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EXISTENCE RESULTS FOR SOBOLEV TYPE FUZZY INTEGRODIFFERENTIAL EVOLUTION EQUATION

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Abstract This paper deals with the existence and uniqueness results for sobolev type fuzzy integrodifferential evolution equation with non-local condition. The invented outputs are derived by contraction principle and fuzzy number. Also fuzzy number values are normal, upper semi-continuous, convex and compactly supported interval in $\mathscr{E}_n$. Finally, an example is provided.

1 Introduction

The theory of existence results for non-local evolution equation in Banach spaces has been studied first by Byszewski [4]. Many research papers related to the linear and non-linear integrodifferential models existing in the literature and written by several authors have studied the integrodifferential equations in Banach spaces [19, 20, 21]. The study of sobolev type nonlinear integrodifferential equations [1, 2, 6, 12, 13, 17, 24, 25] and Radhakrishnan et al. [22] studied existence result of sobolev type nonlinear neutral integrodifferential equations.

The fuzzy set theory concept was first introduced by Zadeh [26], which is primarily based on the fact that "all things happening in real world are unstable and unpredictable". The fuzzy theory was put forward and successfully applied in various fields of science and engineering.

Furthermore, there are few research paper on differential equations in a fuzzy context. The notion of metric space of fuzzy sets was established by Diamand et al. [5]. Kaleva[10, 11] focused on fuzzy differential equations in particular. Radhakrishnan [18] established the controllability results for fuzzy neutral integrodifferential systems by using the variations of parameter formula under semigroup theory concept. Sobolev type differential equations under fuzzy concept was studied by Muslim et al [16]. For more study of different type of fuzzy differential equations, one can see[3, 8, 7, 14, 15, 23].

Here the investigation is based on the existence and uniqueness results for Sobolev type fuzzy integrodifferential evolution equation:

$$(\mathscr{B}x)'(\tau) = \mathscr{A}(\tau)x(\tau) + \mathscr{F}\Big(\tau, x_\tau, \int_0^\tau \mathscr{H}(\tau,\mu,x_\mu)d\mu\Big),\ \tau \neq \tau_i,\ \tau \in \mathcal{J} = [0,b], \quad (1.1)$$

$$x(\mu) + [q(x_{r_1},...,x_{r_p})](\mu) = \varphi(\mu),\ \mu \in [-d,0], \quad (1.2)$$

where $\mathscr{A}(t) : \mathscr{E}_n \to \mathscr{E}_n$ is a linear operator, the coefficient of fuzzy is defined as $\mathscr{B} : \mathcal{J} \to \mathscr{E}_n$ where $\mathscr{E}_n$ is the set of all convex, normal and upper semicontinuous fuzzy numbers on $\mathscr{R}$. The non-linear functions $\mathscr{H} : \mathcal{J} \times \mathcal{J} \times \mathscr{E}_n \to \mathscr{E}_n$ and $\mathscr{F} : \mathcal{J} \times \mathscr{E}_n \times \mathscr{E}_n \to \mathscr{E}_n$ are continuous. The non-local function $q : [\mathcal{PC}([-d,0]);\mathscr{E}_n)] \to \mathscr{E}_n$ are given functions. The history x_t represents the function $x_\tau : (-d,0] \to \mathscr{E}_n$ defined by $x_\tau(\theta) = x(\tau+\theta)$; for $\tau \in [0,b]; \theta \in [-d,0]$.

2 Preliminaries

A membership function is used to build a fuzzy subset of $\mathscr{R}^n$ that assigns a membership grade to each $x \in \mathscr{R}^n$. The function of membership is addressed in this way.

$$w : \mathscr{R}^n \ \text{ to the closed interval } [0,1].$$

The assumption that w maps $\mathscr{R}^n$ onto $[0,1]$, that $[w]^0$ is a bounded subset of $\mathscr{R}^n$, that w is fuzzy convex, and that w is upper semicontinuous was used throughout the study. Here $\mathscr{E}_n$ be the space of all fuzzy subsets w of $\mathscr{R}^n$ which are fuzzy convex, normal and upper semicontinuous fuzzy sets with bounded supports. Particularly, $\mathscr{E}^1$ represents the space of all fuzzy subsets w of $\mathscr{R}$.

