Palestine Journal of Mathematics

Vol. 11(Special Issue I)(2022) , 162–170

APPLICATIONS OF ORTHONORMAL BERNOULLI POLYNOMIALS FOR APPROXIMATE SOLUTION OF SOME VOLTERRA INTEGRAL EQUATIONS

Udaya P. Singh

Communicated by R Udhayakumar

MSC 2010 Classifications: 45A05, 45D05, 45L05, 65R20.

Keywords and phrases: Volterra integral equation, Bernoulli polynomials, orthonormal polynomials.

Abstract A new approach has been developed to obtain numerical solution of linear Volterra type integral equations by obtaining asymptotic approximation to solutions. Using the classical Bernoulli polynomials, a set of orthonormal polynomials have been derived, and these orthonormal polynomials have been used to form an operational matrix of integration which is has been implemented to find numerical or exact solution of non-singular Volterra integral equations. A few numerical examples have been discussed in order to demonstrate the effectiveness of present method. The Obtained approximate solutions have been compared with the exact solutions for numerical values. High degree of accuracy of numerical solutions has established the credibility of the present method.

1 Introduction

Many physical problems are formulated as integral equations. Diffusion problems, heat conduction, concrete problem of physics and mechanics, unsteady Poiseuille flow in a pipe are some such examples. Also, such integral equations arise naturally in different applications of potential theory, continuum mechanics, electricity and magnetism, geophysics, antenna, synthesis problem, population genetics communication theory, mathematical modelling of economics, radiation problems, fluid mechanics, problems of astrophysics concerning transport of particles, and many more. Bulk of literature is available on Volterra and Fredholm integral equations [1, 21, 10, 17, 3]. Bernoulli polynomials and its properties have been discussed by many authors [4, 8, 13].

Volterra integral equations uncover several difficulties referring to mathematical physics such as heat conduction difficulties. In recent years, researchers have focused their attention to find approximate solutions of integral equations. Xu [20] adopted method of variational iteration. Cheon [4] discussed possible applications of Bernoulli polynomials and functions in numerical analysis. Some other latest investigations include uses of Chebyshev polynomials [10], Legendre polynomials [14], Laguerre polynomials and Wavelet Galerkin method [15], Legendre wavelets [21], the operational matrix [16], Bernoulli matrix method [18]. Recently, Bernoulli polynomials were used by Tohidi and Khorsand [3, 19] to solve second-order linear system of partial differential equations, Mohsenyzadeh [11] to solve linear Volterra integral equations, and Samadyar and Mirazee [17] to find numerical solution for singular partial integro-differential equation of fractional order.

However, the numerical methods have certain limits and, therefore, there is always a need for an efficient method to produce more accurate numerical solution of integral equations.

In this work, it is proposed to introduce a new operational matrix of integration for orthonormal polynomials to reduce Volterra type integral equations into a system of algebraic equations. By using operational matrix of these orthonormal polynomials, exact solution for many Volterra integral equations can be obtained. Furthermore, the solutions to the integral equations solved with present method have been compared with exact solution of the problem.

2 Bernoulli Polynomials

The monic polynomials

$$B_n(\zeta) = \sum_{j=0}^{n} \begin{pmatrix} n \\ j \end{pmatrix} B_j(0)\,\zeta^{n-j}, \quad n = 0, 1, 2, ... \,; \quad 0 \le \zeta \le 1 \tag{2.1}$$

were introduced by Jacob Bernoulli in early sixteenth century, where $B_k(0)$ are the Bernoulli numbers. To have a better understanding, first few Bernoulli polynomials are represented as:

$$B_0(\zeta) = 1 \tag{2.2}$$

$$B_1(\zeta) = \zeta - \frac{1}{2} \tag{2.3}$$

$$B_2(\zeta) = \zeta^2 - \zeta + \frac{1}{6} \tag{2.4}$$

$$B_3(\zeta) = \zeta^3 - \frac{3}{2}\zeta^2 + \frac{1}{2}\zeta \tag{2.5}$$

$$B_4(\zeta) = \zeta^4 - 2\zeta^3 + \zeta^2 - \frac{1}{30} \tag{2.6}$$

However, the name *Bernoulli Polynomials* was coined by J. L. Raabe in 1851, a thorough study of these polynomials for arbitrary value of its variable was first done by Leonhard Euler in 1755, who showed in his book "*Foundations of differential calculus*" that these polynomials satisfy the finite difference relation.

