Palestine Journal of Mathematics

Vol. 11(Special Issue I)(2022) , 141–150

APPROXIMATE CONTROLLABILITY OF FRACTIONAL SEMI-LINEAR DELAY DIFFERENTIAL CONTROL SYSTEM WITH RANDOM IMPULSE

M. Tamilarasi, B. Radhakrishnan and P. Anukokila

Communicated by Anurag Shukla

MSC 2010 Classifications: 26A33, 93B05, 93E99, 34K45, 35R11, 35R60.

Keywords and phrases: Approximate controllability, fractional differential equations, semi-linear control system, delay differential equations, random impulse, mild solution.

Abstract The approximate controllability of a class of fractional order semi-linear delay differential control systems with random impulse is investigated in this study under the natural premise that the linear system is approximately controllable. The existence and uniqueness of the mild solution to the above-mentioned system were determined by using Banach contraction principle. To demonstrate our analytical findings, we present an example.

1 Introduction

Fractional differential equations have been received more attention in recent years, and it plays a significant role in real-world applications. Because, during the last few decades, all scientific and engineering research have proved that the dynamics of many systems may be better described by utilizing non-integer order differential equations. In fact, it has been used as an alternate tool to model nonlinear differential equations. Many real-world situations are represented, interpreted, and conveyed using fractional differential equations to ensure that the physical meaning is preserved and that the information is assimilated more correctly and realistically. Fractional integrals and derivatives also emerge in the concept of control of dynamical systems, where fractional differential equations are employed to describe the controlled system and the controller. The fractional order nonlinear differential equations have been explored by many researchers in [1, 7, 9, 12, 15, 19, 20]. The existence and controllability results of fractional differential equations receives very little attention in the study. When the operator (or) the C_0 semigroup $T(\tau)$ is compact, it is noticed in publications [8, 17, 30] that exact controllability of abstract differential equations in infinite-dimensional space is lacking. In [10, 13, 27–29], authors studied the approximate controllability of abstract semilinear differential equations. This encourages researchers to investigate approximate controllability results for fractional differential equations in infinite-dimensional space.

.

On the other hand, there are natural phenomena that can have their states changed instantly at a specific point in time, such as automobile systems with impact, biological systems in particular, flow of blood, heartbeats, population dynamics, radio physics, medicine, and so on. It is crucial to evaluate dynamical systems with discontinuous trajectories, also referred as differential equations with impulses, in order to achieve this idealistic view. Fixed and random impulses are the two types of impulses available. There have already been numerous studies on impulsive differential systems in history, see [14, 26]. The majority of research articles treat the problem of impulses as if it were a fixed-time instance, yet this is not always the case in real-time situations. The solutions are a stochastic process due to the type of impulses as random time. As a result, the random impulse dynamical system is much more realistic when compared with the deterministic impulsive system.

The exponential stability of random impulsive differential equations is investigated in [2]. Pazy [18] has examined possible solutions to nonlinear and semilinear evolution equations using the method of semigroup. Wu and Meng [31] studied random impulsive ordinary differential equations and used Liapunov's direct technique to examine the boundedness of solutions. The existence and stability of differential system with random impulse have been investigated in [3–5]. Wu et al. first introduced the existence and uniqueness of functional differential equations with random impulse and analysed the p-moment stability, almost sure stability of solutions using Liapunovs function connected with the Razumikhin strategy under Lipchitz conditions in their papers [35–37]. The oscillation and boundedness of solutions in the system with the equivalent non-impulsive differential system, as well as the stability with random impulses, were discussed by Wu et al. in [32–34]. Previously in [21–25], Radhakrishnan et al investigated the existence, uniqueness and stability results for semilinear, quasilinear differential equations, inclusions and also integrodifferential equations with random impulsive circumstances. Recently in [6, 11, 16], the authors discusses about the approximate controllability of second order system and hilfer fractional evolution equations and inclusions. The approximate controllability were studied in the previous survey utilising fixed point theory, semigroup compactness, and uniform boundedness for nonlinear term. In our proof, the compactness and uniform boundedness criteria are waived. There is no publication that we are aware of that investigates the approximate controllability of random impulsive fractional differential equations.

This article is structured in the following manner: Formulation of the research problem have been described in section 2. The existence, uniqueness result, and approximate controllability of a random impulsive fractional semilinear control system have been examined in sections 3 and 4, respectively. In the last part, an example of how to use the applicability of the obtained results is given.

2 Problem Formulation

Let the two separable real Hilbert spaces be $\mathscr{U}$ and $\mathscr{V}$ with the property that $\mathscr{V} \to \mathscr{U}$ is dense and continuous. Let κ be a non-empty set and ρ_k is a random variable defined from $\kappa \to D_k \equiv (0, d_k)$, where $0 < d_k < +\infty$ with λ as the parameter for $k = 1, 2, ...$ Also, for $\mu, \nu = 1, 2, ...$, assume that ρ_μ and ρ_ν are independent of each other as $\mu \neq \nu$. Let $\eta, T \in \mathscr{R}$ be two constants that fulfill $\eta < T$.