A fuzzy number c in $\mathscr{R}$ is a fuzzy set characterized by a membership function χ_c is defined from $\mathscr{R}$ to $[0,1]$. Moreover, a fuzzy number c becomes

$$c = \int_{x \in \mathscr{R}} \frac{\chi_c}{x}$$

with $\chi_c(\cdot)$ in the closed interval 0 and 1 that presents the membership grade of $(\cdot)$ in b and the integral sign defines the union of $\frac{\chi_c}{x}$.

Let x in $\mathscr{R}^n$ and $\mathcal{A}$ be a nonempty subsets of $\mathscr{R}^n$. Now the Hausdroff separation of $\mathcal{B}$ from $\mathcal{A}$ is defined by

$$d(x, \mathcal{A}) = \inf\{\|x - a\| : a \in \mathcal{A}\}.$$

Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty subsets of $\mathscr{R}^n$. The Hausdroff separation of $\mathcal{B}$ from $\mathcal{A}$ is defined by

$$\mathcal{H}^d_*(\mathcal{B}, \mathcal{A}) = \sup\{d(b, \mathcal{A}) : b \in \mathcal{B}\}.$$

The Hausdroff distance between $\mathcal{A}$ and $\mathcal{B}$ of $\mathscr{R}^n$ is defined by

$$\mathcal{H}^d(\mathcal{A}, \mathcal{B}) = \max\{\mathcal{H}^d_*(\mathcal{A}, \mathcal{B}), \mathcal{H}^d_*(\mathcal{B}, \mathcal{A})\}.$$

It is symmetric in $\mathcal{A}$ and $\mathcal{B}$. Consequently,

(a) $\mathcal{H}^d(\mathcal{A}, \mathcal{B}) \geq 0$ with $\mathcal{H}^d(\mathcal{A}, \mathcal{B}) = 0$ if and only if $\overline{\mathcal{A}} = \overline{\mathcal{B}}$;

(b) $\mathcal{H}^d(\mathcal{A}, \mathcal{B}) = \mathcal{H}^d(\mathcal{B}, \mathcal{A})$;

(c) $\mathcal{H}^d(\mathcal{A}, \mathcal{B}) \leq \mathcal{H}^d(\mathcal{A}, \mathcal{C}) + \mathcal{H}^d(\mathcal{C}, \mathcal{B})$;

for any nonempty subsets of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ of $\mathscr{R}^n$.

The supremum metric $\mathcal{H}^1$ on $C(\mathcal{J}, \mathscr{E}_n)$ is defined by

$$\mathcal{H}^1(x, y) = \sup\{d^\infty(x(\tau), y(\tau) : \tau \in \mathcal{J}), \text{ for all } x, y \in \mathcal{C}(\mathcal{J}, \mathscr{E}_n)\}.$$

Fuzzy Solution Operators: We adopt the general definition and theorem of operator theory on $\mathscr{E}_n$ in [15]. If $\mathscr{A} : \mathscr{E}_n \to \mathscr{E}_n$ is linear, then

$$\begin{aligned} \mathscr{A}(\tau + \mu) &= \mathscr{A}(\tau) + \mathscr{A}(\mu) \\ \mathscr{A}(\lambda\tau) &= \lambda\mathscr{A}(\tau), \end{aligned} \tag{2.1}$$

for all $\tau, \mu \in \mathcal{J} = [0, b], \lambda \in \mathscr{R}$.

The infinitesimal generator $\{\mathscr{A}(t) : 0 \leq t \leq b\}$ generates a two parameter family of operators $\{\mathscr{U}(t, s) : 0 \leq s \leq t \leq b\}$ and its various properties can be seen in ([9]).

