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SYMMETRY ANALYSIS OF THE KORTEWEG-DE VRIES EQUATION

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Abstract In this paper, we explain how symmetry analysis is applied to the Korteweg-de Vries equation. This equation under study is a nonlinear partial differential equation of third order. We develop the required Lie invariance condition using local inverse theorem, an approach different from the existing Lie-Bäcklund operators and Taylor series method. This condition is then used to obtain the determining equations of the admitted Lie group. The corresponding Lie algebra is found to be solvable. We obtain the invariant surface condition, symmetry algebra and the group classification admitted by this equation. Further, we obtain the exact and similarity solutions of this equation, comment on them and represent them graphically.

1 Introduction

Over hundred years ago, Sophus Lie introduced Lie groups to model the continuous symmetries of differential equations. This was the only unified understanding for solving differential equations analytically. This powerful tool that helps study properties of solutions of differential equations has been gaining wide popularity in recent years as the theory is generalized to classes of several differential equations. A symmetry is a transformation that leaves an object unchanged. In [16] it is explained that the regularity of the laws that are independent of certain inessential circumstances are accounted for by symmetries. The application of the concept of symmetries is in physical and applied sciences. A fascinating application of symmetries is its relationship with conservation laws. A great detail on the development of symmetry analysis can be found in [3, 7, 8].

A highly used equation in modelling water waves, called the Korteweg-de Vries (KdV) equation is given by

$$u_t(t,x) + u(t,x)u_x(t,x) + u_{xxx}(t,x) = 0,$$

where $t,\ x \in \mathbb{R}$ and u is a real valued function.
This equation which contrasts the Burgers' equation, the literature for which can be found in [4], finds practical applications in optical fibres and shallow water theory. By using Harrison method and first integral method, [14] exhibits solutions of the KdV equation.

While several partial differential equations like the semi linear heat equation in [17] and the reaction diffusion equation in [18], to name a few, have been solved and analyzed numerically, a growing research interest to investigate the symmetry properties of differential equations is on the rise. Lie symmetry approach is used to report the admittance of continuous point symmetries of Quispel, Roberts and Thompson type nonlinear partial differential equations by [20]. They also analyze the integrability nature of the partial differntial equations through degree growth of iterates. Further, invariance analysis, exact solutions and conservation laws of (2+1)-dimensional long wave equations are reported in [22]. In addition, [6] reports the generalized group invariant solutions of the (2+1) Date-Jimbo-Kashiwara-Miwa equations using symmetry analysis. The wave equation with delay, a second order partial differential equation is thoroughly investigated by [12] and the invariant solutions of this equation are given. Lot of papers can be found on symmetry analysis for various forms of the KdV equation. The perturbed KdV equation is studied in

[1] using the Lagrange method and the forced KdV equation is analysed in [13]. The symmetry reductions of the (2+1)-dimensional KdV equation with variable coefficients are obtained in [19] and all two-dimensional solvable symmetry subalgebras have been determined. The concept of integrable nonholonomic deformation found for KdV equations is extended by [21] to modified KdV (mKdV) equation. It is noteworthy to mention here that a complete Lie symmetry analysis of the damped wave equation with time dependent coefficients is investigated in [5], and the invariant and exact solutions generated from the symmetries is presented. The Kawahara-KdV equations have been gaining a lot of interest in recent years and [2] introduces a new solution for these equations. Another class of solutions for the Kawahara-KdV type equations are demonstrated by [10], and by using Lie symmetry reductions these equations are reduced to ordinary differential equations. The (2+1)-dimensional KdV is investigated for its residual symmetries in [25] using a truncated Painlevé expansion. Lie symmetry analysis has also been extended to space-time fractional differential equations in [15, 24]. These papers have motivated us to thoroughly analyse the Lie algebraic structure of the KdV equation.

We organize this paper as follows:
In the following section some preliminaries are given. In the next section, the required Lie invariance condition is developed using a new approach that requires local inverse theorem. This condition is then used in the subsequent section to obtain the corresponding symmetry algebra and determine the solvability of the equation under study. The solutions of the KdV equation are then constructed and similarity solutions, exact solutions and soliton solutions motivated by [23] are presented. Such soliton solutions are also constructed by [11] in studying the fifth-order KdV equation. These solutions are represented graphically. The concluding section summarizes the results obtained. This provides a thorough analysis of the KdV equation which is new in literature.