$$B_n(\zeta + 1) - B_n(\zeta) = n\zeta^{n-1}, n \ge 1 \tag{2.7}$$

Bernoulli Polynomials form a complete basis for $\wp_n$ (the set of all polynomials of degree less than or equal to n) over $[0, 1]$ [7] and can also be extracted from its generating function

$$\frac{\gamma e^{\zeta\gamma}}{e^{\gamma} - 1} = \sum_{n=0}^{\infty} B_n(\zeta)\frac{\gamma^n}{n!}\,(|\zeta| < 2\pi) \tag{2.8}$$

Some interesting properties of Bernoulli polynomials [6] are as follows:

$$\left.\begin{aligned} &B'_n(\zeta) = nB_{n-1}(\zeta), \;\; n \ge 1 \\ &\int_0^1 B_n(z)dz = 0, \qquad n \ge 1 \\ &B_n(\zeta + 1) - B_n(\zeta) = n\zeta^{n-1}, \;\; n \ge 1 \end{aligned}\right\} \tag{2.9}$$

Some more properties and generalizations of Bernoulli polynomials can be found in the significant works [8, 13, 9, 5, 12].

3 The Orthonormal Polynomials

It can be easily verified that the polynomials $B_n(x)$ $(n \ge 1)$ given by eq. (2.1) are orthogonal to $B_o(x)$ with respect to standard inner product on $L^2 \in [0, 1]$. Using this property, an orthonormal set of polynomials can be derived for any B_n with Gram-Schmidt orthogonalization. First few orthonormal polynomials derived for $B_9(x)$:

$$\phi_0\,(\zeta) = 1 \tag{3.1}$$

$$\phi_1(\zeta) = \sqrt{3}(-1 + 2\zeta) \tag{3.2}$$

$$\phi_2(x) = \sqrt{5}\left(1 - 6x + 6x^2\right) \tag{3.3}$$

$$\phi_3(\zeta) = \sqrt{7}(-1 + 12\zeta - 30\zeta^2 + 20\zeta^3) \tag{3.4}$$

$$\phi_4(\zeta) = 3(1 - 20\zeta + 90\zeta^2 - 140\zeta^3 + 70\zeta^4) \tag{3.5}$$

4 Approximation of Functions

Let $\phi = \{\phi_0, \phi_1, \phi_2, ..., \phi_n\}$ contains first n+1 orthonormal polynomials derived for Bernoulli polynomial $B_n(x)$. Since $\phi \subset L^2[0,1]$ and $span\{\phi\}$ is a finite dimensional space, any function $f \in L^2[0,1]$ has a unique and best approximation $\hat{f} \in span\{\phi\}$ such that $\forall g \in span\{\phi\}, ||\hat{f} - f|| \leq ||f - g||$, and

$$f = \hat{f} = \lim_{n\to\infty} \sum_{k=0}^{n} c_k\, \phi_k(\zeta) \tag{4.1}$$

where $c_k = \langle f|\phi_k\rangle$, and $\langle .|.\rangle$ is the standard inner product on $L^2[0,1]$ [2].

For numerical approximation, series in eq.(4.1) can be truncated after certain number of terms, say $n = m$ terms, so that

$$f(\zeta) \cong \sum_{k=0}^{m} c_k\, \phi_k = C^T\, \phi(\zeta), \tag{4.2}$$

where $C = (c_0, c_1, c_2, ..., c_m)\,, \phi(\zeta) = (\phi_0, \phi_1, \phi_2, ..., \phi_m)$ are column vectors, and number of terms m is chosen to meet required accuracy.

5 Construction of operational matrix

The orthonormal polynomials (as shown in eq. (3.1-3.5)) can be expressed as:

$$\int_0^{\zeta} \phi_o(\eta)d\eta = \phi_o(\zeta) + \frac{1}{2\sqrt{3}}\phi_1(\zeta) \tag{5.1}$$

$$\begin{aligned}\int_0^{\zeta} \phi_i(x)dx = & \frac{1}{2\sqrt{(2i-1)(2i+1)}}\phi_{i-1}(\zeta) \\ & + \frac{1}{2\sqrt{(2i+1)(2i+3)}}\phi_{i+1}(\zeta), \quad (\,for\, i = 1\,, 2, \ldots, m)\end{aligned} \tag{5.2}$$