Definition 2.1. Let $x(\cdot) \in \mathcal{PC}([-d, a], \mathscr{E}_n)$ is called a mild solution of $(1.1) - (1.2)$ if $x(\mu) + [q(x_{r_1}, ..., x_{r_p})](\mu) = \varphi(\mu), \mu \in [-d, 0]$; the restriction of $x(\cdot)$ to the interval $\mathcal{J}$, is continuous and the following conditions are satisfied:

$$\begin{aligned} (i)\ \ x(\tau) &= \mathscr{B}^{-1}\mathscr{U}(\tau, 0)\mathscr{B}\varphi(0) - \mathscr{U}(\tau, 0)[q(x_{r_1}, ..., x_{r_p})](0) \\ &\quad + \int_0^\tau \mathscr{U}(\tau, \mu)\mathscr{B}\mathscr{F}\Big(\mu, x_\mu, \int_0^\mu \mathscr{H}(\mu, \tau, x_\tau)d\mu\Big)d\tau, \quad \tau \in [0, a], \end{aligned}$$

$$(ii)\ \ x(\mu) + [q(x_{r_1}, ..., x_{r_p})](\mu) = \varphi(\mu), \quad \mu \in [-d, 0).$$

To determine the existence results, the following assumptions are required:

(H1) In $\mathscr{E}_n$, the operator $\mathscr{A}(t)$ yields a two parameter family of operators $\mathscr{U}(\gamma, \nu)$ and $\exists$ a constant $M > 0$, such that

$$\|\mathscr{U}(\gamma, \nu)\| \leq M, \text{ for } 0 \leq \gamma \leq \nu \leq a.$$

(H2) $f : \mathcal{J} \times \mathscr{E}_n \times \mathscr{E}_n \to \mathscr{E}_n$ is continuous and $\exists$ constants $L_B > 0,\ L_0 > 0$, such that

$$\mathcal{H}^d\Big(\Big[\mathscr{F}(\mu, x_\mu, u_\mu)\Big]^\beta, \Big[\mathscr{F}(\mu, y_\mu, v_\mu)\Big]^\beta\Big) \leq L_B(\mathcal{H}^d(x,y)+\mathcal{H}^d(u,v)),\ \text{for } x,y,u,v \in \mathscr{E}_n,\ \mu \in \mathcal{J},$$

$$L_0 = \max_{\mu \in \mathcal{J}} \|\mathscr{F}(\mu, 0, 0)\|.$$

(H3) For each $\mu, \tau \in \mathcal{J}$, $\mathcal{H} : \mathcal{J} \times \mathcal{J} \times \mathscr{E}_n \to \mathscr{E}_n$ is continuous and $\exists$ constants $N_B > 0$ and $N_0 > 0$, such that

$$\mathcal{H}^d\Big(\Big[\int_0^\nu \mathscr{H}(\mu,\tau,u_\mu)d\mu\Big]^\beta, \Big[\int_0^\nu \mathscr{H}(\mu,\tau,v_\mu)d\mu\Big]^\beta\Big) \leq N_B\|u-v\|\ ,\ \text{for } u,v \in \mathscr{E}_n,\ \nu,\mu \in \mathcal{J},$$

$$N_0 = \max\{\int_0^\nu \|\mathscr{H}(\nu,\mu,0)\|ds : \nu,\mu \in \Omega\}.$$

(H4) $q : [\mathcal{PC}([-d,0],\mathscr{E}_n)]^p \to \mathscr{E}_n$ is continuous and $\exists$ a constant $G_B > 0$, such that

$$\mathcal{H}^d\Big(\Big[[q(u_{r_1},...,u_{r_p}](\mu)\Big]^\beta, \Big[[q(v_{r_1},...,v_{r_p}](\mu)\Big]^\beta\Big) \leq G_B\mathcal{H}^d(u,v),$$

for each $u,\ v \in \mathcal{PC}([-d,a],\mathscr{E}_n), \mu \in [-d,0]$,

$$G_0 = \max\{\|[q(u_{r_1},...,u_{r_p}](\mu)\| : u,v \in \mathcal{PC}([-d,a],\mathscr{E}_n), \mu \in [-d,0]\}.$$

(H5) $\exists$ a positive constant $\delta > 0$ such that

$$M(1+bM)\Big[\|\varphi(0)\| + G_0 + b\{L_B[N_0 + (1+N_B)r] + L_0\}\Big] \leq \delta.$$

