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Oscillation of a Delayed Quaternion-Valued Fuzzy Recurrent Neural Networks on Time Scales

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Abstract In this paper, we consider quaternion-valued fuzzy recurrent neural networks with time-varying delays on time scales. Different from the previous literature, we use a direct method to obtain our theoretical results to avoid decomposing the model into real-valued or complex-valued systems. Then, we obtain some sufficient conditions on the existence, uniqueness, and Sp-global exponential stability of weighted Stepanov-like pseudo almost periodic solution on time scales of the considered model by applying inequality analysis techniques on time scales, a fixed point theorem, and composition theorem, and by constructing an appropriate Lyapunov function. At the end of this work, we give a numerical example and simulations to illustrate the effectiveness of the obtained results.

1 Introduction

In recent decades, neural networks that take value in real (RVNNs) and complex (CVNNs) domains have been widely studied. However, sometimes RVNNs or CVNNs are not applicable to certain engineering problems, especially when the data is three-dimensional or more. (For example, four-dimensional signals, color images, and body images.) As a result, more general and advanced NNs than CVNNs, which are quaternion-valued neural networks (QVNNs), have been used. In addition, it should be noted that the quaternion was introduced in 1843 by the British mathematician W.R. Hamilton ([18]) is widely used in several domains, such as physics, computer graphics, and modern mathematics ([16, 24]), to generalize the properties of complex numbers to multidimensional space. Moreover, the quaternion representation is more compact and the calculation speed is faster than the matrix representation. QVNNs can therefore handle multi-level information and require only half of the CVNN connection weight parameters ([20]). Likewise, in the case of quaternion-valued recurrent neural networks (QVRNNs), everything we have described remains valid because it is capable of learning characteristics and sequential data modelling. Furthermore, QVRNNs contain other forms of NNs, such as QVNNs ([28]) and quaternion-valued Hopfield NNs ([19]). Because all of these applications rely heavily on their dynamics, the study of various dynamic behaviors for quaternionic neural networks has piqued the interest of many researchers ([6, 29]).
The fuzzy theory was first conceived by L.A. Zadeh in the 1960s. However, it took almost 20 years before it became more widely used at a practical level. Not long ago, Yang and Yang ([31]) developed a fuzzy neural network (FNNs) based on conventional NNs that incorporates fuzzy logic into the conventional NNs structure and preserves local cell-to-cell connectivity. Besides, it is important to consider both the fuzzy logic and the delay effects on the dynamic behavior of NNs ([21, 30]).
As we know, discrete and continuous recurrent neural networks play a key role in theoretical research and applications. Also, discrete-time neural networks are more beneficial and convenient for numerical simulation and computation than continuous-time NNs. Hence, not only do we need to study continuous-time neural networks, but we also need to study discrete NNs. To avoid the difficulties of studying the dynamic properties of continuous and discrete systems, respectively, it is helpful to study these properties on time scales, which Stefan Hilger ([17]) introduced in his PhD thesis in order to unify continuous and discrete analysis. As a result, using

time scale dynamic systems, subjects such as the existence of a solution, stability, floquet theory, periodicity, and the dynamics of NNs can be studied more precisely and broadly. ([10, 11, 12]). Recently, the existence and stability of the periodic solution on time scales has been one of the most attractive themes in the context of various kinds of abstract dynamic equations ([9]), partial dynamic equations ([13]), integro-dynamic equations ([2]) and general dynamic systems ([25]). For example, in ([22]) the authors studied the existence and global exponential stability of pseudo almost periodic solution for neutral QVNNs with delays in the leakage term, and the authors in ([32]) obtained some sufficient conditions for the global asymptotic stability of FNNs. Very recently, M. Es-saiydy and M. Zitane ([8]) introduced another notion called weighted Stepanov-like pseudo almost periodicity on time scales (WS^pPAP), which naturally generalizes the classical notion of periodicity and its various extensions (anti-periodicity, almost periodicity, weighted pseudo almost periodicity, Stepanov pseudo almost periodicity, etc.).
To our knowledge, no paper has been published on the existence and Stepanov global exponential stability of the WS^pPAP solution of QVFRNNs with time-varying delays on time scales. This is important both in terms of theory and application, which is also a very difficult issue. Motivated by the above statement, we summarized the innovation points of this paper as follows : (I) we integrate fuzzy operations into quaternion-valued RNNs with time-varying delays on time scales. (II) For the time being, this is the first time that the WS^pPAP dynamics of a delayed QVFRNNs are being investigated on time scales, which can unify both continuous time and discrete time cases of RNNs. The QVFRNNs proposed in this work also contain real-VFRNNs and complex-VFCNNs as their special cases. (III) We take into account another oscillation space that has never been taken into account in the different classes of recurrent neural networks.
The organization of this paper is briefly described as follows: In Section 2, we make some preparations for the next sections. In Section 3, we will provide the model of QVFRNNs. In Sections 4 and 5, some sufficient conditions are derived to ensure the existence and S^p-global exponential stability of a unique weighted Stepanov-like pseudo almost periodic solution of considered QVFRNNs. In Section 6, we provide a numerical example to illustrate the feasibility of our abstract results.

2 Preliminaries and functions spaces

In this section, we shall first recall some fundamental definitions, lemmas which are used in what follows. Throughout this paper we fix $p \geq 1$ and $(X, \| \, . \, \|)$ is a Banach spaces. We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the set of positive integers, the set of integers, the set of real and the set of complex numbers respectively.

2.1 The algebra of quaternions

Definition 2.1 ([26]). The algebra of quaternions $\mathbb{Q}$ is an extension of complex numbers defined in a space composed of four elements denoted 1, i, j, and k representing a rotation. Element 1 corresponds to identity. The skew field of the quaternion is determined by

$$\mathbb{Q} := \{x; x = x^R + x^I i + x^J j + x^K k\},$$

where x^R, x^I, x^J and x^K are real numbers and the elements i, j, and k obey the Hamilton's multiplication rules:

- All possible products of i, j, and k:

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j, \quad i^2 = j^2 = k^2 = -1.$$

- The quaternion conjugate is defined by: $\bar{x} = x^R - x^I i - x^J j - x^K k.$
- The norm of x is defined by : $| \, x \, |_{\mathbb{Q}} = \sqrt{x\bar{x}}.$

Lemma 2.2 ([26]). For any $u, v \in \mathbb{Q}$, if $P \in \mathbb{Q}^{n\times n}$ is a positive-definite Hermitian matrix, then

$$\bar{u}v + \bar{v}u \leq \bar{u}Pu + \bar{v}P^{-1}v.$$

2.2 Essentials of time scales

Definition 2.3 ([3]). Let $\mathbb{T}$ be a time scale, that is, a closed and nonempty subset of $\mathbb{R}$.

(i) The forward and backward jump operators $\sigma, \rho : \mathbb{T} \longrightarrow \mathbb{T}$ and the graininess $\mu : \mathbb{T} \longrightarrow \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \quad \mu(t) = \sigma(t) - t.$$

(ii) The point $t \in \mathbb{T}$ is called left-dense, left-scattered, right-dense, or right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, or $\sigma(t) > t$, respectively

(iii) A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is called right-dense continuous or rd-continuous provided that it is continuous at all right-dense points in $\mathbb{T}$ and its left-side limits exist (finite) at left-dense points in $\mathbb{T}$. A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is called continuous if and only if it is both left-dense continuous and right-dense continuous.

(iv) If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$.

(v) A function $p : \mathbb{T} \longrightarrow \mathbb{R}$ is called μ-regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \longrightarrow \mathbb{R}$ will be denoted by $\mathfrak{R} = \mathfrak{R}(\mathbb{T}) = \mathfrak{R}(\mathbb{T};\mathbb{R})$.

(vi) We define the set $\mathfrak{R}^+$ of all positively regressive elements by $\mathfrak{R}^+ = \mathfrak{R}^+(\mathbb{T}) = \mathfrak{R}^+(\mathbb{T};\mathbb{R}) = \{p \in \mathfrak{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

(vii) Let $a, b \in \mathbb{T}$, with $a \leq b$, $[a, b]$, $[a, b)$, $(a, b]$ and (a, b) being the usual intervals on the real line. The intervals $[a, a)$, $(a, a]$, (a, a) are understood as the empty set, and we use the following symbols :

$$[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}, \quad [a, b)_{\mathbb{T}} = [a, b) \cap \mathbb{T}, \quad (a, b]_{\mathbb{T}} = (a, b] \cap \mathbb{T}, \quad (a, b)_{\mathbb{T}} = (a, b) \cap \mathbb{T}.$$

Definition 2.4 ([3]). A time scale $\mathbb{T}$ is called invariant under translations if

$$\Pi = \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}; \forall t \in \mathbb{T}\} \neq \{0\}.$$

Definition 2.5 ([3]). If $p \in \mathfrak{R}$, then we define the exponential function by :

$$e_p(t, s) = \exp\left\{ \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau \right\},$$

for $s, t \in \mathbb{T}$, with the cylinder transformation

$$\xi_m(z) = \begin{cases} \frac{\log(1+hz)}{h}, & if\ h \neq 0, \\ z, & if\ h = 0. \end{cases}$$

Definition 2.6 ([3]). For $p \in \mathfrak{R}$, define a circle minus p by

$$\ominus p = -\frac{p}{1 + \mu p}.$$

Lemma 2.7 ([3]). Let $p, q \in \mathfrak{R}$, So,

1) $e_0(t, s) = 1$ and $e_p(t, t) = 1$;
2) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;
3) $(e_p(t, s))^\Delta = p(t)e_p(t, s)$;
4) $\int_a^b e_p(c, \sigma(t))p(t)\Delta t = e_p(c, a) - e_p(c, b)$, $(a, b, c \in \mathbb{T})$.

Definition 2.8 ([3]). For $f : \mathbb{T} \to X$ and $s \in \mathbb{T} \setminus \{\max \mathbb{T}\}$, $f^\Delta(t) \in X$ is the Δ-derivative of f at s if for $\varepsilon > 0$, there is a neighborhood V of s such that for $t \in V$,

$$\| f(\sigma(s)) - f(t) - f^\Delta(s)(\sigma(s) - t) \| < \varepsilon \mid \sigma(s) - t \mid .$$

Hence, f is Δ-differentiable on $\mathbb{T}$ provided that $f^\Delta(s)$ exists for $s \in \mathbb{T}$.

Definition 2.9 ([1]). $f : \mathbb{T} \to X$ is a delta measurable function if there exists a simple function sequence $\{f_k : k \in \mathbb{N}\}$ such that $f_k(s) \to f(s)$ a.e. in $\mathbb{T}$.

Definition 2.10 ([1]). $f : \mathbb{T} \to X$ is a delta integrable function if there exists a simple function sequence $\{f_k : k \in \mathbb{N}\}$ such that $f_k(s) \to f(s)$ a.e. in $\mathbb{T}$ and,

$$\lim_{k\to\infty} \int_{\mathbb{T}} \parallel f_k(s) - f(s) \parallel \Delta s = 0.$$

Then, the integral of f is defined as

$$\int_{\mathbb{T}} f(s)\Delta s = \lim_{k\to\infty} \int_{\mathbb{T}} f_k(s)\Delta s.$$

Definition 2.11 ([1]). For $p \geq 1$, $f : \mathbb{T} \to X$ is called locally L^p $\Delta-$integrable if f is $\Delta-$measurable and for any compact $\Delta-$measurable set $E \subset \mathbb{T}$, the $\Delta-$integral

$$\int_E \parallel f(s) \parallel^p \Delta s < \infty.$$

The set of all L^p $\Delta-$integrable functions is denoted by $L^p_{loc}(\mathbb{T}; X)$.

2.3 Weighted Stepanov-like pseudo almost periodic functions on $\mathbb{T}$

This subsection is devoted to recall some definitions and the important properties of weighted Stepanov-like pseudo almost periodic functions on time scales introduced by M. Es-saiydy and M. Zitane ([8]).

