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# NUMERICAL METHOD OF ABEL'S TYPE INTEGRAL EQUATIONS WITH EULER'S OPERATIONAL MATRIX OF INTEGRATION

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**Abstract**: In this paper, we describe a new numerical approach for the solution of integral equations with Abel type singularity based on normalized Euler polynomials. Euler's operational matrix of integration is introduced and the Euler polynomials have been used to obtain it. This polynomial is orthonormalized first, and then their operational matrix of integration is obtained. The orthogonality approach is used to convert integral equations into a set of algebraic equations in which it can be calculated fast. We demonstrated the accuracy and utility of the suggested operational matrix of integration using numerical examples. A comparative study with other methods reflects that the proposed algorithm has good degree of accuracy and also easy to compute.

## 1 Introduction

Polynomial series and orthogonal functions have been studied extensively in the context of dynamic systems. Since a long time, approximations based on the orthonormal series of functions have been applied to solve problems in mathematical modelling, simulation, as well as engineering and technology. They have played an essential role in the evaluation of novel strategies to tackle issues such as identification, differential equations, integral equations, and optimum control since previous four decades. The key feature of this technique is that it simplifies these problems by reducing them to a set of algebraic equations.

The main concept behind this method is to use several integrations to convert a differential equation into an integral equation. Following that, the various parameters of the integral equations are then determined by defining them with linear combinations of the orthonormal basis functions and modifying it to the desired accuracy. The integral equation is finally transformed into an algebraic equation through inserting the operational matrix of integration of the basis functions.

Singular integral equations are generally useful in physics and theoretical mechanics, especially in the fields of aerodynamics, elasticity and unstable aerofoil theory. Abel derived the Abel integral equation in 1823 while making assumptions and solving the Tautochrone issue. It makes things simpler to figure out how long it takes for a particle to drop along a certain curve. This integral equation is written as follows:

$$q(r)=\int_0^x \frac{\zeta(t)}{\sqrt{r-t}}dt, \quad 0\le r\le 1 \tag{1.1}$$

$q(r)$ denotes a known function, whereas $\zeta(t)$ represents an unknown function. The exact solution is given as follows:

$$\zeta(r)=\frac{1}{\pi}\int_0^r \frac{\zeta(t)}{\sqrt{r-t}}\frac{dq(t)}{d(t)}dt, \quad 0\le r\le 1. \tag{1.2}$$

It is simply assumed that $q(0)=0$, without limiting consistency.

Abel integral equations can be found in many scientific fields such as electron radiation, atomic scattering, X-ray radiography, radar coverage, plasma diagnostics, seismology, radio astronomy and optical fibre estimation. Abel's equation is an integral equation that is created automatically from a mechanics or physics problem (but without the use of a differential equation). The very first example of an integral equation [10] is Abel's integral equation. Orthogonal functions or polynomials, such as Fourier series, Block Pulse functions, Walsh functions, Chebyshev polynomials, Legendre polynomials, and Laguerre polynomials, have been mostly utilised to estimate solutions of some systems, including the differential equations, integral equations and integro-differential equations, in recent decades. Several numerical different approaches for estimating the solution of integral equations using the polynomials stated above are discussed.

The Tau approach [16], Haar Wavelets operational matrix [6], iteration variation method [31],[28], homotopy perturbation method [30],[8], Sine-Cosine Wavelets method [19], radial basis functions method [11], collocation method [17], Homotopy analysis method [1],[12], Legendre matrix method [20],[21], Homotopy analysis transform method [22], Bernoulli Wavelet method [29, 25], Legendre Wavelets operational matrix [18], Wavelet Galerkin method [23], Bernstein polynomials method [32], were used to analyze the differential, integral, and integro-differential equations. Iterative approaches are used by [7] to transform an integral equation into another system of nonlinear equations that must be easily solved. Baratella and Orsi [2] introduced a numerical solution of weakly singular Volterra integral equations. Singh et al. [24] described and developed a numerical solution to singular Volterra integral equations of Abel type using the Bernstein polynomials. For solving fractional integro-differential equations with weakly singular kernels, Wang et al. [27] proposed Fractional-order Euler functions.