Moreover, let us put

$$\gamma = M(1+bM)[G_B + bL_B + bN_B].$$

3 Fuzzy Sobolev Type Integrodifferential Equation

Consider the following non-local Sobolev type nonlinear fuzzy integrodiferential evolution equation

$$(\mathscr{B}x)'(\tau) = \mathscr{A}(\tau)x(\tau) + \mathscr{F}\Big(\tau, x_\tau, \int_0^\tau \mathscr{H}(\tau,\mu,x_\mu)d\mu\Big),\ \tau \neq \tau_i,\ \tau \in \mathcal{J}, \tag{3.1}$$

$$x(\mu) + [q(x_{r_1},...,x_{r_p})](\mu) = \varphi(\mu),\ \mu \in [-d,0], \tag{3.2}$$

In this case, we used the operator $\mathscr{E}$ on $\mathscr{E}_n$, which is given by

$$\mathscr{E} = \Big[I + q(x_{r_1},...,x_{r_p})(\mu)\mathscr{B}^{-1}\mathscr{U}(\tau_i,0)\mathscr{B}\Big]^{-1}$$

with

$$\mathscr{E}\Big\{\int_0^{\tau_i} \mathscr{B}^{-1}\mathscr{U}(\tau_i,\mu)\Big[\mathscr{F}(\mu, x_\mu(\mu), \int_0^\mu \mathscr{H}(\mu,\tau,x(\tau))d\tau)\Big]d\mu\Big\} \in \mathscr{E}_n.$$

Definition 3.1. A fuzzy solution x in the form of integral equation

$$\begin{aligned} x(\tau) &= \mathscr{B}^{-1}\mathscr{U}(\tau,0)\mathscr{B}\mathscr{E}\phi(0) - q(x_{r_1},...,x_{r_p})(\mu)\mathscr{B}^{-1}\mathscr{U}(\tau,0)\mathscr{B}\mathscr{E} \\ &\times \Big\{\mathscr{B}^{-1}\int_0^{\tau_i} \mathscr{B}^{-1}\mathscr{U}(\tau_i,\mu)\mathscr{F}(\mu,x_\mu,\int_0^\mu \mathscr{H}(\mu,\tau,x_\tau)d\tau)d\mu\Big\}) \\ &+ \int_0^\tau \mathscr{B}^{-1}\mathscr{U}(\tau,\mu)\mathscr{F}(\mu,x_\mu,\int_0^\mu \mathscr{H}(\mu,\tau,x_\tau)d\tau)d\mu \end{aligned} \tag{3.3}$$

is said to be a solution of (3.1)-(3.2) on J.

Remark: 3.1. A fuzzy solution of the sobolev type fuzzy integrodifferential equation (3.1)-(3.2) satisfies the condition (3.2)-(3.3).

$$\begin{aligned} x(\mu) &= \mathscr{E}\phi(\mu) - q(x_{r_1},...,x_{r_p})(\mu)\mathscr{E}\Big\{\int_0^{\tau_i}\mathscr{U}(\tau_i,\mu)\mathscr{B}^{-1}\mathscr{F}\Big(\tau,x_\tau,\int_0^{\mu}\mathscr{H}(\mu,x_\mu)d\mu\Big)d\tau_i\Big\} \\ &+ q(x_{r_1},...x_{r_p})\mathscr{E}\Big\{\mathscr{F}\Big(\mu,x_\tau,\int_0^{\mu}\mathscr{H}(\tau,\mu,x_\tau)d\tau\Big)d\mu\Big\} \end{aligned}$$

and

$$\begin{aligned} x(\tau_j) &= \mathscr{B}^{-1}\mathscr{U}(\tau_j,0)\mathscr{B}\mathscr{E}\phi(0) + q(x_{r_1},...x_{r_p})\mathscr{B}^{-1}\mathscr{U}(\tau_j,0)\mathscr{B}\mathscr{E}\mathscr{F}(\tau,x_\tau,\int_0^{\mu}\mathscr{H}(\tau,\mu,x_\mu)d\mu)d\tau \\ &+ \int_0^{\tau_j}\mathscr{U}(\tau_j,\mu)\mathscr{B}^{-1}\mathscr{F}\Big(\tau,x_\tau,\int_0^{\mu}\mathscr{H}(\tau,\mu,x_\mu)d\mu\Big)d\tau. \end{aligned}$$

