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EXISTENCE OF SOLUTIONS OF FUZZY FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract This article is concerned with the existence and uniqueness of solutions of fuzzy fractional differential equations using fixed point theory. We provide some results answering when we can expect a solution of the problem.

1 Introduction

Fractional order fuzzy differential equations provide us a tool for modelling which appears to be better than that of the ordinary differential equations in the sense that the predictions made using models involving fractional derivatives are more close to nature than the ones done using ordinary differential equations. To work with models involving fuzzy fractional derivatives one must be able to tell whether the model is well posed or not that is whether it has a unique solution or not. Therefore it is necessary to study the restrictions and the properties of the models and to give some sufficient conditions on them so they possess a unique solution.

Fractional differential equations find its applications in the problems arising in the fields including but not limited to electrical and mechanical properties of materials, dynamics of turbulence, electrochemistry, viscoelasticity. Literature for fractional calculus can be found in ([13], [20] [23], [24], [26]). Further articles can be looked into ([16], [14], [15], [22]).

Letting fuzzy sets involve in our model allows us to harness the ability to handle the vagueness present in the nature. Since its introduction in 1965 by Lotfy Zadeh the literature for fuzzy set theory has only grown and a useful amount of it can be found in ([6], [17], [31], [32]). Further studies are referred in ([10], [19], [29]).

In 2010 Agarwal et. al.[2] merged fractional and fuzzy differential equation. Despite being new, this topic is growing very fast and many articles related to this are published. Some early works can be found in ([5], [28]). Useful surveys and collection of the literature for fuzzy fractional differential equations is given in ([1], [4]). While, the literature for fixed point theory and its application is vast, some of which, that is relevant to this paper, are referred in ([18], [3], [12]). Many situations in the study of nonlinear equations, calculus of variations, partial differential equations, optimal control and inverse problems can be formulated in terms of fixed point problems ([8], [7], [9], [21], [11], [30]).

Motivated by the work initiated by Agarwal et al.[2] in which they viewed the set of all fuzzy numbers as a semi-linear space and constructed a fixed point theorem for the space in semi-linear sense. It is known that the set of all fuzzy numbers is also a metric space and hence Banach contraction principle can be straight forwardly used, so observing that the Banach contraction principle is not straight-forwardly used, we study the fuzzy fractional initial value problem given below:

$$ {}^{C}_{t_0}D^{\alpha}_{t}\mathfrak{x}(t) = f(t, \mathfrak{x}(t)), \quad \mathfrak{x}(t_0) = x_0 \in \mathbb{F}_{\mathbb{R}} \tag{1.1} $$

Where, $\mathbb{F}_{\mathbb{R}}$ denotes the collection of all fuzzy numbers with universe $\mathbb{R}$, $t \in [t_0, T]$, $\mathfrak{x}$ is the unknown with codomain $\mathbb{F}_{\mathbb{R}}$, ${}_{t_0}^{C}D_t^{\alpha}$ denotes α order fractional derivative in Caputo sense with $0 < \alpha < 1$, f is also a fuzzy number valued function with the property that:

$$f \in C([t_0, T] \times B(x_0, \eta), \mathbb{F}_{\mathbb{R}}) \tag{1.2}$$

Here, $B(x_0, \eta)$ is a fuzzy ball with center x_0 and radius η.

The paper is designed as follows; this Section 1 is devoted to introduction, Section 2 contains preliminary definition to be used in the paper. The main result is contained in Section 3 and Section 4 has an example supporting the results we established, the literature regarding the example can be found in ([25], [27]). Section 5 concludes this article and then the useful reference are listed.

2 Preliminary Supplements

With an intention to make this paper self-sufficient, the following definitions are given.

Definition 2.1. [4] "A fuzzy number is a fuzzy set P if for its membership function $\mu_P : \mathbb{R} \to [0, 1]$ the following holds:

(i) P is normal. i.e, there exists a real member q_0 such that $\mu_P(q_0) = 1$.

(ii) P is fuzzy convex. i.e,
for two arbitrary real numbers q_1, q_2 and $l \in [0, 1]$ we have,

$$\mu_P(lq_1 + (1 - l)q_2) \geq Min\{\mu_P(q_1), \mu_P(q_2)\}.$$

(iii) P is upper semi-continuous.

(iv) The closure of $Supp(P) = \{q \in \mathbb{R} : \mu_P(q) > 0\}$ is compact."

The $Supp$ represents the support set of the fuzzy set P and is defined as in above.