In this article, an operational matrix of integration based on the Euler polynomial is derived. This operational matrix is generated after the Euler polynomials have been orthonormalized. We will solve Abel's integral equation (1.1), commonly known as the singular Volterra integral equation of first kind, using the formulated operational matrix. We will also solve Volterra's second-order integral equation, which is shown below:

$$\zeta(r) = q(r) + \int_0^r \frac{\zeta(t)}{\sqrt{r-t}} dt, \quad 0 \leq r \leq 1 \tag{1.3}$$

where $q(r)$ is in $L^2(R)$, in the range $0 \leq r \leq 1$.

## 2 The Euler polynomials

Euler polynomials and numbers, first presented in 1740 by Euler, have unique features and applications in domains such as number theory, differential geometry, analysis and algebraic topology [34]. They are closely related to Bernoulli's polynomial theory in several ways [9], [13]. The exponential generating functions are commonly used to determine the classical Euler polynomial $E_n(t)$ (see [5] for further details):

$$\frac{2e^{rt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(r)\frac{t^n}{n!}, \quad (|t| \leq \pi) \tag{2.1}$$

The explicit representation of well-known nth-degree Euler polynomials is written as:

$$E_n(r) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} \left(r - \frac{1}{2}\right)^{n-k}, \quad n \in \mathbb{N} \tag{2.2}$$

Where $E_k$ is Euler's number for $k = 0, 1, 2, \cdots, n$ . Conversely, the Euler numbers are expressed with the Euler polynomials by

$$E_k = 2^k E_k\left(\frac{1}{2}\right)$$

Numerous useful and applicable characteristics and associations using these polynomials and numbers can be further used also in these references [5], [9], [13].

The first Euler polynomial is $E_0(r) = 1$ and the next five which are used in this paper, are listed as follows

$$E_1(r) = r - \frac{1}{2},\ E_2(r) = r^2 - r,\ E_3(r) = r^3 - \frac{3}{2}r^2 + \frac{1}{4}$$

$$E_4(r) = r^4 - 2r^3 + r,\ E_5(r) = r^5 - \frac{5}{2}r^4 + \frac{5}{2}r^2 - \frac{1}{2}$$

Euler polynomials satisfy the following interesting properties [14][15],[24].

$$E'_n(r) = nE_{n-1}(r), \quad n = 1, 2, \cdots \tag{2.3}$$

$$\sum_{k=0}^{n} \binom{n}{k} E_k(r) + E_n(r) = 2r^n \tag{2.4}$$

$$E_n(1-r) = (-1)^n E_n(r) \tag{2.5}$$

$$E_n(r+y) = \sum_{k=0}^{n} \binom{n}{k} E_k(r) y^{n-k} \tag{2.6}$$

$$E_n(r) = \frac{1}{n+1} \sum_{k=1}^{n+1} (2 - 2^{k+1}) \binom{n+1}{k} B_k(0) r^{n+1-k} \tag{2.7}$$

where $B_k(r),\ k = 0, 1, 2, \cdots$ is the order $k$ Bernoulli's polynomial, which is denoted as:

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k(r) = (n+1)r^n.$$

The behavior of certain Euler polynomials in the interval is shown in Figure (1).

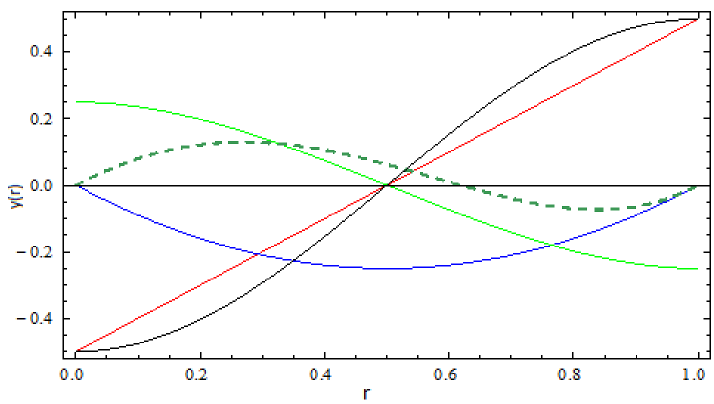


**Figure 1.** Behavior of the first five Euler polynomials.

## 3 The orthonormal Euler's polynomials:

We generate a family of orthonormal polynomials by using the Gram-Schmidt orthonormalization procedure to $E_n$ and normalizing it.They are known as orthonormal Euler polynomials of order $n$ and represented by $E_0, E_1, \cdots, E_n$.