Relations (5.1-5.2) can be represented in combined form as:

$$\int_0^{\zeta} \phi(\eta)d\eta = \Theta_{(m+1)}\, \phi(\zeta), \tag{5.3}$$

where $\zeta \in [0,1]$ and Θ_{m+1} is operational matrix of order $(m+1)$ given as :

$$\Theta_{m+1} = \frac{1}{2}\begin{bmatrix} 1 & \frac{1}{\sqrt{1.3}} & 0 & \cdots & 0 \\ \frac{-1}{\sqrt{1.3}} & 0 & \frac{1}{\sqrt{3.5}} & \cdots & 0 \\ 0 & \frac{-1}{\sqrt{3.5}} & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & \frac{1}{\sqrt{(2m-1).(2m+1)}} \\ 0 & 0 & \cdots & \frac{-1}{\sqrt{(2m-1).(2m+1)}} & 0 \end{bmatrix} \tag{5.4}$$

6 Solution of Linear Volterra Integral Equations

Consider the linear Volterra integral equation of second kind:

$$y(\zeta) = f(\zeta) + \int_0^\zeta \kappa(\zeta, x)\ y(x)dx, \quad 0 \leq \zeta \leq 1 \tag{6.1}$$

where $y(\zeta)$ is some real valued function, $f(\zeta)$ and $k(\zeta, x)$ are continuous functions defined on $I = [0, 1]$ and $S = \{(\zeta,\ x)\ : 0 \leq x \leq \zeta \leq 1\}$ respectively. Following the classical theory of Volterra integral equations, eq. (6.1) possesses a unique solution in $C\,[0, 1]$. Moreover, if $f(\zeta)$ and $k(\zeta, x)$ are continuously $n - differential$ on $[0, 1]$ and S respectively, the unique solution of eq. (6.1) is also continuously $n - differential$ on $[0, 1]$.

Representing $y(\zeta)$ and $f(\zeta)$ as :

$$y(\zeta) = C^T\,\phi(\zeta) \tag{6.2}$$

$$f(\zeta) = F^T\,\phi(\zeta), \tag{6.3}$$

eq. (6.1) can be re-written as:

$$C^T\phi(\zeta) = F^T\phi(\zeta) + C^T \int_0^\zeta \kappa(\zeta, x)\ \phi(x)dx = F^T\phi(\zeta) + C^T\,\Phi_{m+1}\ \phi(\zeta) \tag{6.4}$$

which gives

$$C^T = (I - \Phi_{m+1})^{-1}F^T \tag{6.5}$$

where, $\Phi_{m+1}\ \phi(\zeta)\ =\ \int_0^\zeta \kappa(\zeta, x)\ \phi(x)dx$, and Φ_{m+1} is associate matrix of Θ_{m+1} of order $m+1$, for illustration, it can be readily observed from eq. (5.3), that $\Phi_{m+1} = c\,\Theta_{m+1}$ if $\kappa(\zeta, x) = c$ (constant) and $\Phi_{m+1} = \Theta^j_{m+1}$ if $\kappa(\zeta, x) = (\zeta - x)^j, (j > 0)$.

7 Error Estimate and Convergence Analysis

Theorem 7.1. *Let $y(\zeta)$ be continuous on $[0, 1]$ and $\phi^y_n(\zeta) = \sum\limits_{n=0}^{\infty} c_k\phi_k$ be an approximation of $y(\zeta)$ in terms of orthonormal Bernoulli polynomials (ϕ_k) , and $R_n(\zeta)$ be the remainder due truncation, then following relations hold.*

$$\phi^y_n(\zeta) = y(\zeta) + R_n(\zeta);\ \ \forall\ x \in [0, 1] \tag{7.1}$$

$$\phi^y_n(\zeta) = \int_0^1 y(\eta)d\eta + \sum_{k=0}^{n} \frac{\phi_k(\zeta)}{k} \left(y^{(k-1)}(1) - y^{(k-1)}(0)\right) \tag{7.2}$$

$$R_n(\zeta) = -\frac{1}{n!}\int_0^1 \phi^*_n(\zeta - \eta)y^{(n)}(\eta)d\eta \tag{7.3}$$

where $\phi^*_n(\zeta) = \phi_n(\zeta - [\zeta])$ and $[\cdot]$ is the greatest integer function.

Proof. See Tohidi and Kiliçman [18] or Mahmoud [2].