Definition 2.2. [4] "The parametric form of a fuzzy number P is given by $q_P = [P_l(q), P_u(q)]$ for any $0 \leq q \leq 1$, iff,

(i) $P_l(q) \leq P_u(q)$.

(ii) $P_l(q)$ increases with q and is left continuous function on $[0, 1]$ and right continuous on 0 with respect to q.

(iii) $P_u(q)$ decreases with q and is left continuous function on $[0, 1]$ and right continuous on 0 with respect to q.

(iv) $q_P = [P_l(q), P_u(q)]$ is a compact interval for any $0 \leq q \leq 1$."

Definition 2.3. [4] "A singleton fuzzy number is a real number a, if $q_a = [a_l(q), a_u(q)] = [a, a]$ i.e, the membership function at a is 1 and at other values is zero."

For example, **0** denotes the singleton fuzzy zero with,

$$\mu_{\mathbf{0}}(q) = \begin{cases} 1, & q = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.4. [4] "Let P and Q be two fuzzy numbers in parametric form then the addition R of P and Q is given by

$$P \oplus Q = R,$$
$$q_R = [R_l(q), R_u(q)] = q_P + q_Q = [P_l(q), P_u(q)] + [Q_l(q), Q_u(q)],$$

where,

$$R_l(q) = P_l(q) + Q_l(q),\ R_u(q) = P_u(q) + Q_u(q)."$$

Definition 2.5. [4] "Let P and Q be two fuzzy numbers in parametric form then the generalized Hukuhara difference of P and Q is given by

$$P \ominus_g Q = R \Leftrightarrow \begin{cases} (i) P = Q \oplus R \\ \quad or \\ (ii) Q = P \oplus (-1)R." \end{cases}$$

Definition 2.6. [4] "Let P and Q be two fuzzy numbers in parametric form then the multiplication of P and Q is given by

$$P \odot Q = R,$$

$$q_R = [R_l(q), R_u(q)] = q_P \times q_Q = [P_l(q), P_u(q)] \times [Q_l(q), Q_u(q)],$$

where,

$$R_l(q) = \min\{P_l(q) \times Q_l(q), P_l(q) \times Q_u(q), P_u(q) \times Q_l(q), P_u(q) \times Q_u(q)\},$$

$$R_u(q) = \max\{P_l(q) \times Q_l(q), P_l(q) \times Q_u(q), P_u(q) \times Q_l(q), P_u(q) \times Q_u(q)\}."$$

This is also valid if one of P and Q is a real number.

Definition 2.7. [4] "The Hausdorff distance $\mathfrak{D}_H : \mathbb{F}_\mathbb{R} \times \mathbb{F}_\mathbb{R} \to \mathbb{R}$, between two fuzzy numbers P and Q is given by

$$\mathfrak{D}_H(P, Q) = \sup_{q \in [0,1]} \max\{|P_l(q) - Q_l(q)|, |P_u(q) - Q_u(q)|\}."$$

Following are some properties satisfied by $\mathfrak{D}_H$. Here $P, Q, R, S \in \mathbb{F}_\mathbb{R}$ and $k \in \mathbb{R}$:

(i) $\mathfrak{D}_H(P \oplus R, Q \oplus R) = \mathfrak{D}_H(P, Q)$.

(ii) $\mathfrak{D}_H(k \odot P, k \odot Q) = |k| \mathfrak{D}_H(P, Q)$.

(iii) $\mathfrak{D}_H(P \oplus Q, R \oplus S) \leq \mathfrak{D}_H(P, R) + \mathfrak{D}_H(Q, S)$.

(iv) $(\mathbb{F}_\mathbb{R}, \mathfrak{D}_H)$ is a complete metric space.

(v) $\mathfrak{D}_H(P \ominus_g Q, R \ominus_g S) \leq \mathfrak{D}_H(P, R) + \mathfrak{D}_H(Q, S)$.

(vi) $\mathfrak{D}_H(P \ominus_g Q, \mathbf{0}) = \mathfrak{D}_H(P, Q)$.

Last two properties are very useful as they relate the Hausdorff distance to the generalized Hukuhara difference.

Definition 2.8. [4] "The generalized Hukuhara derivative of a fuzzy number valued function $\mathfrak{x} : [0, T] \to \mathbb{F}_\mathbb{R}$ at $t_0 \in [0, T]$ is given by :

$$\mathfrak{x}'(t_0) = \lim_{h \to 0} \frac{\mathfrak{x}(t_0 + h) \ominus_g \mathfrak{x}(t_0)}{h},$$

provided that the difference $\mathfrak{x}(t_0 + h) \ominus_g \mathfrak{x}(t_0)$ and the limit exists then the function $\mathfrak{x}$ is called gH-differentiable."