Consider the fractional semi-linear delay differential equation with random impulsive

$$\left.\begin{array}{lcl} {}^cD_\tau^\beta u(\tau) & = & Au(\tau) + Bc(\tau) + r(\tau, u_\tau, c(\tau)),\ \tau \neq \sigma_k,\ \tau \in]0, \eta] \\ u(\sigma_k) & = & b_k(\rho_k)u(\sigma_k^-), \quad k = 1, 2, ... \\ u_0\theta & = & \varphi(\theta),\ \theta \in [-h, 0] \end{array}\right\} \tag{2.1}$$

where ${}^cD_t^\beta$ is the Caputo fractional derivative of order $\beta \in (0, 1)$. u is a state variable with values in a Banach space $\mathscr{U}$ and $c(\cdot)$ is a control function with values in a Hilbert space $\mathscr{C}$. The function r is defined as $r : [0, \eta] \times \mathscr{C}([-h, 0]; \mathscr{U}) \times \mathscr{U} \to \mathscr{U}$ in the sequel and B is a linear operator from $\mathscr{V}$ into $\mathscr{C}([-h, 0]; \mathscr{U})$. Let $u_\tau(\theta)$ be the function defined as $u_\tau(\theta) = u(\tau + \theta)$, for τ is fixed and $\theta \in [-h, 0]$ and $\varphi \in \mathscr{C}([-h, 0], \mathscr{U})$ for $h > 0$. $\sigma_0 = \tau_0$ and $\sigma_k = \sigma_{k-1} + \rho_k$, for $k = 1, 2, ...$ A sequence of random variables σ_k is strictly increasing, i.e, $\sigma_0 < \sigma_1 < \sigma_2 < ...\sigma_k < ...T$; $\tau_0 \in \mathscr{R}_\eta$ is any arbitrary real number. Also, $\lim_{k\to\infty} \sigma_k = \infty$; $u(\sigma_k^-) = \lim_{\tau\to\sigma_k} u(\tau)$ with the norm $||u||_\tau = \sup_{\tau - r \le s \le \tau} |u(\tau)|$, for each τ satisfying $0 \le \tau \le T$, $||.||$ is any given norm in $\mathscr{U}$.

The fundamental solution of the linear equation of (2.1) defined as $T(\tau)$ (where $B = 0$ and $r = 0$), is an operator valued function in the form $T(\tau) = S(\tau)$, $\tau \in [0, \eta]$, $T(0) = I$ and $T(\theta) = 0$, for $\theta \in [-h, 0]$. The C_0- semigroup $S(\tau)$ is obviously bounded on $[0, \eta]$, and also that $T(\tau)$ is bounded on $[0, \eta]$.

3 Existence and Uniqueness

It is appropriate to recast the problem (2.1) in the equivalent integral equation using the fractional integral and Caputo fractional derivative definitions [15].

$$\left.\begin{aligned} u(\tau) &= \prod_{\mu=1}^{k} b_\mu(\rho_\mu)\varphi + \sum_{\mu=1}^{k}\prod_{\nu=\mu}^{k} b_\nu(\rho_\nu)\frac{1}{\Gamma(\beta)}\int_{\sigma_\mu-1}^{\sigma_\mu}(\tau-s)^{\beta-1}[Au(s)+Bc(s)+r(s,u_s,c(s))]ds \\ &\quad +\frac{1}{\Gamma(\beta)}\int_{\sigma_k}^{\tau}(\tau-s)^{\beta-1}[Au(s)+Bc(s)+r(s,u_s,c(s))]ds, \end{aligned}\right\}\quad(3.1)$$

provided the integral (3.1) exists. Applying Laplace transform, we get

$$\begin{aligned} u(\tau) &= \prod_{\mu=1}^{k} b_\mu(\rho_\mu)X(\tau)\varphi + \sum_{\mu=1}^{k}\prod_{\nu=\mu}^{k} b_\nu(\rho_\nu)\frac{1}{\Gamma(\beta)}\int_{\sigma_\mu-1}^{\sigma_\mu}(\tau-s)^{\beta-1}Y(\tau-s)[Bc(s)+r(s,u_s,c(s))]d\tau \\ &\quad +\frac{1}{\Gamma(\beta)}\int_{\sigma_k}^{\tau}(\tau-s)^{\beta-1}Y(\tau-s)[Bc(s)+r(s,u_s,c(s))]d\tau, \end{aligned}$$

where $X, Y : \mathcal{U} \to \mathcal{U}_\beta$ are operators such that $X(\tau) = \int_0^\infty \varsigma_\beta(\theta)S(\tau^\beta\theta)d\theta : \mathscr{U} \to \mathscr{U}_\beta$,
$Y(\tau) = \beta\int_0^\infty \theta\varsigma_\beta(\theta)S(\tau^\beta\theta)d\theta$, $\varsigma_\beta = \frac{1}{\beta}\theta^{-1-\frac{1}{\beta}}\rho_\beta(\theta^{\frac{-1}{\beta}}) \geq 0$ and
$\rho_\beta(\theta) = \frac{1}{\pi}\sum_{\alpha=1}^{\infty}(-1)^{\alpha-1}\theta^{-\beta\alpha-1}\frac{\Gamma(\alpha\beta+1)}{\alpha!}sin(\alpha\pi\beta)$.
Here ς_β is a probability density function defined on $(0,\infty)$, that is $\varsigma_\beta(\theta) \geq 0, \theta \in (0,\infty)$ and $\int_0^\infty \varsigma_\beta(\theta)d\theta = 1$. Here $\prod_{\nu=m}^{\alpha}(\cdot) = 1$ as $m > \alpha$, $\prod_{\nu=\mu}^{k} b_\nu(\rho_\nu) = b_k(\rho_k)b_{k-1}(\rho_{k-1})...b_\mu(\rho_\mu)$ and $I_\Xi(.)$ is an index function,