Theorem 7.2. *Suppose that $y(\zeta) \in C^\infty[0, 1]$ and $\phi^y_n(\zeta)$ is an approximation of $y(\zeta)$ using orthonormal Bernoulli polynomials. Then the error bound of approximation can be obtained as:*

$$e(y) = \|y(\zeta) - \phi^y_n(\zeta)\|_\infty \leq \frac{1}{n!}M \tag{7.4}$$

where, $M = \underset{\zeta\in[0,1]}{Max}\,\phi^y_n(\zeta)y(\zeta)$.

Proof. See Tohidi and Kiliçman [18] or Mahmoud [2].

From these theorems, it is clear that the error may be minimized to required level by including ϕ_n of higher degree. Furthermore, it is also obvious that the error vanishes faster with the inclusion of higher degree ϕ_n.

8 Numerical Examples

In order to discuss and establish the accuracy and effectiveness of the present method, following examples have been taken.

Example 8.1. The Volterra integral equation

$$y(\zeta) = 6\zeta + 3\zeta^2 - \int_0^{\zeta} y(\eta)d\eta \tag{8.1}$$

has exact solution $y(\zeta) = 6\zeta$.

Comparing eq. (8.1) to standard eq. (6.1) and taking $m = 5$, equations (6.2-6.5) yield

$$F^T = \left[4, -\frac{3\sqrt{3}}{2}, \frac{1}{2\sqrt{5}}, 0, 0, 0\right] \tag{8.2}$$

$$C^T = \left[3, -\sqrt{3}, 0, 0, 0, 0\right] \tag{8.3}$$

Substituting eqs. (8.2-8.3) and $\phi(\zeta) = [\phi_0,\ \phi_1,\ \phi_2,\ ..., \phi_5]^T$ back into eq. (6.2), the exact solution $y(\zeta) = 6\zeta$ of eq. (8.1) is obtained.

Example 8.2. Let us consider the Volterra integral equation of second kind

$$y(\zeta) = 1 + \zeta - \zeta^2 + \int_0^{\zeta} y(\eta)d\eta,\ \ 0 < \zeta < 1 \tag{8.4}$$

which has exact solution $y(\zeta) = 1 + 2\zeta$.

Applying the present method to eq. (8.4) for $m = 5$ as in $example - 1$, we get:

$$F^T = \left[\frac{7}{6}, 0, -\frac{1}{6\sqrt{5}}, 0, 0, 0\right] \tag{8.5}$$

$$C^T = \left[2, -\frac{393379}{398959\sqrt{3}} - \frac{1860\sqrt{3}}{398959}, 0, 0, 0, 0\right] \tag{8.6}$$

Substituting the values of C^T and F^T from eqs. (8.5-8.6) and $\phi(\zeta)$ into eq. (6.2), the exact solution $y(\zeta) = 1 + 2\zeta$ of eq. (8.4) is obtained.

Example 8.3. Consider the following convolution integral equation

$$y(\zeta) = 2 - 2e^{\zeta} + \zeta + \frac{1}{2}\zeta^2 - \textstyle\int_0^{\zeta} (\zeta - x)\, y(x)dx \tag{8.7}$$

having exact solution $y(\zeta) = 1 - e^{\zeta}$. Application of present method for $m = 9$ to eq. (8.7), C^T and F^T are obtained as :

$$F^T = \left[\begin{array}{l} -\frac{3130455131}{4066070400}, \frac{2002713497}{2129846400\sqrt{3}}, -\frac{1425989}{7260840\sqrt{5}}, -\frac{578590253}{20766002400\sqrt{7}}, \\ -\frac{77072}{116475975}, -\frac{1454399}{13214728800\sqrt{11}}, -\frac{445943}{89453548800\sqrt{13}}, \\ -\frac{35531}{188278421760\sqrt{15}}, \frac{47}{8132140800\sqrt{17}}, -\frac{1}{8821612800\sqrt{19}} \end{array}\right] \tag{8.8}$$

$$C^T = \begin{bmatrix} -0.7182286, 0.4878996, -0.0624901, -0.0063109, \\ -0.0003189, -0.101188 \times 10^{-4}, -2.177076 \times 10^{-7}, \\ 4.442934 \times 10^{-9}, 8.4884008 \times 10^{-10}, -7.7385133 \times 10^{-12} \end{bmatrix} \tag{8.9}$$