Therefore,

$$\begin{aligned} x(\mu) + q(x_{r_1},...,x_{r_p})(\mu) &= \Big[I + q(x_{r_1},...,x_{r_p})\mathscr{B}^{-1}\mathscr{U}(\tau_j,0)\mathscr{B}\Big]\mathscr{E}\phi(0)\Big] \\ &+ \Big[I + q(x_{r_1},...,x_{r_p})\mathscr{B}^{-1}\mathscr{U}(\tau_j,0)\mathscr{B}\Big] \\ &\times g\mathscr{E}\int_0^{\tau_j}\mathscr{U}(\tau_j,\mu)\mathscr{B}^{-1}\mathscr{F}\Big(\tau,x_\tau,\int_0^{\mu}\mathscr{H}(\tau,\mu,x_\mu)d\mu\Big)d\tau \\ &= \phi(\mu). \end{aligned}$$

Theorem: 3.2. If $(H1)$-$(H5)$ hold, then (3.1)-(3.2) has a fuzzy solution on $\mathcal{J}$.

*Proof.*Let $\mathcal{F}$ be a subset of $\mathcal{C}(J,\mathscr{E}_n)$ defined as follows:

$$\mathcal{F} = \{x : x(\tau) \in \mathscr{E}_n, \|x(\tau)\| \le \rho, \text{ for } \tau \in \mathcal{J}\}.$$

A mapping $\tilde{\zeta} : \mathcal{F} \to \mathcal{F}$ is defined by

$$\begin{aligned} (\tilde{\zeta}x)(\tau) &= \mathscr{B}^{-1}\mathscr{U}(\tau,0)\mathscr{B}\mathscr{E}\phi(0) - q(x_{r_1},...,x_{r_p})(\mu)\mathscr{B}^{-1}\mathscr{U}(\tau,0)\mathscr{B}\mathscr{E} \\ &\times \Big\{\mathscr{B}^{-1}\int_0^{\tau_i}\mathscr{B}^{-1}\mathscr{U}(\tau_i,\mu)\mathscr{F}\Big(\mu,x_\mu,\int_0^{\mu}\mathscr{H}(\tau,\mu,x_\tau)d\tau\Big)d\mu\Big\} \\ &+ \int_0^{\tau}\mathscr{B}^{-1}\mathscr{U}(\tau,\mu)\mathscr{F}\Big(\mu,x_\mu,\int_0^{\mu}\mathscr{H}(\tau,\mu,x_\tau)d\tau\Big)d\mu. \end{aligned}$$

Initially prove that the operator $\tilde{\zeta}$ maps $\mathcal{F}$ into itself.

$$\begin{aligned} |(\tilde{\zeta}x)(\tau)| = \; & |\mathscr{B}^{-1}\mathscr{U}(\tau,0)\mathscr{B}\mathscr{E}\phi(0) - q(x_{r_1},...,x_{r_p})(0)\mathscr{B}^{-1}\mathscr{U}(\tau,0)\mathscr{B}\mathscr{E} \\ & \times \Big\{\mathscr{B}^{-1}\int_0^{\tau_i}\mathscr{B}^{-1}\mathscr{U}(\tau_i,\mu)\mathscr{F}\Big(\mu,x_\mu,\int_0^{\mu}\mathscr{H}(\tau,\mu,x_\tau)d\tau\Big)d\mu\Big\} \\ & + \int_0^{\tau}\mathscr{B}^{-1}\mathscr{U}(\tau,\mu)\mathscr{F}\Big(\mu,x_\mu,\int_0^{\mu}\mathscr{H}(\mu,\tau,x_\tau)d\tau\Big)d\mu| \\ \le \; & M|\phi(0)| + MG_0 + M\int_0^{b}|\mathscr{F}(\mu,x_\mu,\int_0^{\mu}\mathscr{H}(\mu,\tau,x_\tau)d\tau)d\mu|. \end{aligned}$$