$$I_\Xi(\tau) = \begin{cases} 1, & whenever\ \ \tau \in \Xi \\ 0, & whenever\ \ t \notin \Xi. \end{cases}$$

Now to prove the existence and uniqueness of the mild solution of the equation (2.1) by assuming the following hypotheses.

(H1): $A : D(A) \subset \mathscr{U} \to \mathscr{U}$ is an infinitesimal generator of $T(\tau), \tau > 0$, a compact analytic semigroup of a uniformly bounded linear operator in $\mathscr{U}$, that is, for all $\tau \geq 0$, there exists a constant $M > 1$ such that $\|T(\tau)\| \leq M$.

(H2): The condition $\max_{\mu,k}\{\prod_{\nu=\mu}^{k}\|b_\nu(\rho_\nu)\|\}$ is uniformly bounded, that is there is some $K > 0$ such that

$$\max_{i,k}\{\prod_{\nu=\mu}^{k}\|b_\nu(\rho_\nu)\|\} \leq K, \text{ for all } \rho_\nu \in D_\nu, \nu = 1, 2, \ldots.$$

(H3): The continuous function $r : [0,\eta]\times\mathscr{C}([-h,0];\mathscr{U})\times\mathscr{U} \to \mathscr{U}$, satisfy the Lipchitz condition. For $u, v \in \mathscr{U}$ and $\tau_0 \leq \tau \leq T$ there exists arbitrary constants $L_r \geq 0$ such that

$$\|r(\tau,u_\tau,c_1(\tau)) - r(\tau,v_\tau,c_2(\tau))\| \leq L_r[\|u_\tau - v_\tau\| + \|c_1(\tau) - c_2(\tau)\|]$$

Theorem: 3.1. Assume that the hypotheses (H1) - (H3) holds. Then, the integral equation (3.1) has a unique mild solution on $[\tau_0, T]$.

Proof. The nonlinear operator $\Phi : \mathscr{C}([-h,0];\mathscr{U}) \to \mathscr{C}([-h,0];\mathscr{U})$ is defined as follows

$$(\Phi u)(\tau) = \varphi(\tau - \tau_0), \text{for } \tau \in [-h, \tau_0].$$

And, for $\tau \in [\tau_0, T]$,

$$(\Phi u)(\tau) = \sum_{k=0}^{+\infty}\Big[\prod_{\mu=1}^{k} b_\mu(\rho_\mu)T(\tau_0)\varphi + \sum_{\mu=1}^{k}\prod_{\nu=\mu}^{k} b_\nu(\rho_\nu)\frac{1}{\Gamma(\beta)}\int_{\sigma_\mu-1}^{\sigma_\mu}(\tau-s)^{\beta-1}T(\tau-s)r(s,u_s,c(s))ds + \frac{1}{\Gamma(\beta)}\int_{\sigma_k}^{\tau}(\tau-s)^{\beta-1}T(\tau-\tau)r(s,u_s,c(s))ds\Big]I_{[\sigma_k,\sigma_{k+1})}(\tau).$$

It is simple to examine the continuity of the operator Φ. Next to show that the mapping Φ is a contraction. For any $u, v \in \mathscr{C}([-h,0];\mathscr{U})$, we have

$$\begin{aligned}
&E\|(\Phi u)(\tau)-(\Phi v)(\tau)\|^2\\
&\le E\|\sum_{k=0}^{+\infty}\Big[\sum_{\mu=1}^{k}\prod_{\nu=\mu}^{k} b_\nu(\rho_\nu)\frac{1}{\Gamma(\beta)}\int_{\sigma_i-1}^{\sigma_i}(\tau-s)^{\beta-1}T(\tau-s)[r(s,u_s,c(s))-r(s,v_s,c(s))]ds\\
&\quad+\frac{1}{\Gamma(\beta)}\int_{\sigma_k}^{\tau}(\tau-s)^{\beta-1}T(\tau-s)[r(s,u_s,c(s))-r(s,v_s,c(s))]ds\Big]I_{[\sigma_k,\sigma_{k+1})}(\tau)\|^2\\
&\le \sum_{k=0}^{+\infty}E[\max\{1,\prod_{\nu=\mu}^{k}\|b_\nu(\rho_\nu)\|^2\}]\Big(\frac{1}{\Gamma(\beta)}\int_{\tau_0}^{\tau}(\tau-s)^{\beta-1}\times\\
&\quad E\|T(\tau-s)[r(s,u_s,c(s))-r(s,v_s,c(s))]\|^2ds)I_{[\sigma_k,\sigma_{k+1})}(\tau)\\
&\le \sum_{k=0}^{+\infty}[\max\{1,K^2\}]\frac{M^2}{\Gamma(\beta)}\int_{\tau_0}^{\tau}(\tau-s)^{\beta-1}\times\\
&\quad E\|r(s,u_s,c(s))-r(s,v_s,c(s))\|^2d\tau I_{[\sigma_k,\sigma_{k+1})}(\tau)\\
&\le M^2L_r[\max\{1,K^2\}]\frac{(\tau-\tau_0)^\beta}{\Gamma(\beta+1)}E\|u(\tau)-v(\tau)\|^2.
\end{aligned}$$