With help of eqs. (8.8-8.9),an approximate solution to eq. (8.7) is obtained as :

$$\begin{aligned} y(\zeta) &= 0.002879 - 1.033944\zeta - 0.416883\zeta^2 - 0.2173745\zeta^3 \\ &-0.048615\zeta^4 - 0.005751\zeta^5 - 0.001212\zeta^6 + 0.000225\zeta^7 \\ &-0.000038\zeta^8 - 0.000002\zeta^9 \end{aligned} \tag{8.10}$$

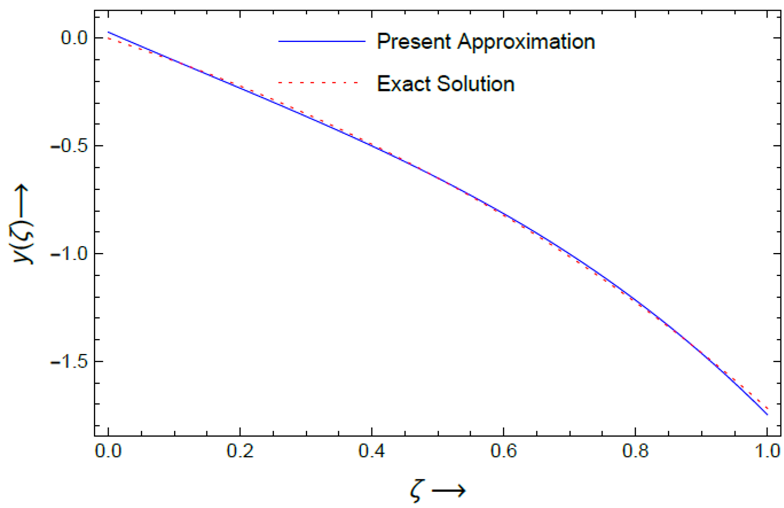

Figure 1. Comparison of Exact Solution and Approximate Solution of Example 3 for $m = 9$.

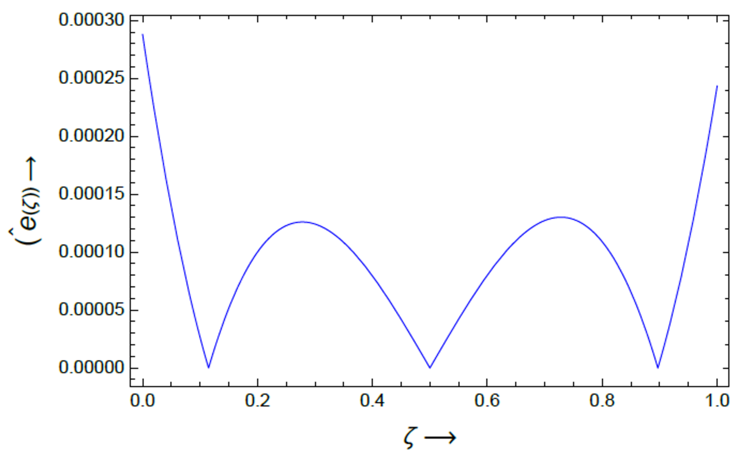

Figure 2. Absolute error, $\hat{e}(\zeta)$, between exact and approximate solution of example 3 for $m = 9$.

Example 8.4. Consider the following integral equation

$$\begin{aligned} y(\zeta) = -1 - \zeta^2 - \tfrac{\zeta^3}{3} + 2\cosh\zeta - \sinh\zeta \\ + \int_0^\zeta (\zeta - \eta)^2 y(\eta) d\eta \; ; \qquad (0 < \zeta < 1) \end{aligned} \tag{8.11}$$

The exact solution of this equation is $y(\zeta) = 1 - \sinh\zeta$.

