2$. Now by Theorem 2.2 and Figures 1, 2, the converse part is completed. ◻

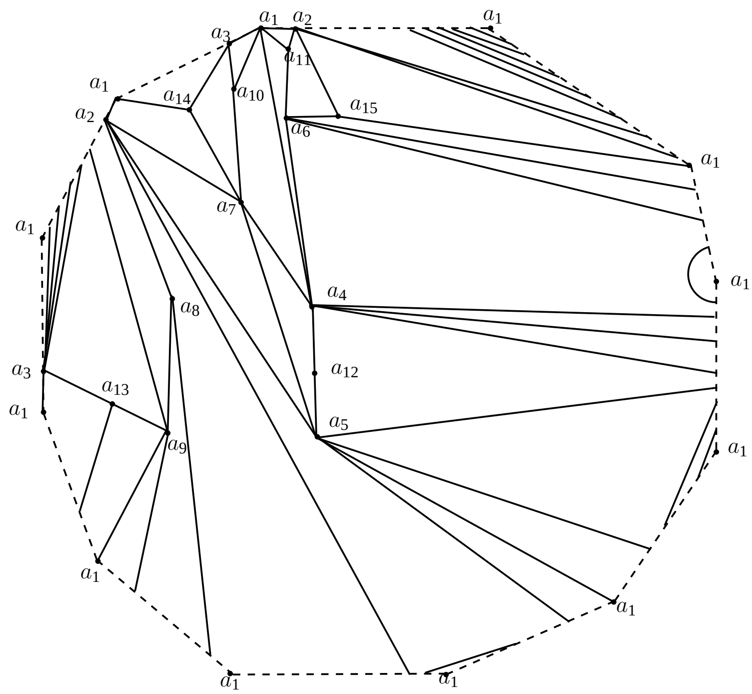

Figure 4. An embedding of $\Gamma_N(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$ in S_3

Let us notate, a_i for $1 \leq i \leq 15$ by $(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (0,0,0,2), (1,1,0,0), (1,0,1,0), (1,0,0,1), (0,1,1,0), (0,1,0,1), (0,0,1,1), (1,1,1,0), (1,0,0,2), (0,1,0,2), (0,0,1,2)$ respectively.

4 Genus three nilpotent graphs

In this section we classify all commutative rings for which $\Gamma_N(R)$ can be embeddable in S_3.

Theorem 4.1. For a local ring R, $g(\Gamma_N(R)) = 3$ if and only if $R = \frac{\mathbb{Z}_4(x)}{(x^2)}$ or $\frac{\mathbb{Z}_4(x)}{(x^2+x+1)}$

Proof. By the proof of Remark 3.2, we have $|\mathfrak{m}^*| \geq 3$, for $g(\Gamma_N(R)) \geq 3$. Suppose $|\mathfrak{m}^*| \geq 4$, then $|R^\times| \geq 20$, which will produce a copy of $K_{4,20}$ in $\Gamma_N(R)$, which has genus greater than 3. Hence $|\mathfrak{m}^*| = 3$. Hence we obtain that, $R = \frac{\mathbb{Z}_4(x)}{(x^2)}$ or $\frac{\mathbb{Z}_4(x)}{(x^2+x+1)}$ and the nilpotent graph of these rings are isomorphic and is $(K_2 \cup K_1) + \overline{K_{12}}$ which will receive genus 3, by Theorem 2.2. ◻

Theorem 4.2. Let$R = \prod_{i=1}^{n} F_i$, where each F_i is a field. Then $g(\Gamma_N(R)) = 3$ if and only if R is any one of the consequent ring:
$\mathbb{Z}_5 \times \mathbb{F}_8, \mathbb{F}_4 \times \mathbb{F}_{13}, \mathbb{Z}_5 \times \mathbb{F}_9, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{13}, \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_7, \mathbb{Z}_2 \times \mathbb{F}_4 \times \mathbb{F}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$

Proof. We will proceed with the following cases.
Case 1: $n \geq 5$.
While taking the least possible case (that is $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$) we can see that the collection $\{\{x_1, x_2, x_3, x_4, z_1, z_2, z_3, z_4, z_5\} - \{x_2x_3, x_2x_4, x_2x_5, x_3x_4, x_3x_5, x_4x_5, x_1x_{10}, x_8x_{10}, x_9x_{10}\}\}$, where $x_1 = (0,1,0,0,0), x_2 = (0,0,1,,0,0), x_3 = (0,0,0,1,0), x_4 = (0,0,0,0,1), z_1 = (1,0,0,0,0), z_2 = (1,1,0,0,0), z_3 = (1,0,0,1,0), z_4 = (1,0,0,0,1), z_5 = (0,1,1,0,0)$ will form a subdivision(each edge subdivided at most once) of $K_{4,5}$. Let Φ_1 be an embedding of $K_{4,5}$ in S_3. By Euler's formula, we have 7 faces in Φ_1. Now, to fit the vertex $(1,0,1,0,0)$ in Φ_1, it is required a face of length at least 10, which is not possible in Φ_1. Hence $n \leq 4$.
Case 2: $n = 4$. Suppose $|F_i| \geq 4$ for some i. Let us choose the least case, $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4$. Let Φ_2 be an embedding of a $K_{3,7}$ in S_3. By Euler's formula, we have 7 faces in Φ_2 and we have the following possibilities:

1.$f_4 = 3, f_6 = 2, f_8 = 1, f_{10} = 1$
2.$f_4 = 4, f_6 = 1, f_{10} = 2$
3.$f_4 = 4, f_8 = 2, f_{10} = 1$
4.$f_4 = 3, f_6 = 1, f_8 = 3$
5.$f_4 = 2, f_6 = 3, f_8 = 2$, where f_i is a face of length i.
Now, we have to add two sets of $K_{3,3}$ to restore $\Gamma_N(R)$. For that, it is essential to have at least 6 faces of length at least six, which is infeasible. Hence $|F_i| \leq 3$ for all i.
Suppose $|F_i| = |F_j| = 3$ for $i \neq j$, then also we cannot insert a $K_{3,4}$ in any of the faces. From all the above arguments we conclude that, $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$.
Case 3: $n = 3$
Suppose $|F_i| > 13$ for some i, then $\Gamma_N(R)$ must contain a $K_{3,15}$ as a subgraph. By Theorem 2.2, $g(\Gamma_N(R)) \geq 4$. Suppose $|F_1| \geq 5, |F_2| \geq 5$, then $\Gamma_N(R)$ posses a $K_{4,9}$ which has genus greater than three. Suppose $|F_1| \geq 3, |F_2| \geq 3$ and $|F_3| \geq 8$, then also we will arrive at a contradiction. By integrating all the above, R lies in the following possibilities:
$\mathbb{Z}_2 \times \mathbb{F}_4 \times \mathbb{F}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{13}, \mathbb{Z}_2 \times \mathbb{F}_4 \times \mathbb{Z}_5, \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5, Z_3 \times \mathbb{F}_4 \times \mathbb{F}_4, \mathbb{F}_4 \times \mathbb{F}_4 \times \mathbb{F}_4, \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_7, \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5, \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_7, \mathbb{F}_4 \times \mathbb{F}_4 \times \mathbb{Z}_5, \mathbb{F}_4 \times \mathbb{F}_4 \times \mathbb{Z}_7$. Clearly $\Gamma_N(Z_3 \times \mathbb{F}_4 \times \mathbb{F}_4)$ is a subgraph of $\Gamma_N(\mathbb{F}_4 \times \mathbb{F}_4 \times \mathbb{F}_4)$ and $\Gamma_N(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5)$ is a subgraph of $\Gamma_N(\mathbb{F}_4 \times \mathbb{F}_4 \times \mathbb{Z}_5)$, $\Gamma_N(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_7)$ and $\Gamma_N(\mathbb{F}_4 \times \mathbb{F}_4 \times \mathbb{Z}_7)$. Also $\Gamma_N(\mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5)$ contains a copy of $K_{4,9}$ and by Theorem 2.2, the above graph will score genus at least 4. In the case of $R = \mathbb{Z}_2 \times \mathbb{F}_4 \times \mathbb{Z}_5$, using the above embedding method, $\Gamma_N(R)$ cannot be embeddable in S_3. Suppose $R = Z_3 \times \mathbb{F}_4 \times \mathbb{F}_4$. Consider $\Psi = \{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2 \ldots \beta_{11}\}$, where $\alpha_i = (0, a_i, 0), 1 \leq i \leq 3, a_i \in \mathbb{F}_4$ and $\beta_1 = (1, 0, 0), \beta_2 = (2, 0, 0), , \beta_j = (0, 0, b_q), 3 \leq j \leq 5, \beta_k = (1, 0, b_q), 6 \leq k \leq 8, \beta_l = (2, 0, b_q), 9 \leq l \leq 11, 1 \leq q \leq 3, b_q \in \mathbb{F}_4$. Let $G_1 = \langle \Psi \rangle - \{\beta_1\beta_3, \beta_1\beta_4, \beta_1\beta_5, \beta_2\beta_3, \beta_2\beta_4, \beta_2\beta_5\}$. Clearly G_2 is $K_{3,11}$. Now, fix a representation for $K_{3,11}$ in S_3. By Euler's formula, G_1 has 15 faces. Let f_i be the number of faces on length i. Then we have the succeeding opportunities:
1.$f_{10} = 1, f_4 = 14$
2.$f_8 = 1, f_6 = 1, f_4 = 13$
3.$f_6 = 3, f_4 = 12$
Now we have to inject the set $\{(1, a_i, 0), (2, a_i, 0), 1 \leq i \leq 3, a_i \in F_4\}$, which has three common neighbors $\beta_3, \beta_4, \beta_5$. Since these three vertices are not adjacent in G_1, one must have at least 6 faces of length at least 6, which cannot be happened. Hence we get a contradiction. Also $\Gamma_N(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5))$ is not embeddable in S_3 due to the very same cause as above. Hence $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{13}, Z_2 \times \mathbb{Z}_3 \times \mathbb{Z}_7$ or $\mathbb{Z}_2 \times \mathbb{F}_4 \times \mathbb{F}_4$.
Case 4: $n = 2$
This case is straightforward using Remark 3.1.
When $n = 2$, the converse part is clear utilizing Theorem 2.2. Also, if $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{13}$, then $\Gamma_N(R) \cong (K_2 \cup K_1) + \overline{K_{12}}$, which will receive genus 3 by Theorem 2.2. Addition with that, Theorem 9 of [5], shows that $g(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_7) = 3$. By Theorem 3.7 and Example 3.6 in [8] we obtain, $g(\mathbb{Z}_2 \times \mathbb{F}_4 \times \mathbb{F}_4) \geq 3$ and $g(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3) \geq 3$ and Figures. 3, 4 implies that, $g(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3) = g(\mathbb{Z}_2 \times \mathbb{F}_4 \times \mathbb{F}_4) = 3$. □