Taking supremum over τ, we have

$$\begin{aligned}
\|\Phi u-\Phi v\|^2 &\le \frac{1}{\Gamma(\beta+1)}\max(1,K^2)M^2L_r(T-\tau_0)^\beta\|u-v\|^2\\
&\le \Lambda\|u-v\|^2.
\end{aligned}$$

Repeating the process, we get

$$\|\Phi^n u-\Phi^n v\|^2 \le \frac{1}{\Gamma(\beta+1)}\max(1,K^2)M^2L_r(T-\tau_0)^{n\beta}\|u-v\|^2.$$

For sufficiently large value of an integer n, Φ^n is a contraction mapping. Also, Φ is a contraction on $\mathscr{C}([-h,0];\mathscr{U})_T$. Hence, by using Banach contraction principle, attains a unique fixed point $u \in \mathscr{C}([-h,0];\mathscr{U})_T$, for the operator Φ, and hence $\Phi u = u$ is a mild solution of the system. This approach can be repeated in finitely many similar steps to extend the solution to the entire interval $[\tau_0, T]$. Therewith finalizing the proof for the existence and uniqueness of mild solutions for the original system on the whole interval $[0, T]$.

4 Approximate Controllability

Assume that, corresponding to the control $c \in \mathscr{C}$, the state value of system (2.1) at time τ with the initial value $\varphi(\tau_0)$, is defined as $u_\tau(\varphi(\tau_0), c)$. The set of all possible trajectories, called as the reachable set of trajectories, for the system (2.1), given by

$$\Delta_\beta(F) = \{u_\beta(\varphi(\tau_0),c) \in \mathscr{C}([\beta?,T],\mathscr{U}) : c \in \mathscr{C}, 0 <?\beta \le T\}. \tag{4.1}$$

At time T,

$$\Delta_T(F) = \{u_T(\varphi(\tau_0),c) : c \in \mathscr{C}\}. \tag{4.2}$$

Whenever $\overline{\Delta_T(F)} = \mathscr{U}$, then the control system is approximately controllable on $[\tau_0, T]$. The following few definitions are given to illustrate the approximate controllable results [18]:

- The solution for the system $\mathscr{W} : Z \to \mathscr{C}([\tau_0, T], \mathscr{U})$ is specified by $(\mathscr{W}c)(\tau) = u(\varphi(\tau_0), c)(\cdot), c \in Z, Z = L_2[\tau, T; \mathscr{U}]$.
- $Q : Z \to \mathscr{C}([\tau_0, T], U)$, is a continuous operator, defined by

$$(Qp)(\tau) = \int_{\tau_0}^{t} T(\tau - s)p(s)d\tau, p \in Z, \tau \in [\tau_0, T];$$

- The function, $\zeta : L_2[\tau_0, T; \mathcal{E}] \to Z$ as $(\zeta u)(\tau) = \int_0^\tau r(s, u(\tau))d\tau$; $u \in L_2[\tau_0, T; \mathcal{E}]$ and $\omega : \mathscr{V} \to L_2[\tau_0, T; \mathscr{U}]$, defined as $(\omega c)(\tau) = Bc(\tau)$.

Definition 4.1. A stochastic process $\{u(\tau) \in \mathcal{Z}, \tau_0 \le t \le T\}$ for a given $T \in (\tau_0, +\infty)$, is called as a mild solution to the equation (2.1) in $(\kappa, P, \{\zeta_t\})$, if $u(\tau_0 + s) = \varphi(s) \in L_2^0(\kappa, \omega)$, when $s \in [-h, 0]$; and

$$\left.\begin{aligned} u(\tau) \;=\; & \sum_{k=0}^{+\infty}\Big[\prod_{\mu=1}^{k} b_\mu(\rho_\mu)T(\tau_0)\varphi \\ & +\sum_{\mu=1}^{k}\prod_{\nu=\mu}^{k} b_\nu(\rho_\nu)\frac{1}{\Gamma(\beta)}\int_{\sigma_\mu-1}^{\sigma_\mu}(\tau-s)^{\beta-1}[Bc(s)+r(s,u_s,c(s))]ds \\ & +\frac{1}{\Gamma(\beta)}\int_{\sigma_k}^{\tau}(\tau-s)^{\beta-1}[Bc(s)+r(s,u_s,c(s))]ds\Big]I_{[\sigma_k,\sigma_{k+1})}(\tau), \quad \tau\in[\tau_0,T] \end{aligned}\right\} \tag{4.3}$$

Now, the following additional hypotheses are introduced for the further discussions:

(H4): There exists some control $c(\cdot) \in \mathscr{V}$, for all $\mathscr{E} > 0$ and $p(\cdot) \in Z$, such that

$$E\|Qp - QBc\|^2 < \mathscr{E};$$

(H5): $\overline{R(\omega)} \supseteq R(\zeta)$.