The five orthonormal polynomials for $n = 5$ are listed as below:

$$\begin{cases} E_0(t) = 1, \\ E_1(t) = \sqrt{12}\left(t - \frac{1}{2}\right), \\ E_2(t) = \sqrt{180}\left(t^2 - t + \frac{1}{6}\right) \\ E_3(t) = \sqrt{2800}\left(t^3 - \frac{3}{2}t^2 + \frac{3}{5}t - \frac{1}{20}\right), \\ E_4(t) = \sqrt{44100}\left(t^4 - 2t^3 + \frac{9}{7}t^2 - \frac{2}{7}t + \frac{1}{70}\right) \\ E_5(t) = \sqrt{698544}\left(t^5 - \frac{5}{2}t^4 + \frac{20}{9}t^3 - \frac{5}{6}t^2 + \frac{5}{42}t - \frac{1}{252}\right). \end{cases} \tag{3.1}$$

## 4 Function approximation

A function $q \in L^2[0,1]$ can be expressed as

$$q(t) = \lim_{n\to\infty} \sum_{k=0}^{n} C_k E_k(t), \tag{4.1}$$

Where, $C_k =< q, E_k >$ and $<,>$ is the normal inner product with $L^2[0,1]$.

If series (4.1) is truncated at $n = m$ then we get

$$q(t) \simeq \sum_{k=0}^{n} C_k E_k = C^T E(t), \tag{4.2}$$

where, $C$ and $E(t)$ are $(m+1)\times 1$ matrices given by

$$C = [C_0, C_1, \cdots, C_m]^T \tag{4.3}$$

and

$$E(t) = [E_0(t), E_1(t), \cdots, E_m(t)]^T. \tag{4.4}$$

## 5 Solution of Abel's integral equation

We used orthonormal Euler polynomials in this section to evaluate Abel's integral equation (1.1) and singular Volterra integral equations (1.3).

Using equation (4.1), we approximate $\zeta(r)$ and $q(r)$ as follows:

$$\zeta(r) = C^T E(r),\ q(r) = G^T E(r) \tag{5.1}$$

where, the matrix $G$ can be obtained by approximating $q(r)$ in terms of the basis given in equation (3.1). Then from equations (1.1), (1.3) and (5.1) we have

For the first kind:

$$G^T E(r) = \int_0^r \frac{C^T E(t)}{\sqrt{r-t}} dt \tag{5.2}$$

and For the second kind:

$$C^T E(r) = G^T E(r) + \int_0^r \frac{C^T E(t)}{\sqrt{r-t}} dt. \tag{5.3}$$

The orthonormal Euler's polynomials operational matrix of order $(m+1)\times(m+1)$ will be derived now. In order to obtain this, consider the following integral:

$$\begin{aligned}\zeta(r) =& \int_0^r \frac{E_m(t)}{\sqrt{r-t}} dt, \quad 0 \le r \le 1\\ =& \sum_{k=0}^{n} C_k E_k(t),\\ =& [C_0, C_1, \cdots, C_m] E(t).\end{aligned} \tag{5.4}$$

Using equations (4.4) and (5.4), we obtain

$$\int_0^r \frac{E_m(t)}{\sqrt{r-t}} dt = P_{m+1} E(r). \tag{5.5}$$

Here $P_{m+1}$ is an operational matrix for singular Volterra integral equations with Abel kernel, which we refer to as the Euler's operational matrix of integration.