Theorem 4.3. Suppose $R = \prod_{i=1}^{n} R_i \times \prod_{j=1}^{m} F_j$, where each R_i is local with maximal ideal $\mathfrak{m}_i$ and F_j is a field and $n, m \geq 1$. Then $g(\Gamma_N(R)) = 2$ if and only if R is any one of the rings: $\mathbb{Z}_2 \times \mathbb{Z}_9, \mathbb{Z}_2 \times \frac{\mathbb{Z}_3(x)}{(x^2)}, \mathbb{Z}_4 \times \mathbb{Z}_7, \frac{\mathbb{Z}_2(x)}{(x^2)} \times \mathbb{Z}_7, \mathbb{Z}_4 \times \mathbb{F}_8$ or $\frac{\mathbb{Z}_2(x)}{(x^2)} \times \mathbb{F}_8$.

Proof. Suppose $n + m \geq 4$. Then the collection$\{(0, 0, 1, 0), (z_1, 0, 1, 0), (0, 0, 0, 1), (z_1, 0, 0, 1), (z_1, 0, 1, 1), (z_1, 0, 0, 0), (0, 1, 0, 0), (z_1, 1, 0, 0), (1, 1, 0, 0), (u_1, 0, 0, 0), (u_1, 1, 0, 0), (1, 0, 0, 0), z_1 \in Z(R_1), u_1 \in R^{\times}\}$ will produce a $K_{5,7}$ which has genus greater than four. Hence $n + m \leq 3$.
Case 1: $n + m = 3$
Suppose $n \geq 2$, then by Theorem 3.3, we will arrive at a contradiction. Hence $R = R_1 \times F_1 \times F_2$. Suppose $|\mathfrak{m}_1^*| \geq 2$, then $\Gamma_N(R)$ possess a $K_{8,9}$, which has genus exceeding 3. Hence $|\mathfrak{m}_1^*| = 1$. Again, if $|F_i| \geq 3, i = 1, 2$, then the resulting graph will contain a $K_{5,8}$, a contradiction. Hence by Theorem 3.4, we have to look over only the ring $\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_2$. For, consider $\Omega = \{c_1, c_2, c_3, c_4, d_1 d_2, d_3, d_4, d_5, d_6, d_7\} - \{d_1d_3, d_1d_4, d_1d_7, d_2d_3, d_2d_4, d_2d_7, d_3d_7, d_4d_7, d_5d_7, d_6d_7\}$, where $c_1 = (0, 1, 0), c_2 = (2, 1, 0), c_3 = (0, 2, 0), c_4 = (2, 2, 0), d_1 = (1, 0, 0), d_2 = (3, 0, 0), d_3 = (0, 0, 1), d_4 = (2, 0, 1), d_5 = (1, 0, 1), d_6 = (3, 0, 1), d_7 = (2, 0, 0)$. We know $\langle \Omega \rangle$ is $K_{4,7}$ and let Φ_3 be an representation of $K_{4,7}$ in S_3. Then it follows any one of the possibilities:

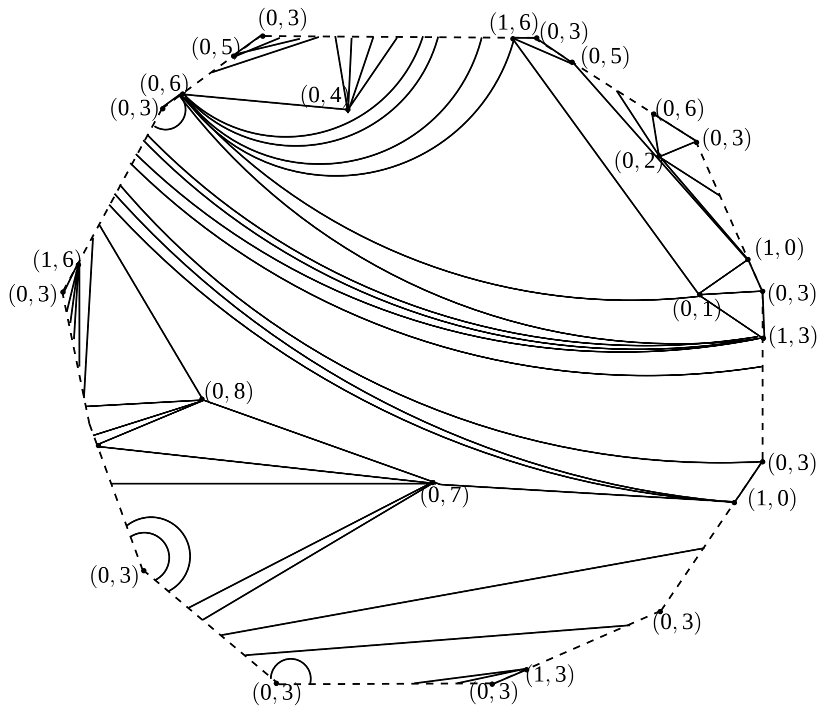

Figure 5. An embedding of $\Gamma_N(\mathbb{Z}_2 \times \mathbb{Z}_9)$ in S_3

In the above figure, we can easily insert the 6 vertices that are adjacent with $(0, 3), (0, 6)$

1.$f_8 = 1, f_4 = 12$
2.$f_6 = 2, f_4 = 11.$
But it is required at least 4 faces of size at least 6 to recover $\Gamma_N(\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_2)$, which is unachievable.

Case 2: $n + m = 2$

Suppose $|\mathfrak{m}_1^*| \geq 3$, then by the argument as above, we conclude that $|\mathfrak{m}_1^*| \leq 2$.

Subcase 1: $|\mathfrak{m}_1^*| = 2$. Suppose $|F_1| \geq 3$, then the graph will have a subgraph isomorphic to $K_{6,8}$, which has genus greater than 3. Hence $R = \mathbb{Z}_2 \times \mathbb{Z}_9$ or $\mathbb{Z}_2 \times \frac{\mathbb{Z}_3(x)}{(x^2)}$.

Subcase 2: $|\mathfrak{m}_1^*| = 1$. Suppose $|F_1| \geq 9$, then the graph will have a subgraph isomorphic to $K_{3,16}$, which has genus greater than 3. Hence $|F_1| \leq 8$. Hence $R = \mathbb{Z}_4 \times \mathbb{Z}_7, \frac{\mathbb{Z}_2(x)}{(x^2)} \times \mathbb{Z}_7, \mathbb{Z}_4 \times \mathbb{F}_8$ or$\frac{\mathbb{Z}_2(x)}{(x^2)} \times \mathbb{F}_8$.

Conversely, $\Gamma_N(\mathbb{Z}_4 \times \mathbb{Z}_7) \cong \Gamma_N(\frac{\mathbb{Z}_2(x)}{(x^2)} \times \mathbb{Z}_7) \cong K_{3,12}$ and $\Gamma_N(\mathbb{Z}_4 \times \mathbb{F}_8) \cong \Gamma_N(\frac{\mathbb{Z}_2(x)}{(x^2)} \times \mathbb{F}_8) \cong K_{3,14}$ and by Theorem 2.2, they will receive genus 3. Suppose $R = \mathbb{Z}_2 \times \mathbb{Z}_9$ or $\mathbb{Z}_2 \times \frac{\mathbb{Z}_3(x)}{(x^2)}$, then $\Gamma_N(R)$ contains $K_{5,6}$. Then $g(\Gamma_N(R)) \geq 3$ and by Figure. 5, $g(\Gamma_N(R)) = 3$. □

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