Lemma 4.1. *Let the hypothesis (H4) be hold. Then $\overline{\Delta_\beta(0)} = \mathscr{U}$.*

Proof. As $D(A)$ is dense in $\mathscr{U}$, it is enough to prove that the domain of A is a subset of $\overline{\Delta_\beta(0)}$. For a given $\mathscr{E} > 0$ and $G \in D(A)$, there exists a control function $c(.) \in \mathscr{V}$ such that

$$E\|G - \overline{G} - QBc\|^2 < \mathscr{E},$$

where $\overline{G} = \sum_{k=0}^{+\infty}\left[\prod_{\mu=1}^{k} b_\mu(\rho_\mu)T(\tau_0)\varphi\right] I(\tau),\ \tau \in [\tau_0, T]$. Take $G \in D(A)$, then $G - \overline{G} \in D(A)$. It can be observed that there exists some control $c(\cdot) \in \mathscr{V}$ such that

$$E\|Qp - QBc\|^2 < \mathscr{E}.$$

As $\mathscr{E}$ is arbitrary, $\Delta_\beta(0)$ is a subset of $D(A)$. The above lemma proves the approximate controllability of the linear system

$$\left.\begin{aligned} {}^cD_t^\beta u(\tau) &= Au(\tau) + Bc(\tau),\ t \neq \sigma_k,\ \tau \in [\tau_0, T] \\ u(\sigma_k) &= b_k(\rho_k)u(\sigma_k^-), \quad k = 1, 2, ... \\ u_{\tau_0} &= \varphi, \end{aligned}\right\} \tag{4.4}$$

For the linear system (4.4), the densed domain $D(A)$ in $\mathscr{U}$ implies the approximate controllability. Next, the following theorem proves the approximate controllability of the fractional random impulsive semilinear control system (2.1).

Theorem: 4.2. Under the hypotheses (H1) - (H5), $\Delta_\beta(0) \subset \overline{\Delta_\beta(f)}$.

Proof. Let $u(\cdot) \in \Delta_\beta(0)$. Then there exists a $c \in \mathscr{V}$, which can be defined as

$$\begin{aligned}
u(\tau) &= \varphi(\tau-\tau_0), \text{ for } \tau \in [-h,\tau_0],\\
\text{For } \tau \in [\tau_0,T],\ u(\tau) &= \sum_{k=0}^{+\infty}\Big[\prod_{\mu=1}^{k} b_\mu(\rho_\mu)T(\tau_0)\varphi\\
&+\sum_{\mu=1}^{k}\prod_{\nu=\mu}^{k} b_\nu(\rho_\nu)\frac{1}{\Gamma(\beta)}\int_{\sigma_\mu-1}^{\sigma_\mu}(\tau-s)^{\beta-1}T(\tau-s)Bc(s)ds\\
&+\frac{1}{\Gamma(\beta)}\int_{\sigma_k}^{\tau}(\tau-s)^{\beta-1}T(\tau-s)Bc(s)ds\Big]I_{[\sigma_k,\sigma_{k+1})}(\tau) \qquad (4.5)
\end{aligned}$$

As $F_u \in \overline{R(B)}$, there exists a $w \in \mathscr{V}$ for any given $\mathscr{E} > 0$, such that

$$E\|F_u - Bw\|_B^2 \le \mathscr{E}$$

Suppose let $v(\tau)$ be an another mild solution of (2.1) with respect to the control $c-w$, then