Definition 2.12 ([27]). $B \subset \mathbb{T}$ is called relatively dense in $\mathbb{T}$ if there exists $l > 0$ such that $[a, a+a]_{\mathbb{T}} \cap B \neq 0$, $a \in \mathbb{T}$. We call l the inclusion length.

Definition 2.13 ([8]). A function $f \in BC(\mathbb{T}, \mathbb{Q})$ is called almost periodic on $\mathbb{T}$ if for every $\varepsilon > 0$, the $\varepsilon-$translation set of f :

$$E(f, \varepsilon) = \{\tau \in \Pi;\ \parallel f(t+\tau) - f(t) \parallel_{\mathbb{Q}} < \varepsilon,\ \forall t \in \mathbb{T}\}$$

is relatively dense in Π. The space of all such functions is denoted by $AP(\mathbb{T}, \mathbb{Q})$.

Let $\mathbb{U}$ denote the collection of functions (weights) $\mu : \mathbb{T} \to (0, \infty)_{\mathbb{T}}$, which are locally integrable over $\mathbb{T}$ such that $\mu > 0$ almost everywhere. Let $\mu \in \mathbb{U}$, for $r \in \Pi$ with $r > 0$, we denote

$$\mu(\Omega_r) = \int_{\Omega_r} \mu(t)\Delta t,$$

where $\Omega_r = [t_0 - r, t_0 + r]_{\mathbb{T}}$ $\big(t_0 = \min\{[0, \infty)_{\mathbb{T}}\}\big)$.

Consequently, we define the space of weights by

$$\mathcal{M} = \left\{\mu \in \mathbb{U} : \inf_{t\in\mathbb{T}} \mu(t) > 0,\ \lim_{t\to\infty} \mu(\Omega_r) = \infty\right\}.$$

Throughout this paper, we fix $1 \leq p < \infty$, $\mu \in \mathcal{M}$, and $\mathbb{T}$ be an almost periodic time scales.

Definition 2.14 ([8]). A function $f \in BC(\mathbb{T}, \mathbb{Q})$ is said to be $\mu-$ergodic $(f \in PAP_0(\mathbb{T}, \mathbb{Q}, \mu))$ if

$$\lim_{r\longrightarrow+\infty} \frac{1}{\mu(\Omega_r)} \int_{\Omega_r} \parallel f(t) \parallel_{\mathbb{Q}} \mu(t)\Delta t = 0.$$

Definition 2.15 ([8]). A function such that $f(.) \in BC(\mathbb{T}, \mathbb{Q})$ is said to be weighted pseudo almost periodic $(f \in PAP(\mathbb{T}, \mathbb{Q}, \mu))$ if f is written in the following form:

$$f = g + \phi,$$

where $g \in AP(\mathbb{T}, \mathbb{Q})$ and $\phi \in PAP_0(\mathbb{T}, \mathbb{Q}, \mu)$.

We set,

$$K = \begin{cases} \inf\{ \mid \tau \mid; \tau \in \mathbb{T}, \tau \neq 0\}, & if\, \mathbb{T} \neq \mathbb{R}, \\ 1, & if\, \mathbb{T} = \mathbb{R}. \end{cases}$$

Let $f \in L^p_{loc}(\mathbb{T}, \mathbb{Q})$. Define :

- $\| \, . \, \|_{S^p}: L^p_{loc}(\mathbb{T}, \mathbb{Q}) \to \mathbb{R}^+$ as : $\| \, f \, \|_{S^p} = \sup_{t \in \mathbb{T}} \left(\frac{1}{K} \int_t^{t+K} \mid f(s) \mid^p_{\mathbb{Q}} \Delta s \right)^{\frac{1}{p}}$.
- $C_{rd}\,(\mathbb{T}; \mathbb{Q}) = \{ f : \mathbb{T} \to \mathbb{Q} : f$ is rd-continuous$\}$.
- $BC_{rd}\,(\mathbb{T}; \mathbb{Q}) = \{ f : T \to \mathbb{Q} : f$ is bounded and rd-continuous$\}$.
- $L^p_{loc}\,(\mathbb{T}; \mathbb{Q}) = \{ f : \mathbb{T} \to \mathbb{Q} : f$ is locally $L^p\ \Delta -$ integrable$\}$.
- $BS^p\,(\mathbb{T}; \mathbb{Q}) = \{ f \in L^p_{loc}\,(\mathbb{T}; \mathbb{Q}) : \| \, f \, \|_{S^p} < \infty \}$.

Definition 2.16 ([8]). A function $f \in BS^p(\mathbb{T}, \mathbb{Q})$ is called Stepanov-like almost periodic on $\mathbb{T}$, $(f \in S^pAP(\mathbb{T}, \mathbb{Q}))$ if for every $\varepsilon > 0$, the $\varepsilon-$translation set of f :

$$T(f, \varepsilon) = \{ \tau \in \Pi; \, \| \, f(t+\tau) - f(t) \, \|_{S^p} < \varepsilon, \, \forall t \in \mathbb{T} \}$$

is relatively dense in Π.

Lemma 2.17 ([27]). For $\varepsilon > 0$, $T(f, \varepsilon)$ is relatively dense in $\mathbb{R}$ if and only if $T(f, \varepsilon)$ is relatively dense in Π.

Definition 2.18 ([8]). A function $f \in BS^p(\mathbb{T}, \mathbb{Q})$ is said to be weighted Stepanov-like ergodic on $\mathbb{T}$ $(f \in WS^pPAP_0(\mathbb{T}, \mathbb{Q}, \mu))$ if :

$$\lim_{r \to +\infty} \frac{1}{\mu(\Omega_r)} \int_{\Omega_r} \left(\frac{1}{K} \int_t^{t+K} \mid f(s) \mid^p_{\mathbb{Q}} \Delta s \right)^{\frac{1}{p}} \mu(t) \Delta t = 0.$$

Definition 2.19 ([8]). A function $f \in BS^p(\mathbb{T}, \mathbb{Q})$ is said to be weighted Stepanov-like pseudo almost periodic on $\mathbb{T}$ or briefly S^p-weighted pseudo almost periodic $(f \in WS^pPAP(\mathbb{T}, \mathbb{Q}, \mu))$ if f is written in the following form :

$$f = g + \phi,$$

where $g \in S^pAP(\mathbb{T}, \mathbb{Q})$ and $\phi \in WS^pPAP_0(\mathbb{T}, \mathbb{Q}, \mu)$.

Now, we recall the Bochner-like transform on time scales.

If $\mathbb{T} \neq \mathbb{R}$, we fix a left scattered point $\omega \in \mathbb{T}$, there is a unique $n_t \in \mathbb{Z}$ such that $t - n_t K \in [\omega, \omega + k)_{\mathbb{T}}$. Let

$$N_t = \begin{cases} t, & \mathbb{T} = \mathbb{R}, \\ n_t & \mathbb{T} \neq \mathbb{R}. \end{cases}$$

Definition 2.20 ([27]). Let $f \in BS^p(\mathbb{T}, \mathbb{Q})$. The Bochner-like transform of f is the function $f^c : \mathbb{T} \times \mathbb{T} \to X$ defined for all $t, s \in \mathbb{T}$ by

$$f^c(t, s) = f(N_t K + s).$$

And we have

$$\| \, f \, \|_{S^p} = \| \, f^c \, \|_{\infty}.$$

Definition 2.21 ([8]). A function $f \in BS^p(\mathbb{T}, \mathbb{Q})$ is said to be μ-Stepanov-like pseudo almost periodic if its Bochner-like transform f^c is μ-pseudo almost periodic in the sense that there exist two functions g, h such that $f^c = g^c + h^c$, where $g^c \in AP(\mathbb{T}, BS^p(\mathbb{T}, \mathbb{Q}))$ and $h^c \in PAP_0(\mathbb{T}, BS^p(\mathbb{T}, \mathbb{Q}), \mu)$.

Lemma 2.22 ([8]). If $f_1, f_2 \in WS^pPAP(\mathbb{T}, \mathbb{Q}^n, \mu)$, then $f_1 + f_2, f_1 f_2 \in WS^pPAP(\mathbb{T}, \mathbb{Q}^n, \mu)$.

Proposition 2.23 ([8]). $(WS^pPAP(\mathbb{T}, \mathbb{Q}^n, \mu), \| \, . \, \|_{S^p})$ is a Banach space.

3 Model Description And Hypotheses

In this paper, we consider the following quaternion-valued fuzzy recurrent neural networks (QVFRNNs) with time-varying delays on time scales which are defined in the following lines:

$$
\begin{aligned}
x_l^{\Delta}(t) = -a_l x_l(t) + \sum_{m=1}^{n} \alpha_{lm}(t) f_m(x_m(t)) + \sum_{m=1}^{n} \beta_{lm}(t) f_m(x_m(t-\delta_m(t)))+ \\
\bigvee_{m=1}^{n} \eta_{lm}(t) h_m\left(x_m(t-\delta_m(t))\right) + \bigwedge_{m=1}^{n} \lambda_{lm}(t) h_m\left(x_m(t-\delta_m(t))\right) + I_l(t), \quad t \in \mathbb{T}.
\end{aligned}
\tag{3.1}
$$

where $l \in \{1,2,...,n\}$, n corresponds to the number of units in neural networks; $\mathbb{T}$ is an almost periodic time scale; $\mathbb{Q}$ is a Quaternion algebra; $x_l(t) \in \mathbb{Q}$ corresponds to the state of the lth unit at time t; $a_l(t) = diag(a_1(t), a_2(t), ..., a_n(t))$ denotes the rate which the ith neuron will reset its potential to the resting state in isolation when disconnected from the network and external input, f_m, and $h_m : \mathbb{Q} \to \mathbb{Q}$ are output transfer functions; $\eta_{lm}(.),\ \lambda_{lm}(.) \in \mathbb{Q}$, are the elements of fuzzy feedback MIN template and fuzzy feedback MAX template, respectively; $\bigvee$, $\bigwedge$ denote the fuzzy AND and fuzzy OR operation, respectively; $\alpha_{lm}(.)$, $\beta_{lm}(.)$, present the connection weights, the discretely delayed connection weights, and the distributively delayed connection weights, of the mth neuron on the l neuron, respectively. $\delta_m(.)$ corresponds to transmission delays at time t and satisfy $t - \delta_m(t) \in \mathbb{T}$ for $t \in \mathbb{T}$; and $I_l(.)$ denote the state input of the lth neuron.

The initial condition of system (3.1) is of the form

$$x_l(s) = \rho_l(s), \quad s \in (-\infty, 0]_{\mathbb{T}},$$

where ρ_l is rd-continuous and $\rho_l \in L^p_{loc}\left((-\infty,0]_{\mathbb{T}}, \mathbb{Q}\right)$ $l = 1,...,n.$

Now, we will list a few hypotheses which will be used for the rest of this article.