The level-wise form of gH-differentiable function $\mathfrak{x}$ is given in following two cases:
CASE I: $\mathfrak{x}'(t, r) = [\mathfrak{x}_l'(t, r), \mathfrak{x}_u'(t, r)]$, if $\mathfrak{x}$ is $i - gH$ differentiable at t.
CASE II: $\mathfrak{x}'(t, r) = [\mathfrak{x}_u'(t, r), \mathfrak{x}_l'(t, r)]$, if $\mathfrak{x}$ is $ii - gH$ differentiable at t.

Definition 2.9. [24] "The fractional integration of $\mathfrak{x} \in L_{1,loc}([t_0, t], \mathbb{R})$ of order $\alpha > 0$ in Riemann-Liouville sense is given by

$${}_{t_0}\mathcal{J}_t^{-\alpha} \mathfrak{x}(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - \tau)^{\alpha - 1} \mathfrak{x}(\tau) d\tau."$$

Definition 2.10. [24] "The fractional derivative of $\mathfrak{x} \in L_{1,loc}([t_0,t],\mathbb{R})$ of order $0 \leq \alpha < 1$ in Riemann-Liouville sense is given by

$$ {}^{RL}_{t_0}\mathcal{D}^{\alpha}_{t}\mathfrak{x}(t) = \frac{d}{dt}\left({}_{t_0}\mathcal{J}_t^{-(1-\alpha)}\mathfrak{x}(t)\right) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{t_0}^{t}(t-\tau)^{-\alpha}\mathfrak{x}(\tau)d\tau." $$

Definition 2.11. [24] "The fractional derivative of a differentiable function $\mathfrak{x}$, such that $\mathfrak{x}' \in L_{1,loc}([t_0,t],\mathbb{R})$, of order $0 < \alpha < 1$ in Caputo sense is given by

$$ {}^{C}_{t_0}\mathcal{D}^{\alpha}_{t}\mathfrak{x}(t) ={}_{t_0} \mathcal{J}_t^{-(1-\alpha)}\frac{d}{dt}\mathfrak{x}(t) = \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}(t-\tau)^{-\alpha}\mathfrak{x}^{'}(\tau)d\tau." $$

Definition 2.12. [24] "The Mittag-Leffler function with two parameter for $z \in \mathbb{C}$ is given by

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \quad (\alpha,\beta > 0)" $$

Definition 2.13. [4] "The fuzzy fractional Caputo derivative of order $0 < \alpha < 1$ of a fuzzy number valued function $\mathfrak{x}$, such that $\mathfrak{x}$ is gH differentiable and $\mathfrak{x}' \in L_{1,loc}([t_0,t],\mathbb{F}_{\mathbb{R}})$, is defined as:

$$ {}^{C}_{t_0}D^{\alpha}_{t}\mathfrak{x}(t) = \frac{1}{\Gamma(1-\alpha)}\odot\int_{t_0}^{t}(t-\tau)^{-\alpha}\odot\mathfrak{x}'(\tau)d\tau." $$

Definition 2.14. [4] "Let $\mathfrak{x}$ be a fuzzy number valued function with parametric form $\mathfrak{x}(t,r) = [\mathfrak{x}_l(t,r),\mathfrak{x}_u(t,r)]$, then length of $\mathfrak{x}$ is defined as:

$$ length(\mathfrak{x}(t,r)) = \mathfrak{x}_u(t,r) - \mathfrak{x}_l(t,r)." $$

Note: The length function is monotonically increasing if the function x is $i - gH$ differentiable and is monotonically decreasing if it is $ii - gH$ differentiable.

Lemma 2.1[4] "Let $\mathfrak{x}(t)$ be a i or $ii - gH$ differentiable fuzzy function. Then $\mathfrak{x}(t)$ is the solution of (1.1) iff, $\mathfrak{x}(t)$ is the solution of the following integral equation:

$$ \mathfrak{x}(t)\ominus_g x_0 = \frac{1}{\Gamma(\alpha)}\odot\int_{t_0}^{t}(t-\tau)^{\alpha-1}\odot f(\tau,\mathfrak{x}(\tau))d\tau." $$

Definition 2.15. [18] "Let $g : X \to X$ and (X,d) a metric space then g is a contraction if there exists a fixed constant $l < 1$ such that

$$ d(g(a),g(b)) \leq ld(a,b), \forall a,b \in X." $$

Theorem 2.16. *[18] "Each contraction map $g : X \to X$ on a complete metric space (X,d) has a unique fixed point."*

Definition 2.17. Let $C([t_0,T],\mathbb{F}_{\mathbb{R}})$ be the set of all continuous functions from $[t_0,T]$ to $\mathbb{F}_{\mathbb{R}}$. Define

$$ H(\mathfrak{x},\mathfrak{y}) = \sup_{t\in[t_0,T]}\mathfrak{D}_H(\mathfrak{x}(t),\mathfrak{y}(t)). $$

Then $(C([t_0,T],\mathbb{F}_{\mathbb{R}}),H)$ is a complete metric space.