$$\begin{aligned}
u(\tau)-v(\tau) &= \sum_{k=0}^{+\infty}\Big[\prod_{\mu=1}^{k} T(\tau_0)b_\mu(\rho_\mu)\varphi\\
&+\frac{1}{\Gamma(\beta)}\sum_{\mu=1}^{k}\prod_{\nu=\mu}^{k} b_\nu(\rho_\nu)\int_{\sigma_\mu-1}^{\sigma_\mu}(\tau-s)^{\beta-1}T(\tau-s)Bc(s)ds\\
&+\frac{1}{\Gamma(\beta)}\int_{\sigma_k}^{\tau}(\tau-s)^{\beta-1}T(\tau-s)Bc(s)ds\Big]I_{[\sigma_k,\sigma_{k+1})}(\tau)\\
&-\sum_{k=0}^{+\infty}\Big[\prod_{\mu=1}^{k} b_\mu(\rho_\mu)T(\tau_0)\varphi\\
&+\frac{1}{\Gamma(\beta)}\sum_{\mu=1}^{k}\prod_{\nu=\mu}^{k} b_\nu(\rho_\nu)\int_{\sigma_\mu-1}^{\sigma_\mu}(\tau-s)^{\beta-1}T(\tau-s)[Fv](s)ds\\
&+\frac{1}{\Gamma(\beta)}\int_{\sigma_k}^{\tau}(\tau-s)^{\beta-1}T(\tau-s)[Fv](s)ds\Big]I_{[\sigma_k,\sigma_{k+1})}(\tau)\\
&\le \sum_{k=0}^{+\infty}\Big[\frac{1}{\Gamma(\beta)}\sum_{\mu=1}^{k}\prod_{\nu=\mu}^{k} b_\nu(\rho_\nu)\int_{\sigma_\mu-1}^{\sigma_\mu}(\tau-s)^{\beta-1}T(\tau-s)[Bw-Fu](s)ds\\
&+\frac{1}{\Gamma(\beta)}\int_{\sigma_k}^{\tau}(\tau-\tau)^{\beta-1}T(\tau-s)[Bw-Fu](s)ds\Big]I_{[\sigma_k,\sigma_{k+1})}(\tau)\\
&+\sum_{k=0}^{+\infty}\Big[\frac{1}{\Gamma(\beta)}\sum_{\mu=1}^{k}\prod_{\nu=\mu}^{k} b_\nu(\rho_\nu)\int_{\sigma_\mu-1}^{\sigma_\mu}(\tau-s)^{\beta-1}T(\tau-s)[Fu-Fv](s)d\tau\\
&+\frac{1}{\Gamma(\beta)}\int_{\sigma_k}^{\tau}(\tau-s)^{\beta-1}T(\tau-s)[Fu-Fv](s)d\tau\Big]I_{[\sigma_k,\sigma_{k+1})}(\tau)
\end{aligned}$$

$E\|u(\tau)-v(\tau)\|^2$

$$\begin{aligned}\leq\quad & E\sum_{k=0}^{+\infty}\Big[\frac{1}{\Gamma(\beta)}\sum_{\mu=1}^{k}\prod_{\nu=\mu}^{k}b_\nu(\rho_\nu)\int_{\sigma_\mu-1}^{\sigma_\mu}(\tau-s)^{\beta-1}\|T(\tau-s)\|\|Bw-Fu\|(s)ds\\ &+\frac{1}{\Gamma(\beta)}\int_{\sigma_k}^{\tau}(\tau-s)^{\beta-1}\|T(\tau-s)\|\|Bw-Fu\|(s)ds\Big]^2 I_{[\sigma_k,\sigma_{k+1})}(\tau)\\ &+E\sum_{k=0}^{+\infty}\Big[\frac{1}{\Gamma(\beta)}\sum_{\mu=1}^{k}\prod_{\nu=\mu}^{k}b_\nu(\rho_\nu)\int_{\sigma_\mu-1}^{\sigma_\mu}(\tau-s)^{\beta-1}\|T(\tau-s)\|\|Fu-Fv\|(s)ds\\ &+\frac{1}{\Gamma(\beta)}\int_{\sigma_k}^{\tau}(\tau-s)^{\beta-1}\|T(\tau-s)\|\|Fu-Fv\|(s)ds\Big]^2 I_{[\sigma_k,\sigma_{k+1})}(\tau)\end{aligned}$$

By using Grownwal's inequality and taking supremum over t, we get

$$\|u-v\|^2\leq N\int_{\tau_0}^{\tau}E\|u-v\|^2ds,$$

where $N>0$ is the constant. It is clear from the inequality above that $\|u-v\|^2$. By selecting the appropriate control w, it can be done arbitrarily small. As a result, Theorem 4.3 is established.

Corollary: 4.3. System (2.1) is approximately controllable on the assumptions of the above theorem.

Proof. At $\beta=T$, the proof is a special case of Theorem 3.1.

Under adequate conditions with fixed time impulses, the system (2.1) is generalized as follows.

Remark 4.4. With the same arguments as Corollary 4, if the impulses exist at fixed times, the system (2.1) is approximately controllable.

If the system does not have an impulsive condition, the problem is reduced to abstract fractional semi-linear differential equations with delays.

Remark 4.5. If no impulses are present, the system (2.1) will be

$$\left.\begin{aligned}{}^cD_\tau^\beta u(\tau)&=Au(\tau)+Bc(\tau)+r(\tau,u_\tau,c(\tau)),\ \tau\neq\sigma_k,\ \tau\in[\tau_0,T]\\ u_{\tau_0}&=\varphi,\end{aligned}\right\}\tag{4.6}$$

Theorem: 4.6. The system (4.6) is approximately controllable under the assumptions (H1)-(H2) and (H4)-(H5).

Proof. At $\beta=T$, the proof is a special instance of Theorem 3.1.