$(\mathcal{H}_1)$: Let $\mu \in \mathcal{M}$, for all $t \in \Pi$

$$\overline{\lim_{|t|\to\infty}} \frac{\mu(t+\tau)}{\mu(t)} < \infty \quad \text{and} \quad \overline{\lim_{|t|\to\infty}} \frac{\mu(\Omega_{r+\tau})}{\mu(\Omega_r)} < \infty.$$

$(\mathcal{H}_2)$: For all $1 \le l, m \le n$, the functions $a_{lm}(.)$, $\alpha_{lm}(.)$, $\beta_{lm}(.) \in WS^pPAP(\mathbb{T}, \mathbb{Q}, \mu)$ and functions $\eta_{lm}(.)$, $\lambda_{lm}(.)$, $\delta_m(.) \in S^pAP(\mathbb{T}, \mathbb{Q}) \cap C^1_{rd}(\mathbb{T}, \mathbb{Q})$, such that

$$0 \le \delta_m(.) \le \overline{\delta}, \quad 0 \le \delta^* - \delta^{\Delta}_m(.) < 1 - \delta^{\Delta}_m(.).$$

$(\mathcal{H}_3)$: There exist positive constants L_{f_l}, L_{h_l} such that for any u, $v \in \mathbb{Q}$, the activity functions f_l, $h_l \in C_{rd}(\mathbb{Q}, \mathbb{Q})$ satisfying

$$
\begin{aligned}
\mid f_l(u) - f_l(v) \mid_{\mathbb{Q}} &\le L_{f_l} \mid u - v \mid_{\mathbb{Q}}, \\
\mid h_l(u) - h_l(v) \mid_{\mathbb{Q}} &\le L_{h_l} \mid u - v \mid_{\mathbb{Q}}.
\end{aligned}
$$

Furthermore, we suppose that $f_l(0) = h_l(0) = 0$.
As a convenience, we have introduced these notations which simplify the writing of the equations:

$$f^+ = \sup_{t\in\mathbb{T}} \mid f(t) \mid_{\mathbb{Q}},\ \overline{f} = \inf_{t\in\mathbb{T}} \mid f(t) \mid_{\mathbb{Q}},\ a_l^* = \sup_{t\in\mathbb{T}} a_l(t) > 0,\ \check{a}_l = \inf_{t\in\mathbb{T}} \overline{a}_l + \inf_{t\in\mathbb{T}} a_l,\ \bar{\mu} = \sup_{t\in\mathbb{T}} \mu(t),$$

and

$$L^* = \max_{1\le l\le n} \left\{ M_l^{\frac{1}{q}}(q) M_l^{\frac{1}{p}}(p) \left[\sum_{m=1}^{n} \alpha^+_{lm} L_{f_m} + \sum_{m=1}^{n} \frac{\beta^+_{lm} L_{f_m}}{(1-\delta^*_m)^{\frac{1}{p}}} + \sum_{m=1}^{n} \frac{\eta^+_{lm} L_{h_m}}{(1-\delta^*_m)^{\frac{1}{p}}} + \sum_{m=1}^{n} \frac{\lambda^+_{lm} L_{h_m}}{(1-\delta^*_m)^{\frac{1}{p}}} \right] \right\},$$

with $M_l(q) = \frac{2+\overline{a}_l \bar{\mu} q}{\overline{a}_l q}$.

4 Weighted Stepanov-Like Pseudo Almost Periodic Solution on $\mathbb{T}$

In this section, we will present a new conditions for the existence, uniqueness, and S^p-global exponential stability of weighted Stepanov-like pseudo almost periodic solution of QVFRNNs (3.1) on time scales based on the Banach fixed point theorem, composition theorem, and the theory of calculus on time scales.

Lemma 4.1 ([5]). Suppose x_m and y_m are two states of system (3.1). Then we have

$$\left|\bigvee_{m=1}^{n} u_{lm}h_m(x_m) - \bigvee_{m=1}^{n} u_{lm}h_m(y_m)\right| \leq \sum_{m=1}^{n} \mid u_{lm} \mid\mid h_m(x_m) - h_m(y_m) \mid \quad 1 \leq m \leq n,$$

and

$$\left|\bigwedge_{m=1}^{n} u_{lm}h_m(x_m) - \bigwedge_{m=1}^{n} u_{lm}h_m(y_m)\right| \leq \sum_{m=1}^{n} \mid u_{lm} \mid\mid h_m(x_m) - h_m(y_m) \mid \quad 1 \leq m \leq n.$$

Lemma 4.2. *Suppose that condition $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ hold. If function $f \in WS^pPAP(\mathbb{T}, \mathbb{Q}, \mu)$, then $f(. - \delta(.)) \in WS^pPAP(\mathbb{T}, \mathbb{Q}, \mu)$.*

Proof. Since $f \in WS^pPAP(\mathbb{T}, \mathbb{Q}, \mu)$, we can write $f = f_1 + f_2$, such that $f_1 \in S^pAP(\mathbb{T}, \mathbb{Q})$ and $f_2 \in WS^pPAP_0(\mathbb{T}, \mathbb{Q}, \mu)$. Obviously, for all $t \in \mathbb{T}$

$$f(t - \delta(t)) = f_1(t - \delta(t)) + f_2(t - \delta(t)) = F_1(t) + F_2(t),$$

where, $F_1(t) = f_1(t - \delta(t))$ and $F_2(t) = f_2(t - \delta(t))$. By the properties of Stepanov-like almost periodic functions, thus $F_1(t) = f_1(t - \delta(t)) \in S^pAP(\mathbb{T}, \mathbb{Q})$.
It remains to show that $F_2 \in WS^pPAP_0(\mathbb{T}, \mathbb{Q}, \mu)$. Indeed, since $f_2 \in WS^pPAP_0(\mathbb{T}, \mathbb{Q}, \mu)$, we get, for all $t \in \mathbb{T}$

$$\Gamma(t) = \left(\frac{1}{K}\int_t^{t+K} \mid f_2(s) \mid_{\mathbb{Q}}^p \Delta s\right)^{\frac{1}{p}} \in WS^pPAP_0(\mathbb{T}, \mathbb{Q}, \mu).$$

Thus,

$$\begin{aligned}
&\lim_{r\to+\infty}\frac{1}{\mu(\Omega_r)}\int_{\Omega_r}\left(\frac{1}{K}\int_t^{t+K} \mid f_2(s-\delta(s)) \mid_{\mathbb{Q}}^p \Delta s\right)^{\frac{1}{p}}\mu(t)\Delta t \\
&\quad= \frac{1}{(1-\delta^{\Delta}(s))^{\frac{1}{p}}}\cdot\lim_{r\to+\infty}\frac{1}{\mu(\Omega_r)}\int_{\Omega_r}\left(\frac{1}{K}\int_{t-\delta(t)}^{t+K-\delta(t+k)} \mid f_2(z) \mid_{\mathbb{Q}}^p \Delta z\right)^{\frac{1}{p}}\mu(t)\Delta t, \\
&\quad\leq \frac{1}{(1-\delta^*)^{\frac{1}{p}}}\cdot\lim_{r\to+\infty}\frac{1}{\mu(\Omega_r)}\int_{\Omega_r}\left(\frac{1}{K}\int_{t-\bar{\delta}}^{t+K-\bar{\delta}} \mid f_2(z) \mid_{\mathbb{Q}}^p \Delta z\right)^{\frac{1}{p}}\mu(t)\Delta t, \\
&\quad\leq \frac{1}{(1-\delta^*)^{\frac{1}{p}}}\cdot\lim_{r\to+\infty}\frac{1}{\mu(\Omega_r)}\int_{\Omega_r} \mid \Gamma(t-\bar{\delta}) \mid_{\mathbb{Q}} \mu(t)\Delta t, \\
&\quad\leq \frac{1}{(1-\delta^*)^{\frac{1}{p}}}\cdot\lim_{r\to+\infty}\frac{1}{\mu(\Omega_r)}\int_{[t_0-r-\bar{\delta}, t_0+r-\bar{\delta}]_{\mathbb{T}}} \mid \Gamma(t) \mid_{\mathbb{Q}} \mu(t+\bar{\delta})\Delta t, \\
&\quad\leq \frac{1}{(1-\delta^*)^{\frac{1}{p}}}\cdot\lim_{r\to+\infty}\frac{1}{\mu(\Omega_r)}\int_{\Omega_{r+\bar{\delta}}} \mid \Gamma(t) \mid_{\mathbb{Q}} \mu(t+\bar{\delta})\Delta t,
\end{aligned}$$

It follows from condition $(\mathcal{H}_1)$ that

$$\begin{aligned}
&\lim_{r\to+\infty}\frac{1}{\mu(\Omega_r)}\int_{\Omega_r}\left(\frac{1}{K}\int_t^{t+K} \mid f_2(s-\delta(s)) \mid_{\mathbb{Q}}^p \Delta s\right)^{\frac{1}{p}}\mu(t)\Delta t \\
&\quad\leq \frac{1}{(1-\delta^*)^{\frac{1}{p}}}\cdot\lim_{r\to+\infty}\frac{1}{\mu(\Omega_r)}\int_{\Omega_{r+\bar{\delta}}} \mid \Gamma(t) \mid_{\mathbb{Q}} \mu(t+\bar{\delta})\Delta t, \\
&\quad= \frac{1}{(1-\delta^*)^{\frac{1}{p}}}\cdot\lim_{r\to+\infty}\frac{\mu(\Omega_{r+\bar{\delta}})}{\mu(\Omega_r)}\cdot\frac{1}{\mu(\Omega_{r+\bar{\delta}})}\int_{\Omega_{r+\bar{\delta}}} \mid \Gamma(t) \mid_{\mathbb{Q}} \mu(t+\bar{\delta})\Delta t, \\
&\quad= 0.
\end{aligned}$$

Which implies that $F_2 \in WS^pPAP_0(\mathbb{T},\mathbb{Q},\mu)$. Consequently, $f(.-\delta(.)) \in WS^pPAP(\mathbb{T},\mathbb{Q},\mu)$. □

Remark 4.3. Our previous composition theorem is more general than [theorem 3.23, ([8])] and [lemma 3, ([5])].

Lemma 4.4. If a function $f \in C_{rd}(\mathbb{Q},\mathbb{Q})$ satisfies condition $(\mathcal{H}_3)$ and $h \in WS^pPAP(\mathbb{T},\mathbb{Q},\mu)$, then $f\circ h \in WS^pPAP(\mathbb{T},\mathbb{Q},\mu)$.

Proof. By the Stepanov-like pseudo almost periodicity of h, one can write $h = h_1 + h_2$ where $h_1 \in S^pAP(\mathbb{T},\mathbb{Q})$ and $h_2 \in WS^pPAP_0(\mathbb{T},\mathbb{Q},\mu)$. For all $t\in\mathbb{T}$. We pose

$$f\circ h(t) = f\circ h_1(t) + f\circ h(t) - f\circ h_1(t) = H_1(t) + H_2(t),$$

where, $H_1(t) = f\circ h_1(t)$ and $H_2(t) = f\circ h(t) - f\circ h_1(t)$. Since $h_1 \in S^pAP(\mathbb{T},\mathbb{Q})$, then for every $\varepsilon > 0$, $T(h_1,\varepsilon) = \{\tau\in\Pi;\ \| h_1(t+\tau) - h_1(t)\|_{S^p} < \varepsilon,\ \forall t\in\mathbb{T}\}$ is relatively dense in Π. From Lemma (2.17) we have, $T(h_1(.),\varepsilon)$ is relatively dense in $\mathbb{R}$ i.e. $\forall\varepsilon>0$, $\exists l>0$, $\forall a\in\mathbb{R}$, $\exists\tau\in[a,a+l]$,

$$\sup_{t\in\mathbb{T}}\left(\frac{1}{K}\int_t^{t+K} | h_1(s+\tau) - h_1(s) |_{\mathbb{Q}}^p \,\Delta s\right)^{\frac{1}{p}} < \frac{\varepsilon}{L_f}.$$

Therefore,

$$\sup_{t\in\mathbb{T}}\left(\frac{1}{K}\int_t^{t+K} | f\circ h_1(s+\tau) - f\circ h_1(s) |_{\mathbb{Q}}^p \,\Delta s\right)^{\frac{1}{p}} < L_f.\frac{\varepsilon}{L_f} < \varepsilon.$$

Then, $T(f\circ h_1(.),\varepsilon)$ is relatively dense in $\mathbb{R}$, which shows that $H_1(.) \in S^pAP(\mathbb{T},\mathbb{Q})$.
Now,

$$\begin{aligned}
&\lim_{r\to+\infty}\frac{1}{\mu(\Omega_r)}\int_{\Omega_r}\left(\frac{1}{K}\int_t^{t+K} | H_2(s) |_{\mathbb{Q}}^p \,\Delta s\right)^{\frac{1}{p}}\mu(t)\Delta t\\
&= \lim_{r\to+\infty}\frac{1}{\mu(\Omega_r)}\int_{\Omega_r}\left(\frac{1}{K}\int_t^{t+K} | f\circ h(s) - f\circ h_1(s) |_{\mathbb{Q}}^p \,\Delta s\right)^{\frac{1}{p}}\mu(t)\Delta t,\\
&\le \lim_{r\to+\infty}\frac{L_f}{\mu(\Omega_r)}\int_{\Omega_r}\left(\frac{1}{K}\int_t^{t+K} | h(s) - h_1(s) |_{\mathbb{Q}}^p \,\Delta s\right)^{\frac{1}{p}}\mu(t)\Delta t,\\
&\le \lim_{r\to+\infty}\frac{L_f}{\mu(\Omega_r)}\int_{\Omega_r}\left(\frac{1}{K}\int_t^{t+K} | h_2(s) |_{\mathbb{Q}}^p \,\Delta s\right)^{\frac{1}{p}}\mu(t)\Delta t,\\
&= 0.
\end{aligned}$$