Since from assumptions (H2) and (H3), we have

$$\begin{aligned}
\|\mathscr{F}(\mu, x_\mu, \int_0^\mu \mathscr{H}(\mu,\tau,x_\tau)d\tau)\| \leq \quad & |\mathscr{F}\Big(\mu, x_\mu, \int_0^\mu \mathscr{H}(\mu,\tau,x_\tau)d\tau\Big) - \mathscr{F}(\mu,0,0) + \mathscr{F}(\mu,0,0)| \\
& \leq \|\mathscr{F}\Big(\mu, x_\mu, \int_0^\mu \mathscr{H}(\mu,\tau,x_\tau)d\tau\Big) - \mathscr{F}(\mu,0,0)\| + \|\mathscr{F}(\mu,0,0)\| \\
& \leq L_B\|x_s\| + \int_0^\tau \|\mathscr{H}(\mu,\tau,x_\tau)d\tau\| + L_0 \\
& \leq L_B\|x_s\| + \int_0^\tau \|\mathscr{H}(\mu,\tau,x_\tau) - \mathscr{H}(\mu,\tau,0)\|d\tau \\
& + \int_0^\mu |\mathscr{H}(\mu,\tau,0)\|d\tau + L_0 \\
& \leq L_B[N_0 + (1+N_B)|x_s|] + L_0
\end{aligned}$$

there holds

$$|\tilde{\zeta}x(\tau)| \leq M|\phi(0)| + MG_0 + bML_B[N_0 + (1+N_B)|x_s| + L_0].$$

From (H5), one gets $|\tilde{\zeta}x(\tau)| \leq \rho$, therefore $\tilde{\zeta}$ maps $\mathcal{F}$ into itself.

$$\begin{aligned}
& \mathcal{H}^d\Big([(\tilde{\zeta}x)(\tau)]^\beta, [(\tilde{\zeta}y)(\tau)]^\beta\Big) \\
& = \mathcal{H}^d\Big(\Big[\mathscr{B}^{-1}\mathscr{U}(\tau,0)\mathscr{B}\mathscr{E}\phi(0) - q(x_{r_1},...,x_{r_p})(\mu)\mathscr{B}^{-1}\mathscr{U}(\tau,0)\mathscr{B}\mathscr{E} \\
& \times \Big\{\mathscr{B}^{-1}\int_0^{\tau_i} \mathscr{B}^{-1}\mathscr{U}(\tau_i,\mu)\mathscr{F}(\mu,x_\mu,\int_0^\mu \mathscr{H}(\tau,x_\tau)d\tau)d\mu\Big\}) \\
& + \int_0^\tau \mathscr{B}^{-1}\mathscr{U}(\tau,\mu)\mathscr{F}(\mu,x_\mu,\int_0^\mu \mathscr{H}(\tau,x_\tau)d\tau)d\mu\Big]^\beta, \\
& \Big[\mathscr{B}^{-1}\mathscr{U}(\tau,0)\mathscr{B}\mathscr{E}\phi(0) - q(y_{r_1},...,y_{r_p})(\mu)\mathscr{B}^{-1}\mathscr{U}(\tau,0)\mathscr{B}\mathscr{E} \\
& \times \Big\{\mathscr{B}^{-1}\int_0^{\tau_i} \mathscr{B}^{-1}\mathscr{U}(\tau_i,\mu)\mathscr{F}(\mu,y_\mu,\int_0^\mu \mathscr{H}(\tau,y_\tau)d\tau)d\mu\Big\}) \\
& + \int_0^\tau \mathscr{B}^{-1}\mathscr{U}(\tau,\mu)\mathscr{F}(\mu,y_\mu,\int_0^\mu \mathscr{H}(\tau,y_\tau)d\tau)d\mu\Big]^\beta\Big) \\
& \leq M(1+bM)[|\phi(0)| + G_0 + M[L_B[N_0 + (1+N_B)r + L_0]\mathcal{H}^d([x(\tau)]^\alpha, [y(\tau)]^\alpha).
\end{aligned}$$