5 Example

Let $\mathscr{X}=L_2(0,\pi)$ and $A=\frac{d^2}{dx^2}$ with the domain $D(A)$ consisting of all $g\in\mathscr{X}$ with the condition $g(0)=0=g(\pi)$. Assume that $\phi_p(x)=(\frac{2}{\pi})^{1/2}sin(px)$; $0\leq x\leq\pi$ is an orthonormal base for $\mathscr{X}$ for $p=1,2,...$and ϕ is the eigenfunction with respect to the eigenvalue $\Lambda_p=-p^2$ of the operator $A, p=1,2,....$ Then the C_0- semigroup $S(\tau)$ has $\exp(\Lambda_p t)$.
Assume that the infinite-dimensional space $\bar{\mathscr{X}}$ is

$$\bar{\mathscr{U}}=\{u|u=\sum_{p=2}^{\infty}u_p\phi_p,\ \text{with}\ \sum_{p=2}^{\infty}u_p^2<\infty\}.$$

Define the norm in $\bar{\mathscr{X}}$ as

$$\|u\|_{\bar{\mathscr{X}}} = (\sum_{p=2}^{\infty} u_p^2)^{1/2}.$$

Let $B : \bar{\mathscr{X}} \to \mathscr{X}$ be a linear continuous map defined as

$$Bu = 2u_2\phi_1 + \sum_{p=2}^{\infty} u_p\phi_p, \text{ for } u = \sum_{p=2}^{\infty} u_p\phi_p \in \overline{\mathscr{X}}.$$

It is obvious that $\|Bu\| \geq \sigma\|u\|$, where $\sigma = 1$.
Consider the semi-linear partial differential random impulsive control system of the form

$$\left.\begin{aligned} \frac{\partial^\gamma}{\partial t^\gamma} g(t,x) &= \frac{\partial^2}{\partial t^2} g(t,x) + (Bu)(t,x) + r(t, g(t,x), u(t,x)),\ 0 < x < \pi \\ g(\sigma_k, x) &= p(k)\rho_k g(\sigma_k^-, x), \quad t = \sigma_k, k = 1, 2, ... \\ g(0,l) &= g(\pi, l) = 0, \\ g(x,t) &= \Phi(x,t),\ \tau \in [-r, 0],\ 0 \leq x \leq \pi \end{aligned}\right\} \tag{5.1}$$

where p is a function of k; $\sigma_k = \sigma_{k-1} + \rho_k$ for $k = 1, 2, ...$; $\sigma_0 = \tau_0 \in \mathscr{R}^+$ is an arbitrary real number. As $i \neq j$, ρ_i and ρ_ν are independent of each other for $i, j = 1, 2, ...$ and $\phi(x,t)$ is continuous.

By our result, if the assumptions (H4) and (H5) are true, then the conditions for the approximate controllability results are obtained. For example, suppose that the function f is given as

$$r(t, g, u) = l[\|g\|\phi + \|u\|\phi],$$

$l \in (0, 1)$. Now r satisfies (H1) with Lipchitz constant $l < 1$. Also, r and B satisfies the assumption (H3). Since $l < \beta$ from Theorem (4.2), the approximate controllability of the system (5.1) follows.

6 Conclusion

In this study, an approximate controllability of fractional random impulsive semi-linear delay differential control system is investigated by assuming that the linear system is approximately controllable. The existence and uniqueness of the mild solution to the above said system were determined using Banach contraction principle. An example is provided at the end to verify the analytical findings.

7 Acknowledgment

The authors would like to express their gratitude to the editor and anonymous referees for their helpful recommendations that improved the manuscript. Tamil Nadu State Council for Science and Technology (TNSCST-RFRS), Chennai, supported the second author. (TNSCST/ RFRS/VR/15/2018-19)

References

[1] B. Ahmad and S. Sivasundaram, Existence results for nonlinear impulsive hybrid boundary value problems involving fractional differential equations, Nonlinear Analysis- Hybrid Systems, **3**, 251-258 (2009).

[2] R. Agarwal, S. Hristova and Donal O'Regan, Exponential stability for differential equations with random impulses at random times, Advances in Difference Equations, **2013**, 372 (2013).

[3] A. Anguraj, S. Wu and A. Vinodkumar, The existence and exponential stability of semilinear functional differential equations with random impulses under non-uniqueness, Nonlinear Analysis, **74**, 331-342 (2011).

[4] A. Anguraj and A. Vinodkumar, Existence, Uniqueness and Stability Results of Random Impulsive Semilinear Differential Systems, Nonlinear Analysis and Hybrid Systems, **4(3)**, 475-483 (2010).

[5] A. Anguraj, A. Vinodkumar and K. Malar, Existence and Stability Results of Random Impulsive Fractional Pantograph Equations, Filomat, **30(14)**, 3839-3854 (2016).

[6] Ankit Kumar, Ramesh K Vats and Avadhesh Kumar, Approximate controllability of second-order nonautonomous system with finite delay, Journal of Dynamical and Control Systems, **26(4)**, 611-627 (2020).

[7] K.Balachandran and J. Y. Park, Nonlocal Cauchy problem for abstract fractional semilinear evolution equations, Nonlinear Analysis: Theory, Method & Applications, **71**, 4471 - 4475 (2009).

[8] K.Balachandran, Controllability of nonlinear systems in Banach space: A survey, Journal of Optimization Theory & Applications, **115**, 7-28 (2002).

[9] M. Benchohra and B.A. Slimani, Existence and uniqueness of solutions to impulsive fractional differential equations, Electronic Journal of Differential Equations, **2009**, 1-11 (2009).

[10] A. Divya, N. Sukavanam and A. Shukla, On the approximate controllability of semilinear control system, Cogent Mathematics, **3**, 1266773 (2016).