Consequently, $H_2(.) \in WS^pPAP_0(\mathbb{T},\mathbb{Q},\mu)$, which ends the demonstration. □

Lemma 4.5. *If* $\rho_m(.) \in WS^pPAP(\mathbb{T},\mathbb{Q},\mu)$, $\eta_{lm}(.), \lambda_{lm}(.) \in S^pAP(\mathbb{T},\mathbb{Q})$, $\delta_m(.) \in S^pAP(\mathbb{T},\mathbb{Q})$ *and* $(\mathcal{H}_3)$ *holds, then*

$$\bigvee_{m=1}^{n}\eta_{lm}(.)h_m\left(\rho_m(.-\delta_m(.))\right),\ \bigwedge_{m=1}^{n}\lambda_{lm}(.)h_m\left(\rho_m(.-\delta_m(.))\right) \in WS^pPAP(\mathbb{T},\mathbb{Q},\mu).$$

Proof. By using Lemmas (4.2) and (4.4), we get

$$E_m(.) = h_m\left(\rho_m(.-\delta_m(.))\right) \in WS^pPAP(\mathbb{T},\mathbb{Q},\mu).$$

So, $E_m(.) = A_m(.) + B_m(.)$, where $A_m(.) \in S^pAP(\mathbb{T},\mathbb{Q})$ and $B_m(.) \in WS^pPAP_0(\mathbb{T},\mathbb{Q},\mu)$. Furthermore,

$$\begin{aligned}
\bigvee_{m=1}^{n}\eta_{lm}(t)E_m(t) &= \bigvee_{m=1}^{n}\eta_{lm}(t)A_m(t) + \bigvee_{m=1}^{n}\eta_{lm}(t)E_m(t) - \bigvee_{m=1}^{n}\eta_{lm}(t)A_m(t),\\
&= U_l(t) + V_l(t).
\end{aligned}$$

Where $U_l(t) = \bigvee_{m=1}^{n} \eta_{lm}(t)A_m(t)$ and $V_l(t) = \bigvee_{m=1}^{n} \eta_{lm}(t)E_m(t) - \bigvee_{m=1}^{n} \eta_{lm}(t)A_m(t)$. First, let us show that $U_l(.) \in S^pAP(\mathbb{T},\mathbb{Q})$. We use the fact that $A_m(.), \eta_{lm}(.) \in S^pAP(\mathbb{T},\mathbb{Q})$, then for all $\varepsilon > 0$ and $n \in \mathbb{N}^*$, the following sets $T\left(A_m(.), \frac{\varepsilon}{2n\bar{\eta}}\right)$, $T\left(A_m(.), \frac{\varepsilon}{2n\bar{A}}\right)$ are relatively dense in $\mathbb{R}$, where $\bar{\eta} = \max_{1\leq l,m\leq n} \sup_{s\in\mathbb{T}} \mid \eta(s) \mid_{\mathbb{Q}}$ and $\bar{A} = \max_{1\leq l,m\leq n} \sup_{s\in\mathbb{T}} \mid A(s+\tau) \mid_{\mathbb{Q}}$. Let

$$D = T\left(A_m(.), \frac{\varepsilon}{2n\bar{\eta}}\right) \cap T\left(A_m(.), \frac{\varepsilon}{2n\bar{A}}\right).$$

Then, D is relatively dense in Π. Let $\tau \in D$, $t, s \in \mathbb{T}$, by using the Minkowski inequality, we have

$$\begin{aligned}
&\sup_{t\in\mathbb{T}}\left(\frac{1}{K}\int_t^{t+K} \mid U_l(s+\tau) - U_l(s) \mid_{\mathbb{Q}}^p \Delta s\right)^{\frac{1}{p}}\\
=\;& \sup_{t\in\mathbb{T}}\left(\frac{1}{K}\int_t^{t+K} \left|\bigvee_{m=1}^{n} \eta_{lm}(s+\tau)A_m(s+\tau) - \bigvee_{m=1}^{n} \eta_{lm}(s)A_m(s)\right|_{\mathbb{Q}}^p \Delta s\right)^{\frac{1}{p}}\\
=\;& \sup_{t\in\mathbb{T}}\left(\frac{1}{K}\int_t^{t+K} \left|\bigvee_{m=1}^{n} \eta_{lm}(s+\tau)A_m(s+\tau) - \bigvee_{m=1}^{n} \eta_{lm}(s)A_m(s+\tau)\right.\right.\\
&\left.\left. + \bigvee_{m=1}^{n} \eta_{lm}(s)A_m(s+\tau) - \bigvee_{m=1}^{n} \eta_{lm}(s)A_m(s)\right|_{\mathbb{Q}}^p \Delta s\right)^{\frac{1}{p}},\\
=\;& \sup_{t\in\mathbb{T}}\left(\frac{1}{K}\int_t^{t+K} \left|\bigvee_{m=1}^{n} \eta_{lm}(s+\tau)A_m(s+\tau) - \bigvee_{m=1}^{n} \eta_{lm}(s)A_m(s+\tau)\right|_{\mathbb{Q}}^p \Delta s\right)^{\frac{1}{p}}\\
&+\sup_{t\in\mathbb{T}}\left(\frac{1}{K}\int_t^{t+K} \left|\bigvee_{m=1}^{n} \eta_{lm}(s)A_m(s+\tau) - \bigvee_{m=1}^{n} \eta_{lm}(s)A_m(s)\right|_{\mathbb{Q}}^p \Delta s\right)^{\frac{1}{p}}.
\end{aligned}$$

According to Lemma (4.1), we obtain

$$\begin{aligned}
&\sup_{t\in\mathbb{T}}\left(\frac{1}{K}\int_t^{t+K} \mid U_l(s+\tau) - U_l(s) \mid_{\mathbb{Q}}^p \Delta s\right)^{\frac{1}{p}}\\
\leq\;& \sum_{m=1}^{n} \sup_{t\in\mathbb{T}}\left(\frac{1}{K}\int_t^{t+K} \mid \eta_{lm}(s+\tau) - \eta_{lm}(s) \mid_{\mathbb{Q}}^p \mid A_m(s+\tau) \mid_{\mathbb{Q}}^p \Delta s\right)^{\frac{1}{p}}\\
&+\sum_{m=1}^{n} \sup_{t\in\mathbb{T}}\left(\frac{1}{K}\int_t^{t+K} \mid A_m(s+\tau) - A_m(s) \mid_{\mathbb{Q}}^p \mid \eta_{lm}(s) \mid_{\mathbb{Q}}^p \Delta s\right)^{\frac{1}{p}},\\
\leq\;& \bar{A}\sum_{m=1}^{n} \sup_{t\in\mathbb{T}}\left(\frac{1}{K}\int_t^{t+K} \mid \eta_{lm}(s+\tau) - \eta_{lm}(s) \mid_{\mathbb{Q}}^p \Delta s\right)^{\frac{1}{p}}\\
&+\bar{\eta}\sum_{m=1}^{n} \sup_{t\in\mathbb{T}}\left(\frac{1}{K}\int_t^{t+K} \mid A_m(s+\tau) - A_m(s) \mid_{\mathbb{Q}}^p \Delta s\right)^{\frac{1}{p}},\\
\leq\;& n\bar{A}\frac{\varepsilon}{2n\bar{A}} + n\bar{\eta}\frac{\varepsilon}{2n\bar{\eta}},\\
<\;& \varepsilon.
\end{aligned}$$

This implies that $D \subset T\left(U_l(.), \varepsilon\right)$. Accordingly, $T\left(U_l(.), \varepsilon\right)$ is relatively dense in $\mathbb{R}$, and $U_l(.) \in S^pAP(\mathbb{T},\mathbb{Q})$. It remains to show that $V_l(.) \in WS^pPAP_0(\mathbb{T},\mathbb{Q},\mu)$. Indeed,

$$\frac{1}{\mu(\Omega_r)}\int_{\Omega_r}\left(\frac{1}{K}\int_t^{t+K} \mid V_l(s) \mid_{\mathbb{Q}}^p \Delta s\right)^{\frac{1}{p}} \mu(t)\Delta t$$

$$= \frac{1}{\mu(\Omega_r)}\int_{\Omega_r}\left(\frac{1}{K}\int_t^{t+K}\left|\bigvee_{m=1}^{n}\eta_{lm}(s)E_m(s)-\bigvee_{m=1}^{n}\eta_{lm}(s)A_m(s)\right|_{\mathbb{Q}}^{p}\Delta s\right)^{\frac{1}{p}}\mu(t)\Delta t,$$

$$\leq \sum_{m=1}^{n}\sup_{t\in\Omega_r}\sup_{t\leq s\leq t+K}\mid\eta_{lm}(s)\mid_{\mathbb{Q}}\frac{1}{\mu(\Omega_r)}\int_{\Omega_r}\left(\frac{1}{K}\int_t^{t+K}\mid B_m(s)\mid_{\mathbb{Q}}^{p}\Delta s\right)^{\frac{1}{p}}\mu(t)\Delta t.$$

Notice that $B_m(.)\in WS^pPAP_0(\mathbb{T},\mathbb{Q},\mu)$. Thus,

$$\lim_{r\to\infty}\frac{1}{\mu(\Omega_r)}\int_{\Omega_r}\left(\frac{1}{K}\int_t^{t+K}\mid V_l(s)\mid_{\mathbb{Q}}^{p}\Delta s\right)^{\frac{1}{p}}\mu(t)\Delta t$$

$$\leq C_1\lim_{r\to\infty}\frac{1}{\mu(\Omega_r)}\int_{\Omega_r}\left(\frac{1}{K}\int_t^{t+K}\mid B_m(s)\mid_{\mathbb{Q}}^{p}\Delta s\right)^{\frac{1}{p}}\mu(t)\Delta t,$$

$$= 0.$$

Where $C_1=\sum_{m=1}^{n}\sup_{t\in\Omega_r}\sup_{t\leq s\leq t+K}\mid\eta_{lm}(s)\mid_{\mathbb{Q}}$. Hence, $V_l(.)\in WS^pPAP_0(\mathbb{T},\mathbb{Q},\mu)$. Ultimately, for $1\leq l,m\leq m$

$$\bigvee_{m=1}^{n}\eta_{lm}(.)h_m\left(\rho_m(.-\delta_m(.))\right)\in WS^pPAP(\mathbb{T},\mathbb{Q},\mu).$$

Similarly, we can get

$$\bigwedge_{m=1}^{n}\lambda_{lm}(.)h_m\left(\rho_m(.-\delta_m(.))\right)\in WS^pPAP(\mathbb{T},\mathbb{Q},\mu).$$