Let $M(1+bM)[|\phi(0)| + G_0 + M[L_B[N_0 + (1+N_B)r + L_0] = \Delta$. Now

$$\begin{aligned}
\mathcal{H}^d\Big([(\tilde{\zeta}x)(\tau)]^\beta, [(\tilde{\zeta}y)(\tau)]^\beta\Big) &= \Delta\mathcal{H}^d([x(\tau)]^\beta, [y(\tau)]^\beta) \\
d^\infty([\tilde{\zeta}x(\tau)]^\beta, [\tilde{\zeta}y(\tau)]^\beta) &\leq \sup_{\beta\in(0,1)} \mathcal{H}^d([\tilde{\zeta}x(\tau)]^\beta, [\tilde{\zeta}y(\tau)]^\beta) \\
&\leq \sup_{\beta\in(0,1)} \Delta\mathcal{H}^d([x(\tau)]^\beta, [y(\tau)]^\beta) \\
d^\infty([\tilde{\zeta}x(\tau)]^\beta, [\tilde{\zeta}y(\tau)]^\beta) &\leq \Delta d^\infty(x(\tau), y(\tau)).
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathcal{H}^1([\tilde{\zeta}x(\tau)]^\beta, [\tilde{\zeta}y(\tau)]^\beta) &\leq \sup_{\beta\in(0,1)} d^\infty([\zeta x(\tau)]^\beta, [\zeta y(\tau)]^\beta) \\
&\leq \Delta\mathcal{H}^1(x(\tau), y(\tau)).
\end{aligned}$$

Since $\Delta < 1$, from the above it is easy to observe that the operator $\tilde{\zeta}$ is contraction on $\mathscr{E}_n$ and as a result of the Banach contraction principle, there exists a single fixed point $x \in \mathcal{F}$ where $(\tilde{\zeta}x)(\tau) = x(\tau)$. The solution to the problem (3.1)-(3.2) is then this fixed point.

4 Case Study

Consider the fuzzy neutral indegrodifferential equation with nonlocal condition

$$(x(\eta+h))' = \tilde{\mathbf{2}}[x(\eta+h)+\tau x^2(\eta+h)]+\tilde{\mathbf{3}}\eta x(\eta+h)^2 \quad (4.1)$$

$$x(\eta) = \sum_{i=1}^{n} c_i x((\eta_k)). \quad (4.2)$$

Let $\mathscr{F}(\eta,x_\eta)=\tilde{\mathbf{2}}x^2(\eta+h), \int_0^\eta \mathscr{K}(\eta,\mu,x_\eta)d\mu=\tilde{\mathbf{3}}\eta x^2(\eta+h)$.

The β level set of fuzzy numbers

$$\tilde{\mathbf{2}} : [2]^\beta = [\beta+1, 3-\beta];$$

$$\tilde{\mathbf{3}} : [3]^\beta = [\beta+2, 4-\beta];$$

Now β level set of functions are

$$\begin{aligned}[\mathscr{F}(\eta,x_\eta)]^\beta &= [\tilde{\mathbf{2}}\eta x^2(\eta+h)]^\beta\\ &= \eta\Big[(\beta+1)x_l^\beta(\eta+h)^2,(2-\beta)x_r^\beta(\eta+h)^2\Big].\end{aligned}$$

$$\begin{aligned}[\int_0^\eta \mathscr{H}(\eta,\mu,x(\mu))d\mu]^\beta &= [\tilde{\mathbf{3}}\eta x(\eta)^2]^\beta\\ &= \eta\Big[(\beta+2)x_l^\beta(\eta)^2,(4-\beta)x_r^\beta(\eta)^2\Big].\end{aligned}$$