The matrix $P_6$ is given by $P$ and is presented as follows for $m = 5$ :

$$P = \begin{bmatrix} \frac{4}{3} & \frac{4}{5\sqrt{3}} & -\frac{4}{21\sqrt{5}} & \frac{4}{45\sqrt{7}} & -\frac{4}{231} & \frac{4}{117\sqrt{11}} \\ -\frac{4}{5\sqrt{3}} & \frac{4}{7} & \frac{4}{3\sqrt{15}} & -\frac{4}{11\sqrt{21}} & \frac{4}{65\sqrt{3}} & -\frac{4}{35\sqrt{33}} \\ -\frac{4}{21\sqrt{5}} & -\frac{4}{3\sqrt{15}} & \frac{100}{231} & \frac{28\sqrt{\frac{7}{5}}}{117} & -\frac{12}{77\sqrt{5}} & \frac{44\sqrt{\frac{11}{5}}}{1989} \\ -\frac{4}{45\sqrt{7}} & -\frac{4}{11\sqrt{21}} & -\frac{28\sqrt{\frac{7}{5}}}{117} & \frac{4}{11} & \frac{308\sqrt{7}}{3315} & -\frac{52}{95\sqrt{77}} \\ -\frac{4}{231} & -\frac{4}{65\sqrt{3}} & -\frac{12}{77\sqrt{5}} & -\frac{308\sqrt{7}}{3315} & \frac{468}{1463} & \frac{44\sqrt{11}}{663} \\ -\frac{4}{117\sqrt{11}} & -\frac{4}{35\sqrt{33}} & -\frac{44\sqrt{\frac{11}{5}}}{1989} & -\frac{52}{95\sqrt{77}} & -\frac{44\sqrt{11}}{663} & \frac{884}{3059} \end{bmatrix}. \tag{5.6}$$

Substituting (5.5) in (5.2) and (5.3), we get

$$C^T = G^T P^{-1} \text{ (for the first kind)} \tag{5.7}$$

and

$$C^T = G^T (I - P)^{-1} \text{ (for the second kind)}. \tag{5.8}$$

Therefore, by substituting the values of $C^T$ from (5.7) and (5.8) into (5.1), the approximate solutions $\zeta(t)$ of Abel's integral equation (1.1) and the second kind singular Volterra integral equation (1.3) are finally found.

## 6 Illustrative examples

This section includes three examples that illustrate the efficiency, applicability and accuracy of the proposed numerical scheme, all of which have been created through using Wolfram Mathematica mathematical software.The approximate solutions to the examples are given here, and they are compared to the exact solutions of the singular Volterra integral equation. The absolute error between the exact and approximated solution has also been calculated. Consider absolute error as generally follows:

$$\text{Absolute error} = |Y_0(r) - Y_1(r)|, \quad a \le r \le b$$

$Y_0(r)$ and $Y_1(r)$ denote for exact and approximate solutions, respectively.

**Example 6.1.** Consider the singular Volterra integral equation below:

$$Y(r) = r^2 + \frac{16}{15} r^{5/2} - \int_0^r \frac{Y(t)}{\sqrt{r-t}} dt, \text{ (From [31])} \tag{6.1}$$

$Y_0(r) = r^2$ is the exact solution to this problem. On applying proposed scheme with $m = 5$ we obtain

$$C^T = \left[\frac{1}{3}, \frac{1}{2\sqrt{3}}, \frac{1}{6\sqrt{5}}, 0, 0, 0\right],$$

$$G^T = \left[\frac{67}{105}, \frac{127}{126\sqrt{3}}, \frac{551}{1386\sqrt{5}}, \frac{32}{1287\sqrt{7}}, -\frac{32}{45045}, \frac{32}{69615\sqrt{11}}\right].$$

Equations (5.1) and (5.8) give the approximate solution $Y_1(r) = C^T E(r) = r^2$ which is the exact solution i.e. using this method we have no error while in many other methods in literature has at least some error.

**Example 6.2.** Consider the singular Volterra integral equation:

$$Y(r) = r + \frac{4}{3}r^{3/2} - \int_0^r \frac{Y(t)}{\sqrt{r-t}}dt, \text{ (From [3])} \tag{6.2}$$

and that has $Y_0(r) = r$ as an exact solution. Using the same procedure as used in example.1, we get the following result as

$$C^T = \left[\frac{1}{2}, \frac{1}{2\sqrt{3}}, 0, 0, 0, 0\right],$$

$$G^T = \left[\frac{31}{30}, \frac{83}{70\sqrt{3}}, \frac{8}{63\sqrt{5}}, -\frac{8}{195\sqrt{7}}, \frac{8}{5005}, -\frac{8}{4095\sqrt{11}}\right].$$

The approximate solution of this problem is $Y_1(r) = C^T E(r) = r$, which again gives exact solution.