□

Theorem 4.6. *Let* $\rho=(\rho_1,...,\rho_n)\in WS^pPAP(\mathbb{T},\mathbb{Q},\mu)$. *Under assumptions* $(\mathcal{H}_1-\mathcal{H}_3)$, *the nonlinear operator defined by,*

$$(\Lambda_\rho)_l(t)=\int_{-\infty}^{t}\hat{e}_{\ominus a_l}(t,\sigma(s))\Upsilon_l(s)\Delta s,\quad l=1,...,n.$$

where

$$\Upsilon_l(t)=\sum_{m=1}^{n}\alpha_{lm}(t)f_m(\rho_m(t))+\sum_{m=1}^{n}\beta_{lm}(t)f_m(\rho_m(t-\delta_m(t)))$$
$$+\bigvee_{m=1}^{n}\eta_{lm}(t)h_m\left(\rho_m(t-\delta_m(t))\right)+\bigwedge_{m=1}^{n}\lambda_{lm}(t)h_m\left(\rho_m(t-\delta_m(t))\right)+I_l(t)$$

maps $WS^pPAP(\mathbb{T},\mathbb{Q},\mu)$ *into itself.*

Proof. From previous assumptions one can easily see that $(\Lambda_\rho)_l$ is well defined and continuous. Applying the composition theorem of weighted Stepanov-like pseudo almost periodic functions (Lemma (4.2)) and Lemma (4.5) it follows that the function $\Upsilon_l(.)$ belongs to $WS^pPAP(\mathbb{T},\mathbb{Q},\mu)$. Now, let $\Upsilon_l=W_l+Z_l$ with $W_l\in S^pAP(\mathbb{T},\mathbb{Q})$, $Z_l(.)\in WS^pPAP_0(\mathbb{T},\mathbb{Q},\mu)$. So, $(\Lambda_\rho)_l(t)=(\mathcal{A}_\rho)_l(t)+(\mathcal{B}_\rho)_l(t)$ with

$$(\mathcal{A}_\rho)_l(t)=\int_{-\infty}^{t}\hat{e}_{\ominus a_l}(t,\sigma(s))W_l(s)\Delta s,$$

and

$$(\mathcal{B}_\rho)_l(t)=\int_{-\infty}^{t}\hat{e}_{\ominus a_l}(t,\sigma(s))Z_l(s)\Delta s.$$

To complete the proof, we break the proof in two steps.
Step 1 : We will prove that $(\mathcal{A}_\rho)_l(.) \in S^pAP(\mathbb{T},\mathbb{Q})$. Since $W_l(.)$ is Stepanov-like almost periodic then for all $\varepsilon > 0, T(W_l(.),\varepsilon)$ is relatively dense in Π. Let $t,s \in \mathbb{T}$ and τ be a Stepanov-like almost period of $W_l(.)$ then,

$$\begin{aligned}
\mid (\mathcal{A}_\rho)_l(t+\tau) - (\mathcal{A}_\rho)_l(t) \mid_{\mathbb{Q}} &= \Big| \int_{-\infty}^{t} \hat{e}_{\ominus a_l}(t,\sigma(s))W_l(s+\tau)\Delta s - \int_{-\infty}^{t} \hat{e}_{\ominus a_l}(t,\sigma(s))W_l(s)\Delta s \Big|_{\mathbb{Q}}, \\
&\leq \int_{-\infty}^{t} \hat{e}_{\ominus \overline{a}_l}(t,\sigma(s))\big|W_l(s+\tau) - W_l(s)\big|_{\mathbb{Q}}\Delta s, \\
&\leq \int_{-\infty}^{0} \hat{e}_{\ominus \overline{a}_l}(0,\sigma(s))\big|W_l(t+s+\tau) - W_l(t+s)\big|_{\mathbb{Q}}\Delta s, \\
&\leq M_l^{\frac{1}{q}}(q) \left(\int_{-\infty}^{0} \hat{e}_{\ominus(\frac{\overline{a}_l p}{2})}(0,\sigma(s))\big|W_l(t+s+\tau) - W_l(t+s)\big|_{\mathbb{Q}}^p\Delta s \right)^{\frac{1}{p}}.
\end{aligned}$$

By using Fubini's theorem, we have

$$\begin{aligned}
&\sup_{t_1\in\mathbb{T}} \left(\frac{1}{K}\int_{t_1}^{t_1+K} \Big|(\mathcal{A}_\rho)_l(t+\tau) - (\mathcal{A}_\rho)_l(t)\Big|_{\mathbb{Q}}^p \Delta t \right)^{\frac{1}{p}} \\
\leq\ & \sup_{t_1\in\mathbb{T}} \left(\frac{1}{K}\int_{t_1}^{t_1+K} M_l^{\frac{p}{q}} . \int_{-\infty}^{0} \hat{e}_{\ominus(\frac{\overline{a}_l p}{2})}(0,\sigma(s))\Big|W_l(s+t+\tau) - W_l(s+t)\Big|_{\mathbb{Q}}^p \Delta s\Delta t \right)^{\frac{1}{p}}, \\
\leq\ & M_l^{\frac{1}{q}}(q) \left(\int_{-\infty}^{0} \hat{e}_{\ominus(\frac{\overline{a}_l p}{2})}(0,\sigma(s)) \sup_{t_1\in\mathbb{T}} \frac{1}{K}\int_{t_1}^{t_1+K} \Big|W_l(s+t+\tau) - W_l(s+t)\Big|_{\mathbb{Q}}^p \Delta t\Delta s \right)^{\frac{1}{p}}, \\
\leq\ & M_l^{\frac{1}{q}}(q) \left(\int_{-\infty}^{0} \hat{e}_{\ominus(\frac{\overline{a}_l p}{2})}(0,\sigma(s)) \sup_{\bar{t}\in\mathbb{T}} \frac{1}{K}\int_{\bar{t}}^{\bar{t}+K} \Big|W_l(t+\tau) - W_l(t)\Big|_{\mathbb{Q}}^p \Delta t\Delta s \right)^{\frac{1}{p}}, \\
\leq\ & M_l^{\frac{1}{q}}(q) M_l^{\frac{1}{p}}(p)\varepsilon < M\varepsilon.
\end{aligned}$$

Where $M = \max_{1\leq l\leq n} M_l^{\frac{1}{q}}(q).M_l^{\frac{1}{p}}(p)$, which implies that $T\left((\mathcal{A}_\rho)_l(.), M\varepsilon\right)$ is relatively dense in Π. Hence, $(\mathcal{A}_\rho)_l(.) \in S^pAP(\mathbb{T},\mathbb{Q})$.
Step 2 : The next step consists of showing that $(\mathcal{B}_\rho)_l(.) \in WS^pPAP_0(\mathbb{T},\mathbb{Q},\mu)$. By using the Hölder's inequality and Fubini's theorem we get,

$$\int_{\Omega_r} \left(\frac{1}{K}\int_{t}^{t+K} \mid (\mathcal{B}_\rho)_l(s) \mid_{\mathbb{Q}}^p \Delta s \right)^{\frac{1}{p}} \mu(t)\Delta t \leq (\mu(\Omega_r))^{\frac{1}{q}} \left[\int_{\Omega_r} \left(\frac{1}{K}\int_{t}^{t+K} \mid (\mathcal{B}_\rho)_l(s) \mid_{\mathbb{Q}}^p \Delta s \right) \mu(t)\Delta t \right]^{\frac{1}{p}}.$$

In addition,

$$\Big| \int_{-\infty}^{s} \hat{e}_{\ominus a_l}(s,\sigma(z))Z_l(z)\Delta z \Big|_{\mathbb{Q}}^p \leq M_l^{\frac{p}{q}}(q) \int_{-\infty}^{0} \hat{e}_{\ominus(\frac{\overline{a}_l p}{2})}(0,\sigma(z)) \mid Z_l(z+s) \mid_{\mathbb{Q}}^p \Delta z.$$

Which implies that

$$\begin{aligned}
&\int_{\Omega_r} \left(\frac{1}{K}\int_{t}^{t+K} \mid (\mathcal{B}_\rho)_l(s) \mid_{\mathbb{Q}}^p \Delta s \right)^{\frac{1}{p}} \mu(t)\Delta t \\
\leq\ & (\mu(\Omega_r))^{\frac{1}{q}} \left[\int_{\Omega_r} \left(\frac{1}{K}\int_{t}^{t+K} M_l^{\frac{p}{q}}(q) \int_{-\infty}^{0} \hat{e}_{\ominus(\frac{\overline{a}_l p}{2})}(0,\sigma(z)) \mid Z_m(s+z) \mid_{\mathbb{Q}}^p \Delta z\Delta s \right) \mu(t)\Delta t \right]^{\frac{1}{p}}, \\
\leq\ & (\mu(\Omega_r))^{\frac{1}{q}} M_l^{\frac{1}{q}}(q) \left[\int_{-\infty}^{0} \hat{e}_{\ominus(\frac{\overline{a}_l p}{2})}(0,\sigma(z)) \int_{\Omega_r} \frac{1}{K}\int_{t}^{t+K} \mid Z_l(s+z) \mid_{\mathbb{Q}}^p \Delta s\mu(t)\Delta t\Delta z \right]^{\frac{1}{p}}.
\end{aligned}$$

So, one has

$$\begin{aligned}
&\frac{1}{\mu(\Omega_r)}\int_{\Omega_r}\left(\frac{1}{K}\int_t^{t+K}\mid(\mathcal{B}_\rho)_l(s)\mid_{\mathbb{Q}}^p\Delta s\right)^{\frac{1}{p}}\mu(t)\Delta t\\
\leq\ &\frac{1}{\mu(\Omega_r)}\left(\mu(\Omega_r)\right)^{\frac{1}{q}}\left(\mu(\Omega_r)\right)^{\frac{1}{p}}M_l^{\frac{1}{q}}(q)\\
&\times\left(\int_{-\infty}^0\hat{e}_{\ominus(\frac{\overline{a}_lp}{2})}(0,\sigma(z))\frac{1}{\mu(\Omega_r)}\int_{\Omega_r}\frac{1}{K}\int_t^{t+K}\mid Z_l(s+z)\mid_{\mathbb{Q}}^p\Delta s\mu(t)\Delta t\Delta z\right)^{\frac{1}{p}},\\
\leq\ &M_l^{\frac{1}{q}}(q)\left[\int_{-\infty}^0\hat{e}_{\ominus(\frac{\overline{a}_lp}{2})}(0,\sigma(z))\frac{1}{\mu(\Omega_r)}\int_{\Omega_r}\frac{1}{K}\int_t^{t+K}\mid Z_l(s+z)\mid_{\mathbb{Q}}^p\Delta s\mu(t)\Delta t\Delta z\right]^{\frac{1}{p}},\\
\leq\ &M_l^{\frac{1}{q}}(q)\left[\int_{-\infty}^0\hat{e}_{\ominus(\frac{\overline{a}_lp}{2})}(0,\sigma(z))\frac{1}{\mu(\Omega_r)}\int_{\Omega_r}\left(\frac{1}{K}\int_t^{t+K}\mid Z_l(s+z)\mid_{\mathbb{Q}}^q\Delta s\right)^{\frac{p}{q}}\mu(t)\Delta t\Delta z\right]^{\frac{1}{p}},\\
\leq\ &M_l^{\frac{1}{q}}(q)\Bigg[\int_{-\infty}^0\hat{e}_{\ominus(\frac{\overline{a}_lp}{2})}(0,\sigma(z))\frac{1}{\mu(\Omega_r)}\int_{\Omega_r}\left(\frac{1}{K}\int_t^{t+K}\mid Z_l(s+z)\mid_{\mathbb{Q}}^q\Delta s\right)^{\frac{1}{q}}\\
&\times\left(\frac{1}{K}\int_t^{t+K}\mid Z_l(s+z)\mid_{\mathbb{Q}}^q\Delta s\right)^{\frac{p-1}{q}}\mu(t)\Delta t\Delta z\Bigg]^{\frac{1}{p}},\\
\leq\ &M_l^{\frac{1}{q}}(q)\mid Z_l\mid_\infty^{\frac{p-1}{p}}\\
&\times\left\{\int_{-\infty}^0\hat{e}_{\ominus(\frac{\overline{a}_lp}{2})}(0,\sigma(z))\left[\frac{1}{\mu(\Omega_r)}\int_{\Omega_r}\left(\frac{1}{K}\int_t^{t+K}\mid Z_l(s+z)\mid_{\mathbb{Q}}^q\Delta s\right)^{\frac{1}{q}}\mu(t)\Delta t\right]\Delta z\right\}^{\frac{1}{p}}.
\end{aligned}$$

The Lebesgue dominated convergence theorem and $Z_l(.)\in WS^pPAP_0(\mathbb{T},\mathbb{Q},\mu)$ lead to

$$\lim_{r\to\infty}\frac{1}{\mu(\Omega_r)}\int_{\Omega_r}\left(\frac{1}{K}\int_t^{t+K}\mid(\mathcal{B}_\rho)_l(s)\mid_{\mathbb{Q}}^p\Delta s\right)^{\frac{1}{p}}\mu(t)\Delta t=0.$$

Therefore, $(\mathcal{B}_\rho)_l(.)\in WS^pPAP_0(\mathbb{T},\mathbb{Q},\mu)$. Thus, the nonlinear operator $(\Lambda_\rho)_l(.)$ maps $WS^pPAP(\mathbb{T},\mathbb{Q},\mu)$ into itself. This completes the proof of Theorem (4.6). □

Lemma 4.7. Suppose that the assumptions $(\mathcal{H}_1-\mathcal{H}_3)$ hold. Then $\parallel\rho_0\parallel_{S^p}\leq C$. Where $\rho_0=\{(\rho_0)_l\}_{l=1}^n$, and $(\rho_0)_l(t)=\int_{-\infty}^t\hat{e}_{\ominus a_l}(t,\sigma(s))I_l(s)\Delta s\quad 1\leq l\leq n$.