Applying the present method for $m = 9$, we get,

$$F^T = \begin{bmatrix} \frac{1417609}{3628800}, -\frac{1025707}{1478400\sqrt{3}}, -\frac{205349}{1995840\sqrt{5}}, -\frac{322261}{18532800\sqrt{7}}, \\ \frac{19}{54600}, -\frac{17}{5896800\sqrt{11}}, \frac{19}{7257600\sqrt{13}}, -\frac{1}{130690560\sqrt{15}}, \\ \frac{1}{345945600\sqrt{17}}, -\frac{1}{17643225600\sqrt{19}} \end{bmatrix} \tag{8.12}$$

$$C^T = \begin{bmatrix} 0.45687367, \ -0.33369513, \ -0.0197673, \ -0.00360093, \\ -0.00010456, \ -0.00001135, \ -2.18846562 \times 10^{-7}, \\ -1.69010986 \times 10^{-8}, \ -2.33670044 \times 10^{-10}, \ 0 \end{bmatrix} \tag{8.13}$$

and the solution $y(\zeta)$ is obtained as-

$$y(\zeta) = C^T.\phi(\zeta) \tag{8.14}$$

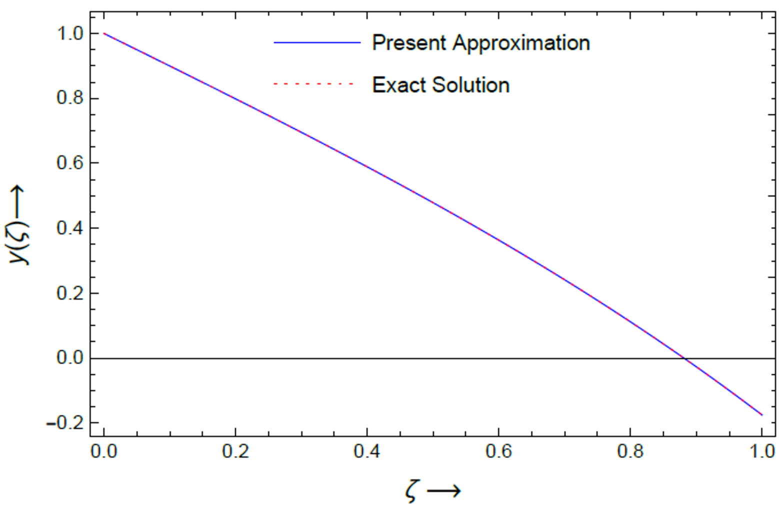

Figure 3. Comparison of Exact and Approximate Solutions of Example 4 for $m = 9$.

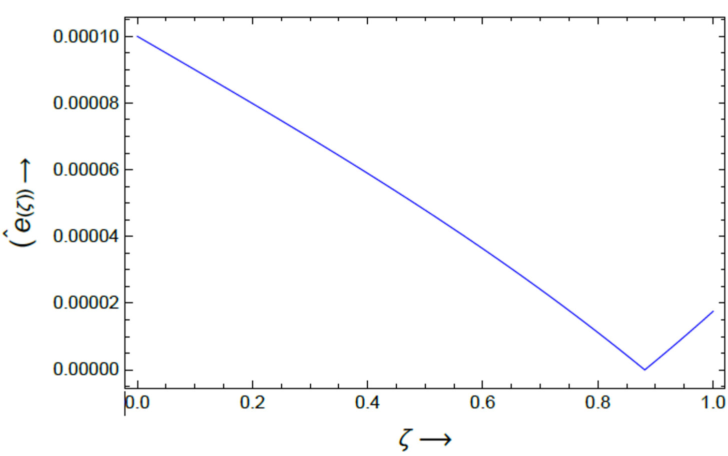

Figure 4. Absolute error, $\hat{e}(\zeta)$, between exact and approximate solutions of example 4 for $m = 9$.

9 Conclusion

In this work, we have discussed a newly developed method to find approximate solution of linear Volterra integral equations of second kind by using Bernoulli polynomials. The process includes the derivation of an operation matrix and orthonormal polynomials. With the present process, an integral equation is converted into a system of algebraic equations with unknown coefficients, which are easily obtained with the help of coefficients generated from known part of the integral equation and operational matrix. With the help of four examples, it has been demonstrated that this method gives either exact solution of an integral equation or an approximation in series form. Required accuracy of solution can be attained with approximation series by taking Bernoulli Polynomials of appropriate order.

In examples 1 and 2, present method gives the exact solution with just 5 orthonormal polynomials. While, in examples 3 and 4, an approximate solution was derived with help of first nine orthonormal polynomials. The errors in examples 3 and 4 are very small in magnitude, which establish the efficacy of the present method.

The beauty of this method lies in that the method is easy for computer programming due to trigonal operational matrix, which enables to employ desired number of orthonormal Bernoulli polynomials to increase the accuracy of numerical solution.

References

[1] K. E. Atkinson, W. Han and D. Stewart, *Numerical Solution of Ordinary Differential Equations*, Wiley, New Jersey, USA, (2011).