Theorem 2.18. *[18] "Let $g : X \to X$ and (X,d) a complete metric space with the property that for some positive integer n, g^n is contraction on X. Then, g has a unique fixed point."*

Proof Let a be the fixed point of g^n, then,

$$ g^n(a) = a \implies g^{n+1}(a) = g(a) \implies g^n(g(a)) = g(a) $$

This means that $g(a)$ is also a fixed point of g^n. But the fixed point is unique, hence $g(a) = a$ that is a is a unique fixed point of g.

3 Main Result

In this section, we give existence and uniqueness results on the solutions of the fuzzy fractional differential equation using Banach fixed point theorem or contraction principle.

Theorem 3.1. *Let f be a continuous function in system (1.1) such that for any $\mathfrak{z}, \mathfrak{w} \in C([t_0, T], B(x_0, \eta))$ and $L > 0$,*

$$\mathfrak{D}_H(f(t, \mathfrak{z}(t)), f(t, \mathfrak{w}(t))) \leq L\mathfrak{D}_H(\mathfrak{z}(t), \mathfrak{w}(t)), \tag{3.1}$$

then the IVP (1.1) has a unique solution.

Proof We'll exploit the Banach contraction principle to prove this result. Define, $\mathfrak{T} : C([t_0, T], \mathbb{F}_{\mathbb{R}}) \to C([t_0, T], \mathbb{F}_{\mathbb{R}})$ as

$$\mathfrak{T}\mathfrak{x}(t) = x_0 \oplus \frac{1}{\Gamma(\alpha)} \odot \int_{t_0}^{t} (t-\tau)^{\alpha-1} \odot f(\tau, \mathfrak{x}(\tau))d\tau.$$

Since $t \in [t_0, T]$, the right hand side is a continuous fuzzy number valued function on $[t_0, T]$ and hence is well defined. Now, consider the following,

$$\begin{aligned}\mathfrak{D}_H(\mathfrak{T}\mathfrak{x}(t), \mathfrak{T}\mathfrak{y}(t)) = \mathfrak{D}_H(x_0 \oplus \frac{1}{\Gamma(\alpha)} \odot \int_{t_0}^{t} (t-\tau)^{\alpha-1} \odot f(\tau, \mathfrak{x}(\tau))d\tau,\\ x_0 \oplus \frac{1}{\Gamma(\alpha)} \odot \int_{t_0}^{t} (t-\tau)^{\alpha-1} \odot f(\tau, \mathfrak{y}(\tau))d\tau),\end{aligned}$$

using properties of the Hausdorff distance and assumptions on f,

$$\begin{aligned}&\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} \mathfrak{D}_H((\tau, \mathfrak{x}(\tau)), f(\tau, \mathfrak{y}(\tau)))d\tau\\ &\leq \frac{L}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} \mathfrak{D}_H(\mathfrak{x}(\tau), \mathfrak{y}(\tau))d\tau,\end{aligned}$$

taking supremum over $[t_0, T]$ both the sides,

$$\begin{aligned}\sup_{[t_0,T]} \mathfrak{D}_H(\mathfrak{T}\mathfrak{x}(t), \mathfrak{T}\mathfrak{y}(t)) &\leq \frac{L}{\Gamma(\alpha)} \int_{t_0}^{t} \sup_{\tau\in[t_0,T]} (t-\tau)^{\alpha-1} \mathfrak{D}_H(\mathfrak{x}(\tau), \mathfrak{y}(\tau))d\tau\\ &\leq \frac{L}{\Gamma(\alpha)} H(\mathfrak{x}, \mathfrak{y}) \left[\int_{t_0}^{t} (t-\tau)^{\alpha-1} d\tau\right]\end{aligned}$$

$$\begin{aligned}H(\mathfrak{T}\mathfrak{x}, \mathfrak{T}\mathfrak{y}) &\leq \frac{L}{\Gamma(\alpha+1)} H(\mathfrak{x}, \mathfrak{y})(t-t_0)^{\alpha}\\ &\leq \frac{L}{\Gamma(\alpha+1)} H(\mathfrak{x}, \mathfrak{y})(T-t_0)^{\alpha}.\end{aligned} \tag{3.2}$$