Proof. By using the Hölder's inequality, we get

$$\begin{aligned}
\parallel\rho_0(t)\parallel_{S^p}\ &=\ \sup_{t_1\in\mathbb{T}}\left(\frac{1}{K}\int_{t_1}^{t_1+K}\left|\int_{-\infty}^t\hat{e}_{\ominus a_l}(t,\sigma(z))I_l(z)\right|_{\mathbb{Q}}^p\Delta z\Delta t\right)^{\frac{1}{p}}\\
&\leq\ M_l^{\frac{1}{q}}(q)\sup_{t_1\in\mathbb{T}}\left(\int_{-\infty}^0\hat{e}_{\ominus(\frac{\overline{a}_lp}{2})}(0,\sigma(z))\frac{1}{K}\int_{t_1}^{t_1+K}\mid I_l(z+t)\mid_{\mathbb{Q}}^p\Delta t\Delta z\right)^{\frac{1}{p}}\\
&\leq\ M_l^{\frac{1}{q}}(q)\sup_{t_2\in\mathbb{T}}\left(\int_{-\infty}^0\hat{e}_{\ominus(\frac{\overline{a}_lp}{2})}(0,I(z))\frac{1}{K}\int_{t_2}^{t_2+K}\mid I_l(\bar{t})\mid_{\mathbb{Q}}^p\Delta\bar{t}\Delta z\right)^{\frac{1}{p}},\\
&\leq\ M_l^{\frac{1}{q}}(q)M_l^{\frac{1}{p}}(p).\parallel I\parallel_{S^p}\leq C.
\end{aligned}$$

with $C=\max_{1\leq l\leq n}M_l^{\frac{1}{q}}(q)M_l^{\frac{1}{p}}(p).\parallel I\parallel_{S^p}$. □

Lemma 4.8. Under assumptions $(\mathcal{H}_1 - \mathcal{H}_3)$. For all $l = 1, ..., n$, the nonlinear operator $(\varLambda_\rho)_l$ is a self-mapping from $\mathfrak{D}^0$ to $\mathfrak{D}^0$. Where

$$\mathfrak{D}^0 = \left\{ \rho : \rho \in WS^pPAP(\mathbb{T}, \mathbb{Q}, \mu), \| \rho - \rho_0 \|_{S^p} \leq \frac{CL^*}{1 - L^*} \right\}.$$

Proof. Let $\rho \in \mathfrak{D}^0$ and $q > 1$ such that $\frac{1}{q} + \frac{1}{p} = 1$. It follows from Hölder's inequality that

$$\begin{aligned} | (\varLambda_\rho)_l(t) - \rho_0(t) |_{\mathbb{Q}} &= \Bigg| \int_{-\infty}^{t} \hat{e}_{\ominus a_l}(t, \sigma(z)) \Bigg[\sum_{m=1}^{n} \alpha_{lm}(z) f_m(\rho_m(z)) + \\ & \sum_{m=1}^{n} \beta_{lm}(z) f_m(\rho_m(z - \delta_m(z))) + \bigvee_{m=1}^{n} \eta_{lm}(z) h_m(\rho_m(z - \delta_m(z))) + \\ & \bigwedge_{m=1}^{n} \lambda_{lm}(z) h_m(\rho_m(z - \delta_m(z))) \Bigg] \Delta z \Bigg|_{\mathbb{Q}}, \\ &\leq M_l^{\frac{1}{q}}(q) \Bigg[\int_{-\infty}^{0} \hat{e}_{\ominus(\frac{\underline{a}_l p}{2})}(0, \sigma(z)) \Bigg| \sum_{m=1}^{n} \alpha_{lm}(z + t) f_m(\rho_m(z + t)) \\ & + \sum_{m=1}^{n} \beta_{lm}(z + t) f_m(\rho_m(z + t - \delta_m(z + t))) \\ & + \bigvee_{m=1}^{n} \eta_{lm}(z + t) h_m(\rho_m(z + t - \delta_m(z + t))) \\ & + \bigwedge_{m=1}^{n} \lambda_{lm}(z + t) h_m(\rho_m(z + t - \delta_m(z + t))) \Bigg|_{\mathbb{Q}}^{p} \Delta z \Bigg]^{\frac{1}{p}}. \end{aligned}$$

Which implies that

$$\begin{aligned} & \| (\varLambda_\rho)_l(t) - \rho_0(t) \|_{S^p} \\ = & \sup_{t_1 \in \mathbb{T}} \Bigg[\frac{1}{K} \int_{t_1}^{t_1 + K} \Bigg| \int_{-\infty}^{t} \hat{e}_{\ominus a_l}(t, \sigma(z)) \Bigg(\sum_{m=1}^{n} \alpha_{lm}(z) f_m(\rho_m(z)) \\ & + \sum_{m=1}^{n} \beta_{lm}(z) f_m(\rho_m(z - \delta_m(z))) + \bigvee_{m=1}^{n} \eta_{lm}(z) h_m(\rho_m(z - \delta_m(z))) \\ & + \bigwedge_{m=1}^{n} \lambda_{lm}(z) h_m(\rho_m(z - \delta_m(z))) \Bigg] \Delta z \Bigg|_{\mathbb{Q}}^{p} \Delta t \Bigg]^{\frac{1}{p}}, \\ \leq & M_l^{\frac{1}{q}}(q) \sup_{t_1 \in \mathbb{T}} \Bigg[\int_{-\infty}^{0} \hat{e}_{\ominus(\frac{\underline{a}_l p}{2})}(0, \sigma(z)) \frac{1}{K} \int_{t_1}^{t_1 + K} \Bigg| \sum_{m=1}^{n} \alpha_{lm}(z + t) f_m(\rho_m(z + t)) \\ & + \sum_{m=1}^{n} \beta_{lm}(z + t) f_m(\rho_m(z + t - \delta_m(z + t))) + \bigvee_{m=1}^{n} \eta_{lm}(z + t) h_m(\rho_m(z + t - \delta_m(z + t))) \\ & + \bigwedge_{m=1}^{n} \lambda_{lm}(z + t) h_m(\rho_m(z + t - \delta_m(z + t))) \Bigg|_{\mathbb{Q}}^{p} \Delta(z + t) \Delta z \Bigg]^{\frac{1}{p}}, \\ \leq & M_l^{\frac{1}{q}}(q) \sup_{t_2 \in \mathbb{T}} \Bigg[\int_{-\infty}^{0} \hat{e}_{\ominus(\frac{\underline{a}_l p}{2})}(0, \sigma(z)) \frac{1}{K} \int_{t_2}^{t_2 + K} \Bigg| \sum_{m=1}^{n} \alpha_{lm}(\hat{t}) f_m(\rho_m(\hat{t})) + \\ & \sum_{m=1}^{n} \beta_{lm}(\hat{t}) f_m(\rho_m(\hat{t} - \delta_m(\hat{t}))) \end{aligned}$$

$$
\begin{aligned}
&+\bigvee_{m=1}^{n}\eta_{lm}(\hat{t})h_m\left(\rho_m(\hat{t}-\delta_m(\hat{t}))\right)+\bigwedge_{m=1}^{n}\lambda_{lm}(\hat{t})h_m\left(\rho_m(\hat{t}-\delta_m\hat{t})\right)\Big|_{\mathbb{Q}}^{p}\Delta\hat{t}\Delta z\Big]^{\frac{1}{p}}.\\
\leq\ & M_l^{\frac{1}{q}}(q)M_l^{\frac{1}{p}}(p)\Bigg(\sup_{t_2\in\mathbb{T}}\left[\frac{1}{K}\int_{t_2}^{t_2+K}\sum_{m=1}^{n}|\ \alpha_{lm}(\hat{t})\ |_{\mathbb{Q}}^{p}|\ f_m\left(\rho_m(\hat{t})\right)|_{\mathbb{Q}}^{p}\ \Delta\hat{t}\right]^{\frac{1}{p}}\\
&+\sup_{t_2\in\mathbb{T}}\left[\frac{1}{K}\int_{t_2}^{t_2+K}\sum_{m=1}^{n}|\ \beta_{lm}(\hat{t})\ |_{\mathbb{Q}}^{p}|\ f_m(\rho_m(\hat{t}-\delta_m(\hat{t})))\ |_{\mathbb{Q}}^{p}\ \Delta\hat{t}\right]^{\frac{1}{p}}\\
&+\sup_{t_2\in\mathbb{T}}\left[\frac{1}{K}\int_{t_2}^{t_2+K}\sum_{m=1}^{n}|\ \eta_{lm}(\hat{t})\ |_{\mathbb{Q}}^{p}|\ h_m(\rho_m(\hat{t}-\delta_m(\hat{t})))\ |_{\mathbb{Q}}^{p}\ \Delta\hat{t}\right]^{\frac{1}{p}}\\
&+\sup_{t_2\in\mathbb{T}}\left[\frac{1}{K}\int_{t_2}^{t_2+K}\sum_{m=1}^{n}|\ \lambda_{lm}(\hat{t})\ |_{\mathbb{Q}}^{p}|\ h_m(\rho_m(\hat{t}-\delta_m(t_2)))\ |_{\mathbb{Q}}^{p}\ \Delta\hat{t}\right]^{\frac{1}{p}}\Bigg),\\
\leq\ & \max_{1\leq l\leq n}\Bigg\{M_l^{\frac{1}{q}}(q).M_l^{\frac{1}{p}}(p)\Bigg[\sum_{m=1}^{n}\alpha_{lm}^{+}L_{f_m}+\sum_{m=1}^{n}\frac{\beta_{lm}^{+}L_{f_m}}{(1-\delta_m^{*})^{\frac{1}{p}}}+\sum_{m=1}^{n}\frac{\eta_{lm}^{+}L_{h_m}}{(1-\delta_m^{*})^{\frac{1}{p}}}\\
&+\sum_{m=1}^{n}\frac{\lambda_{lm}^{+}L_{h_m}}{(1-\delta_m^{*})^{\frac{1}{p}}}\Bigg]\Bigg\}\parallel\rho\parallel_{S^p},\\
\leq\ & L^{*}.\parallel\rho\parallel_{S^p}.
\end{aligned}
$$

In addition, for any $\rho\in\mathfrak{D}^0$, we get
$\parallel\rho\parallel_{S^p}\leq\parallel\rho-\rho_0\parallel_{S^p}+\parallel\rho_0\parallel_{S^p}\leq\frac{CL^*}{1-L^*}+C=\frac{C}{1-L^*}$. Otherwise,

$$
\parallel(\Lambda_\rho)_l(t)-\rho_0(t)\parallel_{S^p}\leq\frac{CL^*}{1-L^*}.
$$

which implies that $(\Lambda_\rho)_l\in\mathfrak{D}^0$, so the mapping $(\Lambda_\rho)_l$ is a self-mapping from $\mathfrak{D}^0$ to $\mathfrak{D}^0$. □

Theorem 4.9. Assume that $(\mathcal{H}_1)$-$(\mathcal{H}_5)$ hold. Then, system (3.1) has a unique weighted Stepanov-like pseudo almost periodic solution in the region $\mathfrak{D}^0$, provided that $L^*<1$.