**Example 6.3.** Consider Abel's integral equation, which is written as:

$$\int_0^r \frac{Y(t)}{\sqrt{r-t}}dt = r^s, \quad 0 < r < 1\text{(From [4])} \tag{6.3}$$

where $s$ is a positive number.This is a weakly singular first-order Volterra integral equation. The exact solution of integral equation (6.3) is as described in the following:

$$Y_0(r) = \frac{2^{2s-1}}{\pi} s \frac{(\Gamma(s))^2}{\Gamma(2s)} r^{s-\frac{1}{2}} \tag{6.4}$$

We generally apply the proposed technique to try to analyze the Abel's integral problem considering $s = 1, 1.5, 5$.
The given function $q(r) = r$ having exact inverse Abel transform $Y_0(r) = \frac{2}{\pi}\sqrt{r}$, when $r = 1$. Furthermore, using the same procedure as in example (6.1), we obtain

$$C^T = [0.424560, 0.147281, -0.026761, 0.011154, -0.004904, 0.004176],$$

$$G^T = [0.5, 0.288675, 0, 0, 0, 0].$$

Equations (5.1) and (5.7) give the approximate solution, which is

$$Y_1(r) = 0.051550 + 1.933127r - 5.477178r^2 + 10.406638r^3 - 9.756223r^4 + 3.490560r^5.$$

The exact solution (solid line) and the approximate solution (dashed line) are shown in Figure (2). Figure (3) depicts the difference in errors between exact and approximate solutions.The two remaining cases for the presented function $q(r)$ with $s = 1.5$ and 5 contain exact inversions as $Y_0(r) = \frac{3}{4}r$ and $Y_0(r) = \frac{1280}{315\pi}r^{9/2}$ respectively, and have also been considered in the same manner as $s = 1$.
For $s = 1.5$,

$$C^T = [0.375, 0.216506, 3.61274 \times 10^{-8}, -2.70615 \times 10^{-8}, 1.10093 \times 10^{-8}, -1.77776 \times 10^{-8}],$$

$$G^T = [0.4, 0.296923, 0.0425918, -0.00458139, 0.0011988, -0.000441775].$$

For $s = 5$,

$$C^T = [0.235175, 0.364063, 0.169899, 0..059130, 0.010594, 0.000570]$$

$$G^T = [0.166667, 0.206197, 0.133099, 0.052495, 0.011905, 0.001196].$$

Hence the approximate solutions for s=1.5 and 5 are

$$\begin{aligned} Y_1(r) =& 0.00000022 + 0.73741922r - 0.00884293r^2 + 0.00001487r^3 + 0.00000010r^4 \\ & - 0.00001486r^5, \end{aligned}$$

$$Y_1(r) = 0.000087 - 0.004184r + 0.049793r^2 - 0.262794r^3 + 1.034751r^4 + 0.475985r^5$$

respectively. Figures (4),(5),(6),(7) demonstrate the comparison of exact and approximate solutions, as well as their absolute error, for s=1.5 and s=5.

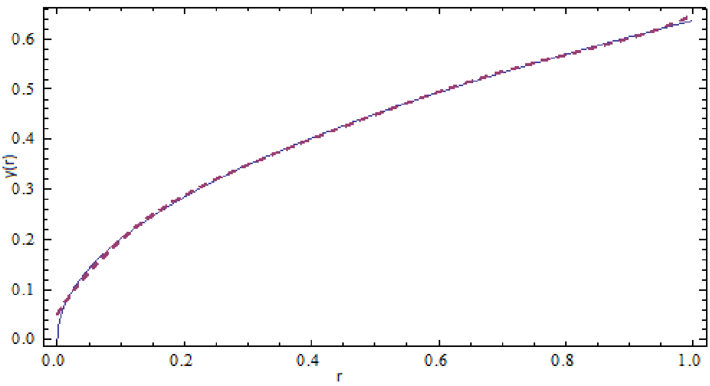


**Figure 2.** Comparison between the exact solution $Y_0(r)$ (solid line) and the approximate solutions $Y_1(r)$ (dashed line) of the Abel's integral equation (6.3) at $s = 1$.