[2] M. Behroozifar and N. Habibi, A numerical approach for solving a class of fractional optimal control problems via operational matrix Bernoulli polynomials, *Journal of Vibration and Control* **24(12)**, 2494–2511 (2018).

[3] A. H. Bhrawy, E. Tohidi and F. Soleymani, A new Bernoulli matrix method for solving high-order linear and nonlinear Fredholm integro-differential equations with piecewise intervals, *Applied Mathematics and Computation* **219(2)**, 482–497 (2012).

[4] G. S. Cheon, A note on the Bernoulli and Euler polynomials, *Applied Mathematics Letters* **16(3)**, 365–368 (2003).

[5] F. A. Costabile and F. Dell'Accio, A new approach to Bernoulli polynomials, *Rendiconti di Matematica, Serie VII* **26**, 1–12 (2006).

[6] F. A. Costabile and F. Dell'Accio, Expansion over a rectangle of real functions in bernoulli polynomials and applications, *BIT Numerical Mathematics* **51(3)**, 451–464 (2001).

[7] E. Kreyszig, *Introductory functional analysis with applications*, John Wiley and Sons Press, New York, USA, (1978).

[8] B. Kurt and Y. Simsek, Notes on generalization of the Bernoulli type polynomials, *Applied Mathematics and Computation* **218(3)**, 906–911 (2011).

[9] D. Q. Lu, Some properties of Bernoulli polynomials and their generalizations, *Applied Mathematics Letters* **24(5)**, 746–751 (2011).

[10] K. Maleknejad, S. Sohrabi and Y. Rostami, Numerical solution of nonlinear Volterra integral equations of the second kind by using Chebyshev polynomials, *Applied Mathematics and Computation* **188(1)**, 123–128 (2007).

[11] M. Mohsenyzadeh, Bernoulli operational Matrix method of linear Volterra integral equations, *Journal of Industrial Mathematics* **8(3)**, 201–207 (2016).

[12] M. Momenzadeh, Approximation Properties of q-Bernoulli Polynomials, *Abstract and Applied Analysis* **2017(Article ID 9828065)**, 1–6 (2017).

[13] P. Natalini and A. Bernardini, A generalization of the Bernoulli polynomials, *Journal of Applied Mathematics* **3(3)**, 155–163 (2003).

[14] S. Nemati, Numerical solution of Volterra-Fredholm integral equations using Legendre collocation method, *Journal of Computational and Applied Mathematics* **278**, 29-36 (2015).

[15] M. A. Rahman, M. S. Islam and M. M. Alam, Numerical Solutions of Volterra Integral Equations Using Laguerre Polynomials, *Journal of Scientific Research* **4(2)**, 357–364 (2012).

[16] P. K. Sahu and B. Mallick, Approximate Solution of Fractional Order Lane–Emden Type Differential Equation by Orthonormal Bernoulli's Polynomials, *International Journal of Applied and Computational Mathematics* **5(89)** (2019). https://doi.org/10.1007/s40819-019-0677-0

[17] N. Samadyar and F. Mirzaee. Numerical scheme for solving singular fractional partial integro-differential equation via orthonormal Bernoulli polynomials, *International Journal of Numerical Modelling: Electronic Networks, Devices and Fields* **32(6)** (2019). https://doi.org/10.1002/jnm.2652

[18] E. Tohidi, A. H. Bhrawy and K. Erfani, A collocation method based on Bernoulli operational matrix for numerical solution of generalized pantograph equation, *Applied Mathematical Modelling* **37(6)**, 4283–4294 (2013).

[19] E. Tohidi and A. Kiliçman, A collocation method based on the bernoulli operational matrix for solving nonlinear BVPs which arise from the problems in calculus of variation, *Mathematical Problems in Engineering* **2013(Article ID 757206)**, 1–9 (2013).

[20] L. Xu, Variational iteration method for solving integral equations, *Computers and Mathematics with Applications* **54(7-8)**, 1071–1078 (2007).

[21] S. A. Yousefi. Numerical solution of Abel's integral equation by using Legendre wavelets, *Applied Mathematics and Computation* **175(1)**, 575–580 (2006).

Author information

Udaya P. Singh, Department of Applied Sciences (Mathematics), Rajkiya Engineering College, Sonbhadra, U.P., India.
E-mail: `upsingh1980@gmail.com`