Proof. For any $\rho,\phi\in\mathfrak{D}^0$ and $m=1,...,n$ we have

$$
\begin{aligned}
&\parallel(\Lambda_\rho)_l(t)-(\Lambda_\phi)_l(t)\parallel_{S^p}\\
=\ & \sup_{t_1\in\mathbb{T}}\left[\frac{1}{K}\int_{t_1}^{t_1+K}\left|\int_{-\infty}^{t}\hat{e}_{\ominus a_l}(t,\sigma(z))\Bigg(\sum_{m=1}^{n}\alpha_{lm}(z)\left(f_m(\rho_m(z))-f_m(\phi_m(z))\right)\right.\right.\\
&+\sum_{m=1}^{n}\beta_{lm}(z)\left(f_m(\rho_m(z-\delta_m(z)))-f_m(\phi_m(z-\delta_m(z)))\right)\\
&+\bigvee_{m=1}^{n}\eta_{lm}(z)h_m\left(\rho_m(z-\delta_m(z))\right)-\bigvee_{m=1}^{n}\eta_{lm}(z)h_m\left(\phi_m(z-\delta_m(z))\right)\\
&\left.\left.+\bigwedge_{m=1}^{n}\lambda_{lm}(z)h_m\left(\rho_m(z-\delta_m(z))\right)-\bigwedge_{m=1}^{n}\lambda_{lm}(z)h_m\left(\phi_m(z-\delta_m(z))\right)\Bigg)\Delta z\right|_{\mathbb{Q}}^{p}\Delta t\right]^{\frac{1}{p}},\\
&\leq\max_{1\leq l\leq n}\Bigg\{M_l^{\frac{1}{q}}(q).M_l^{\frac{1}{p}}(p)\Bigg[\sum_{m=1}^{n}\alpha_{lm}^{+}L_{f_m}+\sum_{m=1}^{n}\frac{\beta_{lm}^{+}L_{f_m}}{(1-\delta_m^{*})^{\frac{1}{p}}}+\sum_{m=1}^{n}\frac{\eta_{lm}^{+}L_{h_m}}{(1-\delta_m^{*})^{\frac{1}{p}}}\\
&+\sum_{m=1}^{n}\frac{\lambda_{lm}^{+}L_{h_m}}{(1-\delta_m^{*})^{\frac{1}{p}}}\Bigg]\Bigg\}\parallel\rho-\phi\parallel_{S^p},
\end{aligned}
$$

$$\leq L^*. \parallel \rho - \phi \parallel_{S^p} < 1.$$

According to the well-known contraction principle there exists a unique fixed point $x^*(.)$ such that $(\Lambda_\rho)_l x^*(t) = x^*(t)$. Besides, $x^*(.)$ is a weighted Stepanov-like pseudo almost periodic solution of the system (3.1) on time scales in $\mathfrak{D}^0$. This completes the proof of Theorem (4.9). □

Remark 4.10. The models studied in ([23, 26]) are considered without fuzzy effects. If $\mu = 1$ and $\mathbb{T} = \mathbb{R}$, then, our results is clearly more general and more sophisticated than results in previous references.

5 S^p-Exponential Stability Of μ-Stepanov-like pseudo-almost periodic solution

In this section, we will study the S^p-global exponential stability of weighted Stepanov-like pseudo-almost periodic solution of QVFRNNs (3.1) on time scales.

Definition 5.1. The dynamical networks (3.1) is said to be S^p-globally exponentially stable, if there exist positive constants α with $\ominus\alpha \in \mathfrak{R}^+$ and $R > 0$ such that

$$\parallel y(t) - x(t) \parallel_{S^p} \leq R\hat{e}_{\ominus\alpha}(t, 0), \quad \forall t \in (0, \infty)_{\mathbb{T}}.$$

Where $x(.) = (x_1(.), x_2(.), ..., x_n(.))$ is a weighted Stepanov-like pseudo almost periodic solution of QVFRNNs (3.1) on $\mathbb{T}$ and $y(.) = (y_1(.), y_2(.), ..., y_n(.))$ is an arbitrary solution of QVFRNNs (3.1) on $\mathbb{T}$.

Theorem 5.2. Suppose that assumptions $(\mathcal{H}_1)$-$(\mathcal{H}_5)$ hold, and $L^* < 1$. Then the unique weighted S^p-pseudo almost periodic solution of system (3.1) is S^p-globally exponentially stable on $\mathbb{T}$ whenever

$$E_l = -\frac{3}{a_l^*} + \check{a}_l - \sum_{m=1}^{n} a_l^* \left(\alpha_{lm}^+\right)^2 (L_{f_l})^2$$
$$- \sum_{m=1}^{n} \frac{1}{1-\delta^*} \left(a_l^* \left(\beta_{lm}^+\right)^2 (L_{f_l})^2 + a_l^* \left(\lambda_{lm}^+\right)^2 (L_{h_l})^2 + a_l^* \left(\eta_{lm}^+\right)^2 (L_{h_l})^2 \right) > 0.$$

Proof. Let $v \in [0, \infty)$, we consider the function $v \mapsto \Theta_l(v)$ defined by

$$\Theta_l(v) = v + \frac{3}{a_l^*} - \check{a}_l + \sum_{m=1}^{n} a_l^* \left(\alpha_{lm}^+\right)^2 (L_{f_l})^2 + \sum_{m=1}^{n} \frac{\exp(v\bar{\mu})\exp(v\bar{\delta})}{1-\delta^*}$$
$$\times \left(a_l^* \left(\beta_{lm}^+\right)^2 (L_{f_l})^2 + a_l^* \left(\lambda_{lm}^+\right)^2 (L_{h_l})^2 + a_l^* \left(\eta_{lm}^+\right)^2 (L_{h_l})^2 \right) < 0.$$

We have

$$\Theta_l(0) = \frac{3}{a_l^*} - \check{a}_l + \sum_{m=1}^{n} a_l^* \left(\alpha_{lm}^+\right)^2 (L_{f_l})^2 + \sum_{m=1}^{n} \frac{1}{1-\delta^*}$$
$$\times \left(a_l^* \left(\beta_{lm}^+\right)^2 (L_{f_l})^2 + a_l^* \left(\lambda_{lm}^+\right)^2 (L_{h_l})^2 + a_l^* \left(\eta_{lm}^+\right)^2 (L_{h_l})^2 \right) < 0.$$

Since the function $\Theta_l(.)$ is continuous on $[0, \infty)$. So, we can choose the positive constant $0 < \alpha < \min_{1 \leq l \leq n} \overline{a}_l$, such that

$$\Theta_l(\alpha) = \alpha + \frac{3}{a_l^*} - \check{a}_l + \sum_{m=1}^{n} a_l^* \left(\alpha_{lm}^+\right)^2 (L_{f_l})^2 + \sum_{m=1}^{n} \frac{\exp(\alpha\bar{\mu})\exp(\alpha\bar{\delta})}{1-\delta^*}$$
$$\times \left(a_l^* \left(\beta_{lm}^+\right)^2 (L_{f_l})^2 + a_l^* \left(\lambda_{lm}^+\right)^2 (L_{h_l})^2 + a_l^* \left(\eta_{lm}^+\right)^2 (L_{h_l})^2 \right) < 0.$$

Let $x(.)$ be the weighted S^p-pseudo almost periodic solution on $\mathbb{T}$, and let $y(.)$ be an arbitrary solution of QVFRNNs (3.1) on $\mathbb{T}$. We set $Y_l(.) := x_l(.)-y_l(.)$ and construct a Lyapunov function as follows :

$$V(t) = \sum_{l=1}^{n} \mid Y_l(t) \mid_{\mathbb{Q}}^2 \hat{e}_\alpha(t,0) + \sum_{l=1}^{n}\sum_{m=1}^{n} \frac{\exp(\alpha\bar{\mu})\exp(\alpha\bar{\delta})}{1-\delta^*}$$
$$\times \left(a_l^*\left(\beta_{lm}^+\right)^2\left(L_{f_l}\right)^2 + a_l^*\left(\lambda_{lm}^+\right)^2\left(L_{h_l}\right)^2 + a_l^*\left(\eta_{lm}^+\right)^2\left(L_{h_l}\right)^2\right)\int_{t-\delta(t)}^{t} \mid Y_l(z) \mid_{\mathbb{Q}}^2 \hat{e}_\alpha(\sigma(z),0)\Delta z.$$

Computing the Δ-derivative of $V(.)$, we get

$$V^\Delta(t) = \alpha\hat{e}_\alpha(t,0)\sum_{l=1}^{n} \mid Y_l(t) \mid_{\mathbb{Q}}^2 + \hat{e}_\alpha(t,0)\sum_{l=1}^{n} Y_l^\Delta(t)\overline{Y_l(t)}$$
$$+\sum_{l=1}^{n}\sum_{m=1}^{n} \frac{\exp(\alpha\bar{\mu})\exp(\alpha\bar{\delta})}{1-\delta^*}\left(a_l^*\left(\beta_{lm}^+\right)^2\left(L_{f_l}\right)^2 + a_l^*\left(\lambda_{lm}^+\right)^2\left(L_{h_l}\right)^2 + a_l^*\left(\eta_{lm}^+\right)^2\left(L_{h_l}\right)^2\right)$$
$$\times\left[\hat{e}_\alpha(t,0) \mid Y_l(t) \mid_{\mathbb{Q}}^2 - (1-\delta_m^\Delta(t))\exp(-\alpha\delta_m(t))\hat{e}_\alpha(t,0) \mid Y_l(t-\delta_m^\Delta(t)) \mid_{\mathbb{Q}}^2\right].$$

We know that $1-\delta_m^\Delta \geq 1-\delta^*$ and $\exp(\alpha\bar{\delta}).\exp(-\alpha\delta_m(t)) > 1$, then it follows from Lemma (2.2) that

$$V^\Delta(t) \leq \alpha\hat{e}_\alpha(t,0)\sum_{l=1}^{n} \mid Y_l(t) \mid_{\mathbb{Q}}^2 - \hat{e}_\alpha(t,0)\sum_{l=1}^{n} \check{a}_l \mid Y_l(t) \mid_{\mathbb{Q}}^2$$
$$+\hat{e}_\alpha(t,0)\sum_{l=1}^{n}\left(\sum_{m=1}^{n}\alpha_{lm}(t)\left(f_m(x_m(t))-f_m(y_m(t))\right).\overline{\alpha_{lm}(t)\left(f_m(x_m(t))-f_m(y_m(t))\right)}.a_l^*\right.$$
$$\left.+\frac{Y_l(t)\overline{Y_l(t)}}{a_l^*}\right)+\hat{e}_\alpha(t,0)\sum_{l=1}^{n}\left(\sum_{m=1}^{n}\overline{\beta_{lm}(t)\left(f_m(x_m(t-\delta_m(t)))-f_m(y_m(t-\delta_m(t)))\right)}\right.$$
$$\left.\times\beta_{lm}(t)\left(f_m(x_m(t-\delta_m(t)))-f_m(y_m(t-\delta_m(t)))\right).a_l^* + \frac{Y_l(t)\overline{Y_l(t)}}{a_l^*}\right)$$
$$+\hat{e}_\alpha(t,0)\sum_{l=1}^{n}\left(\bigvee_{m=1}^{n}\eta_{lm}(t)h_m(x_m(t-\delta_m(t))) - \bigvee_{m=1}^{n}\eta_{lm}(t)h_m(y_m(t-\delta_m(t)))\right)$$
$$\times\overline{\left(\bigvee_{m=1}^{n}\eta_{lm}(t)h_m(x_m(t-\delta_m(t))) - \bigvee_{m=1}^{n}\eta_{lm}(t)h_m(y_m(t-\delta_m(t)))\right)}a_l^* + \hat{e}_\alpha(t,0)\sum_{l=1}^{n}\frac{\overline{Y_l(t)}Y_l(t)}{a_l^*}$$
$$+\hat{e}_\alpha(t,0)\sum_{l=1}^{n}\left(\bigwedge_{m=1}^{n}\lambda_{lm}(t)h_m(x_m(t-\delta_m(t))) - \bigwedge_{m=1}^{n}\lambda_{lm}(t)h_m(y_m(t-\delta_m(t)))\right)$$
$$\times\overline{\left(\bigwedge_{m=1}^{n}\lambda_{lm}(t)h_m(x_m(t-\delta_m(t))) - \bigwedge_{m=1}^{n}\lambda_{lm}(t)h_m(y_m(t-\delta_m(t)))\right)}a_l^* + \hat{e}_\alpha(t,0)\sum_{l=1}^{n}\frac{\overline{Y_l(t)}Y_l(t)}{a_l^*}$$
$$+\sum_{l=1}^{n}\sum_{m=1}^{n}\frac{\exp(\alpha\bar{\mu})}{1-\delta^*}\left(a_l^*\left(\beta_{lm}^+\right)^2\left(L_{f_l}\right)^2 + a_l^*\left(\lambda_{lm}^+\right)^2\left(L_{h_l}\right)^2 + a_l^*\left(\eta_{lm}^+\right)^2\left(L_{h_l}\right)^2\right)$$
$$\times\left(\exp(\alpha\bar{\delta})\hat{e}_\alpha(t,0) \mid Y_l(t) \mid_{\mathbb{Q}}^2 - (1-\delta_m^\Delta(t))\hat{e}_\alpha(t,0) \mid Y_l(t-\delta_m^\Delta(t)) \mid_{\mathbb{Q}}^2\right)$$
$$\leq \hat{e}_\alpha(t,0)\sum_{l=1}^{n}\left(\alpha+\frac{3}{a_l^*}-\check{a}_l+\sum_{m=1}^{n}a_l^*\left(\alpha_{lm}^+\right)^2\left(L_{f_l}\right)^2\right.$$
$$\left.+\sum_{m=1}^{n}\frac{\exp(\alpha\bar{\mu})\exp(\alpha\bar{\delta})}{1-\delta^*}\left(a_l^*\left(\beta_{lm}^+\right)^2\left(L_{f_l}\right)^2 + a_l^*\left(\lambda_{lm}^+\right)^2\left(L_{h_l}\right)^2 + a_l^*\left(\eta_{lm}^+\right)^2\left(L_{h_l}\right)^2\right)\right) \mid Y_l(t) \mid_{\mathbb{Q}}^2,$$
$$< 0.$$