[11] K. Kavitha, V. Vijayakumar, R. Udhayakumar, N. Sakthivel and K.S.Nisar, A note on approximate controllability of the Hilfer fractional neutral differential inclusions with infinite delay, Mathematical Methods in the Applied Sciences, **44(6)**, 4428-4447 (2020).

[12] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.

[13] S. Kumar and N. Sukavanam, Approximate controllability of fractional order neutral control systems with delay, International Journal of Nonlinear Science, **13**, 454-462 (2012).

[14] V.Lakshmikantham, D.D. Bainov and P.S Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.

[15] K.S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993.

[16] M. Mohan Raja, V. Vijayakumar, R. Udhayakumar and Y. Zhou, A new approach on the approximate controllability of fractional differential evolution equations of order $1 < r < 2$ in Hilbert spaces, Chaos, Solitons & Fractals, **141**, 110-310 (2020).

[17] K. Naito, Controllability of semilinear control systems dominated by the linear part, SIAM Journal of Control and Optimization, **25(3)**, 715-722 (1987).

[18] A.Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.

[19] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999.

[20] I. Podlubny, Geometric & physical interpretation of fractional integration fractional differentiation, Fractional Calculus Applications & Analysis, **5** (2002), 367-386.

[21] B. Radhakrishnan and K. Balachandran, Controllability of impulsive neutral functional evolution integrodifferential systems with infinite delay, Nonlinear Analysis: Hybrid Systems, **5**, 655-670 (2011).

[22] B. Radhakrishnan and M. Tamilarasi, Existence of solutions for quasilinear random impulsive neutral differential evolution equation. Arab Journal of Mathematical Sciences, **24**, 235-246 (2018).

[23] B Radhakrishnan and M. Tamilarasi, Existence results for quasilinear random impulsive abstract differential inclusions in Hilbert space, The Journal of Analysis, **27(2)**, 327-345 (2019).

[24] B. Radhakrishnan, M. Tamilarasi and P. Anukokila, Existence, uniqueness and stability results for semilinear integrodifferential non-local evolution equations with random impulse, Filomat, **32(19)**, 6615-6626 (2018).

[25] B. Radhakrishnan and M. Tamilarasi, Existence, Uniqueness and Stability Results for Fractional Hybrid Pantograph Equation with Random Impulse, Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications & Algorithms **28(3)**, 165-181 (2021).

[26] A.M.Samoilenko and N.A.Perestyuk, Impulsive Differential Equations, World Scientific, Singapore, 1995.

[27] N. Sukavanam and N. Tomar, Approximate controllability of semilinear delay control systems, Nonlinear Functional Analysis and Applications, **12(1)**, 53-59 (2007).

[28] N. Sukavanam and A. Divya, Approximate controllability of abstract semilinear deterministic control system, Bulletin Calcutta Mathematical Society, **96(3)**, 195-202 (2004).

[29] N. Tomar and N. Sukavanam, Approximate controllability of non-densely defined semilinear delayed control systems, Nonlinear Studies, **18(2)**, 229-234 (2011).

[30] R. Triggiani, A note on the lack of exact controllability for mild solutions in Banach spaces, SIAM Journal of Control and Optimization, **15**, 407-411 (1977).

[31] S.J.Wu and X.Z.Meng, Boundedness of nonlinear differential systems with impulsive effect on random moments, Acta Mathematica Applicatea Sinica, **20** , 147-154 (2004).

[32] S.J. Wu and D.Han, Exponential stability of functional differential systems with impulsive effect on Random Moments, Computers Mathematical Applications, **50**, 321-328 (2005).

[33] S.J.Wu and Y.R.Duan, Oscillation, stability, and boundedness of second-order differential systems with random impulses, Computational Mathematics & Applications, **49**, 1375-1386 (2005).

[34] S.J.Wu, X.L.Guo and Y.Zhou, p-moment stability of functional differential equations with random impulses, Computational Mathematics & Applications, **52**, 1683-1694 (2006).

[35] S.J.Wu, X.Guo and Z.S.Lin, *Existence and uniqueness of solutions to random impulsive differential systems*, Acta Mathematica Applicatea Sinica **4**, 627-632 (2006).

[36] S.J. Wu, X.Guo and R. Zhai, Almost sure stability of functional differential equations with random impulses, Dynamics of Continuous, Discrete Impulsive Systems, **15**, 403-415 (2008).

[37] Z.Yong and S.J.Wu, Existence and Uniqueness of solutions to Stochastic differential equations with Random Impulsive under Lipchitz conditions, Chinese Journal Applications of Probability and Statistics, **28**, 347-356 (2010).

Author information

M. Tamilarasi, Department of Mathematics, PSG College of Technology, Peelamedu, Coimbatore, Tamilnadu, INDIA.
E-mail: tamilqueenmv@gmail.com

B. Radhakrishnan, Department of Mathematics, PSG College of Technology, Peelamedu, Coimbatore, Tamilnadu, INDIA.
E-mail: radhakrishnanb1985@gmail.com

P. Anukokila, Department of Mathematics, PSG College of Arts & Science College, Coimbatore, Tamilnadu, INDIA.
E-mail: anuparaman@gmail.com