Now,

$$\begin{aligned}\mathfrak{D}_H(\mathfrak{T}^2\mathfrak{x}(t), \mathfrak{T}^2\mathfrak{y}(t)) &= \mathfrak{D}_H(\mathfrak{T}(\mathfrak{T}\mathfrak{x}(t)), \mathfrak{T}(\mathfrak{T}\mathfrak{y}(t)))\\ &\leq \frac{L}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} \mathfrak{D}_H(\mathfrak{T}\mathfrak{x}(t), \mathfrak{T}\mathfrak{y}(t)),\end{aligned}$$

taking supremum over $[t_0, T]$ both the sides,

$$\begin{aligned}\sup_{[t_0,T]} \mathfrak{D}_H(\mathfrak{T}^2\mathfrak{x}(t), \mathfrak{T}^2\mathfrak{y}(t)) &\leq \frac{L}{\Gamma(\alpha)} \frac{L}{\Gamma(\alpha+1)} \int_{t_0}^{t} (t-\tau)^{\alpha-1}(\tau-t_0)^{\alpha} H(\mathfrak{x},\mathfrak{y})d\tau \\ &= \frac{L^2}{\Gamma(\alpha+1)} H(\mathfrak{x},\mathfrak{y}) \Big(\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1}(\tau-t_0)^{\alpha} d\tau\Big) \\ &= \frac{L^2}{\Gamma(\alpha+1)} H(\mathfrak{x},\mathfrak{y}) \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+\alpha)} (t-t_0)^{\alpha+\alpha} \\ &= \frac{L^2}{\Gamma(2\alpha+1)} (t-t_0)^{2\alpha} H(\mathfrak{x},\mathfrak{y})\end{aligned}$$

$$H(\mathfrak{T}^2\mathfrak{x}, \mathfrak{T}^2\mathfrak{y}) \leq \frac{L^2}{\Gamma(2\alpha+1)} (T-t_0)^{2\alpha} H(\mathfrak{x},\mathfrak{y}).$$

Inductively,

$$H(\mathfrak{T}^n\mathfrak{x}, \mathfrak{T}^n\mathfrak{y}) \leq \frac{L^n}{\Gamma(n\alpha+1)} (T-t_0)^{n\alpha} H(\mathfrak{x},\mathfrak{y}).$$

But since $\dfrac{L^n(T-t_0)^{n\alpha}}{\Gamma(n\alpha+1)} \to 0$ as $n \to \infty$, $\exists n \in \mathbb{N}$, sufficiently large such that $\dfrac{L^n(T-t_0)^{n\alpha}}{\Gamma(n\alpha+1)} < 1$. This implies that $\mathfrak{T}^n$ is a contraction.

Now, Theorem 2.18 implies that $\mathfrak{T}$ has a fixed point which assures the existence of a unique solution of 1.1 by Theorem 2.16.

4 Example

We are considering an example which satisfies the conditions of our Theorem 3.1 and works as an illustration of the Theorem 3.1. For $t \in [0, T]$, we have:

$$ {}_0^C D_t^{\alpha} \mathfrak{x}(t) = \mathbf{p} \odot \mathfrak{x}(t) \ominus_g \mathfrak{x}^2(t), \qquad \mathfrak{x}(0) = [0.1 \ \ 0.5 \ \ 1] \in \mathbb{F}_{\mathbb{R}}, \tag{4.1}$$

here, $\mathbf{p}$ is a singleton fuzzy number. Since,

$$f(t, \mathfrak{x}(t)) = \mathbf{p} \odot \mathfrak{x}(t) \ominus_g \mathfrak{x}^2(t), \ \ \mathfrak{x}(t) \in B([0.1 \ \ 0.5 \ \ 1], \eta)$$

is a continuous function with fuzzy ball $B([0.1 \ \ 0.5 \ \ 1], \eta)$ as its codomain and $[0, T]$ as its domain, where $\eta > 0$ is the radius of the fuzzy ball B with centre $[0.1 \ \ 0.5 \ \ 1]$, then it can be shown that the natural map produced by 4.1 is a contraction and hence it has a unique solution.

5 Conclusion

Under the suitable restrictions on the function f we provide a result that tells when to expect a unique solution for the fuzzy fractional $IVP(1.1)$ using a variant 2.18 of the Banach contraction principle stated in theorem 2.16. An example to support the theory is presented.

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