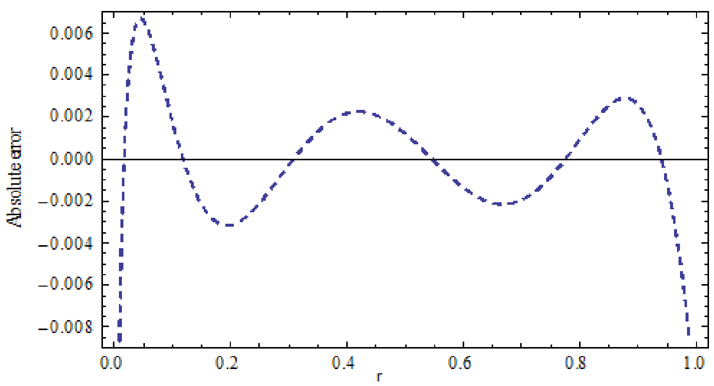


**Figure 3.** The absolute error for the Abel's integral equation (6.3) at $s = 1$.

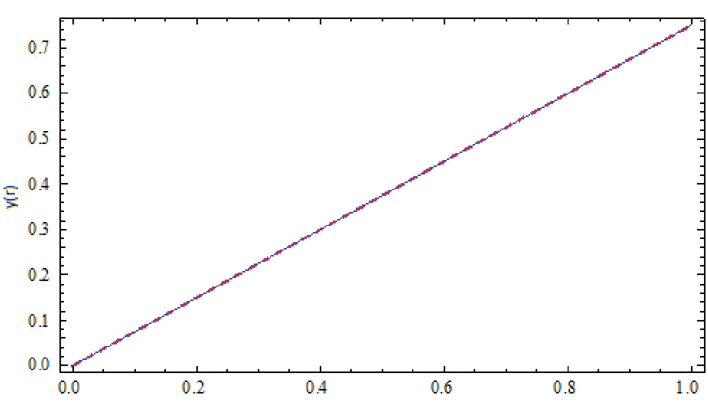


**Figure 4.** Comparison between the exact solution $Y_0(r)$ (solid line) and the approximate solutions $Y_1(r)$ (dashed line) of the Abel's integral equation (6.3) at $s = 1.5$.

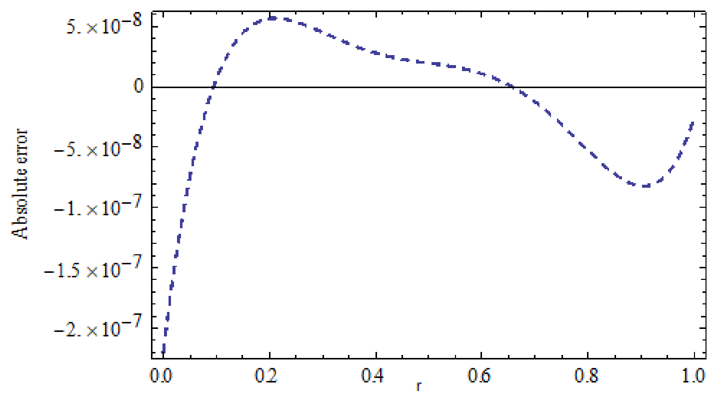


**Figure 5.** The absolute error for the Abel's integral equation (6.3) at $s = 1.5$

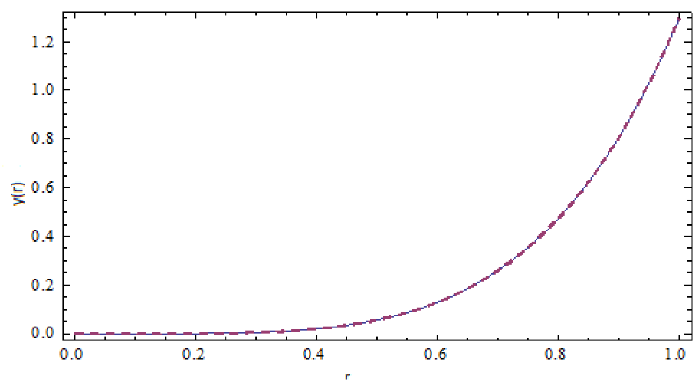


**Figure 6.** Comparison between the exact solution $Y_0(r)$ (solid line) and the approximate solutions $Y_1(r)$ (dashed line) of the Abel's integral equation (6.3) at $s = 5$.