Otherwise, for $l=1,...,n$, we have $\sum_{l=1}^{n} \mid Y_l(t) \mid_{\mathbb{Q}}^2 \leq \hat{e}_{\ominus\alpha}(t,0)V(0)$, where $\ominus\alpha \in \mathfrak{R}^+$. Then

$$\sum_{l=1}^{n} \mid Y_l(t) \mid_{\mathbb{Q}}^p \leq \hat{e}_{\ominus p\alpha}(t,0)V(0)^p, \quad p \geq 2.$$

Consequently,

$$\sum_{l=1}^{n} \frac{1}{K} \mid Y_l(t) \mid_{\mathbb{Q}}^p \leq \frac{\hat{e}_{\ominus p\alpha}(t,0)V(0)^p}{K}, \quad p \geq 2.$$

Hence,

$$\sum_{l=1}^{n} \frac{1}{K} \int_{t-\bar{\delta}}^{t-\bar{\delta}+K} \mid Y_l(z) \mid_{\mathbb{Q}}^p \Delta z \leq \sum_{l=1}^{n} \frac{1}{K} \int_{t}^{t+K} \mid Y_l(z) \mid_{\mathbb{Q}}^p \Delta z \leq \int_{t}^{t+K} \frac{\hat{e}_{\ominus p\alpha}(t,0)V(0)^p}{K}.$$

Then,

$$\sum_{l=1}^{n} \frac{1}{K} \int_{t}^{t+K} \mid Y_l(z) \mid_{\mathbb{Q}}^p \Delta z \leq \frac{V(0)^p \hat{e}_{\ominus\alpha p}(t,0)}{K} \frac{(\exp(-\alpha pK) - 1)}{\ominus\alpha p}.$$

According to the previous inequality, we can obtain

$$\max_{l=1,\dots,n} \sup_{t_1\in\mathbb{T}} \left(\frac{1}{K} \int_{t_1}^{t_1+K} \mid x_l(z) - y_l(z) \mid_{\mathbb{Q}}^p \Delta z \right)^{\frac{1}{p}} \leq \frac{V(0)\hat{e}_{\ominus\alpha}(t,0)}{K^{\frac{1}{p}}} \left(\frac{\exp(-\alpha pK) - 1}{\ominus\alpha p} \right)^{\frac{1}{p}}.$$

Finally,

$$\parallel x - y \parallel_{S^p} \leq \frac{V(0)\hat{e}_{\ominus\alpha}(t,0)}{K^{\frac{1}{p}}} \left(\frac{\exp(-\alpha pK) - 1}{\ominus\alpha p} \right)^{\frac{1}{p}} \leq R\hat{e}_{\ominus\alpha}(t,0).$$

Where $R = \frac{V(0)}{K^{\frac{1}{p}}} \left(\frac{\exp(-\alpha pK)-1}{\ominus\alpha p} \right)^{\frac{1}{p}}$. Therefore, the weighted S^p-pseudo almost periodic solution x of QVFRNNs (3.1) is S^p-globally exponentially stable on time scales. This completes the proof.

Remark 5.3. To our knowledge, there is no results concentrated on Stepanov-like almost periodic, Stepanov-like pseudo almost periodic, and weighted Stepanov-like pseudo-almost periodic solution for QVFRNNs with time varying delays on time scales. As a consequence, the obtained results in this work are essentially new and the methods used in this paper can also be applied to study the WS^pPAP dynamic on time scales for some other models of dynamical neural networks.

□

6 Numerical Example

In this section, we give an example to illustrate the feasibility and effectiveness of our results derived in the previous sections.

Example 6.1. Let $n = l = m = 1$, $p = q = 2$, $\alpha = 1$ and the coefficients are taken as follows:

$$f_1(x_1) = \frac{1}{2}t + i\frac{1}{4}\sin(t) + \frac{1}{4}k, \quad \delta_1 = 1, \quad a_1 = 8 + 2j\cos(\sqrt{7}t),$$

$$h_1(x_1) = \frac{1}{40}x_1^I + \frac{1}{40}i\cos(\pi t) + \frac{1}{40}j\sin(x_1^R + x_1^I + x_1^J + x_1^K) + \frac{1}{40}k\tan(x_1^J + x_1^K),$$

$$\alpha_{11}(t) = \frac{1}{2}j\sin(\sqrt{7}t) + \frac{1}{2}j\exp(-t^4), \quad \beta_{11}(t) = i\cos(\pi t) + i\exp(-t),$$

$$\eta_{11}(t) = \frac{1}{40}k(\cos(\pi t) + \cos(\sqrt{3}t)), \quad \lambda_{11}(t) = -\frac{1}{60}k(\cos(\pi t) + \sin(\sqrt{5}t) + \frac{1}{1+t^2}),$$

$$I_{11} = \frac{1}{2} i \cos(\sqrt{2}t) + \frac{3}{8} j \sin(\pi t) + k \sin(t), \quad \mu(t) = 2 + \sin(t).$$

By calculating, we have

$$L_{f_1} = \frac{1}{8},\ L_{h_1} = \frac{1}{40},\ \delta_1^* = 0,$$

$$\alpha_{11}^{+} = 1 \quad \beta_{11}^{+} = 2, \quad \eta_{11}^{+} = \lambda_{11}^{+} = \frac{1}{20},$$

$$\overline{a}_1 = 6,\ a_1^* = \check{a}_1 = 10.$$

Case 1 : if $\mathbb{T} = \mathbb{R}$ we have $\mu(t) = \sigma(t) - t = t - t = 0 \quad \forall t \in \mathbb{T}$, then $\bar{\mu} = 0$. Moroever, $L^* \simeq 0.125 < 1$.

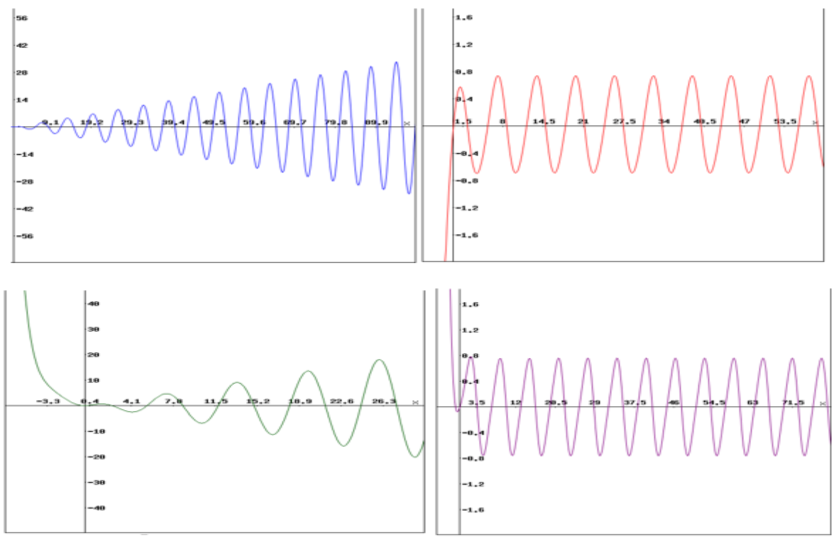

Figure 1. Behavior of the state variables x_1^R, x_1^I, x_1^J and x_1^K of QVFRNNs (3.1) on $\mathbb{T} = \mathbb{R}$.

Case 2 : if $\mathbb{T} = \mathbb{Z}$ we have $\mu(t) = \sigma(t) - t = t + 1 - t = 1 \quad \forall t \in \mathbb{T}$, then $\bar{\mu} = 1$. In addition, we have $L^* \simeq 0.157 < 1$. Also, $E_1 \simeq 6.375 > 0$. Finally, according to Theorem (4.9) and Theorem (5.2), the system (3.1) has a unique $(2 + \sin(t))$-Stepanov-like pseudo almost periodic solution in the region $\mathfrak{D}^0$, which is S^p-globally exponentially stable.

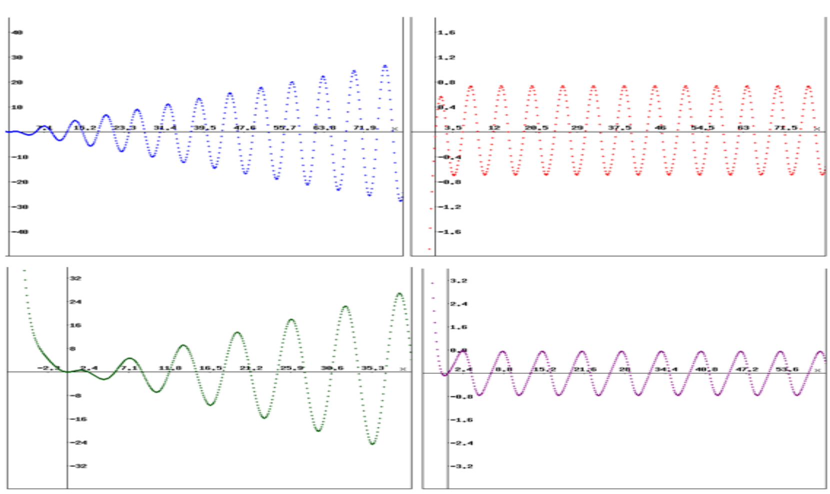

Figure 2. Behavior of the state variables x_1^R, x_1^I, x_1^J and x_1^K of QVFRNNs (3.1) on $\mathbb{T} = \mathbb{Z}$

Remark 6.2. For all we know, this is the first paper to study the weighted Stepanov-like pseudo almost periodic dynamics of Quaternion-valued fuzzy recurrent neural networks with time scale delays. There are no known outcomes that could lead to the conclusion of the example (6.1).

Conclusion

In this paper, some sufficient conditions are obtained by applying, the theory of time scales calculations, the Banach fixed point theorem, and by constructing an appropriate Lyapunov function to ensure the existence, uniqueness and stability of positive WS^pPAP solution of QVFRNNs (3.1).

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