The β- level set of $\sum_{i=1}^n c_i x(\eta_i)$: $\Big[\sum_{i=1}^n c_i x_l^\beta(\eta_i)\Big]^\beta = \Big[\sum_{i=1}^n c_i x_l^\beta(\eta_i), \sum_{i=1}^n c_i x_r^\beta(\eta_i)\Big]$

$$\begin{aligned}\mathcal{H}^d([\mathscr{F}(\eta,x_\eta)]^\beta,[\mathscr{F}(\eta,y_\eta)]^\beta) &= H^d(\eta[(\beta+1)(x_l^\beta(\eta)),(3-\beta)(x_r^\beta(\eta))],\\ &\quad \eta[(\beta+1)(y_l^\beta(\eta+h)),(3-\beta)(y_r^\beta(\eta+h))])\\ &= \eta\max\{(\beta+1)|(x_l^\beta(\eta))-(y_l^\beta(\eta))|,(3-\beta)|(x_r^\beta(\eta))-(y_r^\beta(\eta))|\}\\ &= \eta\max\{(\beta+1)|x_l^\beta(\eta+h)+y_l^\beta(\eta+h)||x_l^\beta(\eta)-y_l^\beta(\eta)|,\\ &\quad (3-\beta)|x_r^\beta(\eta+h))+(y_r^\beta(\eta))||x_r^\beta(\eta+h)-y_r^\beta(\eta+h)|\}\\ &\leq (3-\beta)t|x_r^\beta(\eta+h))+y_r^\beta(\eta+h)|\\ &\times \max\{|x_l^\beta(\eta+h)-y_r^\beta(\eta+h)|,|x_r^\beta(\eta+h)-y_r^\beta(\eta+h)|\}\\ &\leq (3-\beta)b|x_r^\beta(\eta))+y_r^\beta(\eta+h)|\\ &\times \max\{|x_l^\beta(\eta+h)-y_l^\beta(\eta+h)|,|x_r^\beta(\eta)-y_r^\beta(\eta)|\}\\ &\leq 3b|x_r^\beta(\eta))+y_r^\beta(\eta)|\max\{|x_l^\beta(\eta)-y_l^\beta(\eta)|,|x_r^\beta(\eta)-y_r^\beta(\eta)|\}\\ &= L_B\mathcal{H}^d([x(\eta+h)]^\beta,[y(\eta+h)]^\beta),\end{aligned}$$

where $L_B=3b|x_r^\beta(\eta+h))+y_r^\beta(\eta+h)|$ fulfills the inequality specified in condition $(H2)$.

$$\begin{aligned}\mathcal{H}^d([\sum_{i=1}^n c_i x(\eta_i)]^\beta,[\sum_{i=1}^n c_i y(\eta_i)]^\beta) &= H^d\Big(\Big[\sum_{i=1}^n c_i x_l^\beta(\eta_i),\sum_{i=1}^n c_i x_r^\beta(\eta_i)\Big],\Big[\sum_{i=1}^n c_i y_l^\beta(\eta_i),\sum_{i=1}^n c_i y_r^\beta(\eta_i)\Big]\Big)\\ &\leq G_B\mathcal{H}^d([x(\eta_i]^\beta,y(\eta_i]^\beta),\end{aligned}$$

where $G_B = |\sum_{i=1}^{n} c_i|$ satisfies the inequality which is given in condition $(H4)$.

$$\begin{aligned}
\mathcal{H}^d([\int_0^\eta \mathscr{H}(\eta,\mu,x_\mu)]^\beta,[\int_0^\eta \mathscr{K}(\eta,\mu,x_\mu)]^\beta) = \quad & \mathcal{H}^d(\eta[(\beta+2)(x_l^\beta(\eta)^2),(4-\beta)(x_r^\beta(\eta)^2)],\\
& \eta[(\beta+1)(y_l^\beta(\eta)^2),(3-\beta)(y_r^\beta(\eta))^2])\\
= \quad & \eta \max\{(\beta+2)|(x_l^\beta(\eta))-(y_l^\beta(\eta))|,\\
& (4-\beta)|(x_r^\beta(\eta))-(y_r^\beta(\eta))|\}\\
\leq \quad & 4bH^d([x(\eta_i)]^\beta,[y(\eta_i)]^\beta) = N_B\mathcal{H}^d([x(\eta_i)]^\beta,[y(\eta_i)]^\beta),
\end{aligned}$$

where $N_B = 4b|x_r^\beta(\eta)) + y_r^\beta(\eta)|$, and there by the condition $(H3)$ which satisfies the inequality.

Thus all conditions stated in theorem 3.2 are addressed. Therefore the system (4.1)-(4.2) has a unique fuzzy solution.

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