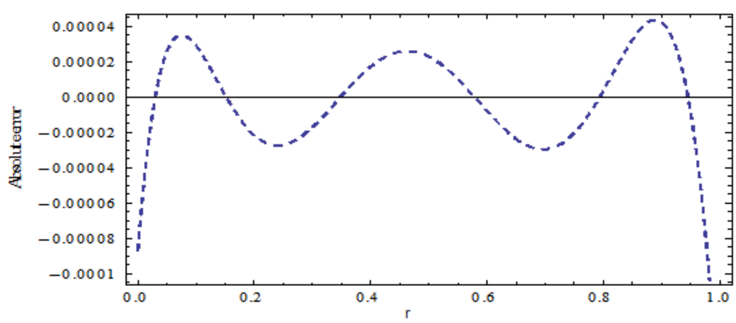


**Figure 7.** The absolute error for the Abel's integral equation (6.3) at $s = 5$

**Table 1.** Numerical results of example (6.3) by using proposed method

| r | Exact solution | | | Proposed method with | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | n=5 | | | n=7 | | |
| | s =1 | s =1.5 | s =5 | s =1 | s =1.5 | s = 5 | s =1 | s =1.5 | s = 5 |
| 0.0 | 0.0 | 0.0 | 0.0 | 0.0515499 | 0.0 | 0.000087 | 0.038851 | 0.0 | 0.0 |
| 0.1 | 0.201317 | 0.075 | 0.000041 | 0.199556 | 0.075 | 0.000012 | 0.203329 | 0.075 | 0.000041 |
| 0.2 | 0.284705 | 0.15 | 0.000926 | 0.287847 | 0.15 | 0.000947 | 0.284235 | 0.15 | 0.000926 |
| 0.3 | 0.348691 | 0.225 | 0.005738 | 0.348976 | 0.225 | 0.005756 | 0.347688 | 0.225 | 0.005738 |
| 0.4 | 0.402634 | 0.30 | 0.020942 | 0.400459 | 0.30 | 0.020925 | 0.403541 | 0.30 | 0.020942 |
| 0.5 | 0.450158 | 0.375 | 0.057163 | 0.448962 | 0.375 | 0.057140 | 0.450719 | 0.375 | 0.057163 |
| 0.6 | 0.493124 | 0.45 | 0.129846 | 0.494492 | 0.45 | 0.129854 | 0.492159 | 0.45 | 0.129846 |
| 0.7 | 0.532634 | 0.525 | 0.259831 | 0.534584 | 0.525 | 0.259860 | 0.532373 | 0.525 | 0.259830 |
| 0.8 | 0.569410 | 0.60 | 0.473865 | 0.568490 | 0.60 | 0.473861 | 0.570618 | 0.60 | 0.473866 |
| 0.9 | 0.603951 | 0.675 | 0.805083 | 0.601367 | 0.675 | 0.805041 | 0.602691 | 0.675 | 0.805083 |
| 1.0 | 0.636620 | 0.75 | 1.293450 | 0.648468 | 0.75 | 1.293640 | 0.644333 | 0.75 | 1.293460 |

For different values of $r$ , Table 1 represents the exact and approximate values of $Y(r)$.

## 7 Conclusions

A numerical approach that is based on Normalized Euler polynomials has been developed in this study for resolving singular Volterra integral equations of Abel type.

The unknown function is approximated with orthonormalized Euler's polynomials in the given technique, and also the integral equation is transformed to a system of algebraic equations. Some examples are used to test its applicability and accuracy. When the approximate solution is compared to the exact solution in such examples, it is absolutely clear that the Euler orthonormal matrix method can achieve very accurate and significant results. Examples (6.1) and (6.2) gives exact solution and Example (6.3) gives very less error in comparison to exact solutions. In Table (1), we can see as the order of operational matrix increases error decreases rapidly. This numerical method has the benefit of being quite simple and clear to put into action on a computer. One may apply this operational matrix technique to solve, partial differential equations, integro-differential equations and differential equations of fractional order also.

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