2 Preliminaries

In this section, we introduce the terminologies and results required to understand the solvability of differential equations. We start by defining one-parameter groups.

Definition 2.1. A one-parameter group of transformations, $\bar{t}_i = g_i(t_j, \delta)$, is a set of transformations satisfying the following:

(i) (*Closure*) Successive application of two transformations yields another transformation of the set.

(ii) (*Identity*) Every transformation has an identity.

(iii) (*Inverse*) Every transformation has an inverse.

Lie groups are defined as:

Definition 2.2. A group that is also a differentiable manifold is known as a Lie group. Consider the binary operation,

$$\mu : G \times G \to G, \quad \mu(t, x) = tx.$$

For a Lie group, the mapping $(t, x) \mapsto t^{-1}x$ is a smooth mapping of the product manifold into G.

Example 2.3. Consider the Lie group of rotation matrices denoted by $SO(2, \mathbb{R})$. These form a subgroup of the group (under multiplication) of 2×2 real invertible matrices denoted by $GL(2, \mathbb{R})$. Let the parameter δ denote the rotation angle. Then we can parametrize this group as follows:

$$SO(2, \mathbb{R}) = \left\{ \begin{pmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{pmatrix} : \delta \in \mathbb{R}/2\pi\mathbb{Z} \right\}.$$

Multiplying any two elements of $SO(2, \mathbb{R})$ yields another element of $SO(2, \mathbb{R})$ with the rotation angle as the addition of the two angles, and on inversion, we see that we get the opposite angle. Therefore both multiplication and inversion are differentiable maps.

For each $i, j = 1, 2, \cdots, n$, the functions $g_i(t_j, \delta)$, are referred to as the *global form* of the group.
For three variables (the case for partial differential equations, one being dependent while the other two being independent), we shall denote the variables by t, x and u. Thus, we consider the transformations

$$\bar{t} = f(t, x, u, \delta), \quad \bar{x} = g(t, x, u, \delta), \quad \bar{u} = h(t, x, u, \delta) \tag{2.1}$$

Representing equation (2.1) as a Taylor series about $\delta = 0$, we get

$$\bar{t} = t + \delta \left(\frac{d\bar{t}}{d\delta}\right)_{\delta=0} + O(\delta^2),\ \bar{x} = x + \delta \left(\frac{d\bar{x}}{d\delta}\right)_{\delta=0} + O(\delta^2),\ \bar{u} = u + \delta \left(\frac{d\bar{u}}{d\delta}\right)_{\delta=0} + O(\delta^2), \tag{2.2}$$

where $O(\delta^2)$ indicates terms involving only powers of δ greater than or equal to two.
Let,

$$\left(\frac{d\bar{t}}{d\delta}\right)_{\delta=0} = T(t, x, u), \quad \left(\frac{d\bar{x}}{d\delta}\right)_{\delta=0} = X(t, x, u), \quad \left(\frac{d\bar{u}}{d\delta}\right)_{\delta=0} = U(t, x, u). \tag{2.3}$$

Then we get,

$$\bar{t} = t + \delta T(t, x, u) + O(\delta^2), \quad \bar{x} = x + \delta X(t, x, u) + O(\delta^2), \quad \bar{u} = u + \delta U(t, x, u) + O(\delta^2). \tag{2.4}$$

Equation (2.4) is referred to as *infinitesimal transformations*.
Further, T, X and U are called *coefficients of the infinitesimal transformations* or simply *infinitesimals*.

A property that is intrinsic to determining equations is that a set of solutions of any determining equations forms what is called a *Lie algebra.*
So now we turn to define Lie algebras and related concepts. (This term was introduced by H. Weyl; Sophus Lie himself used the term *infinitesimal group.*)

Definition 2.4. Let L be a vector space over some field $\mathbb{F}$. Suppose that there is a binary operation "[,]" satisfying

$$[G_\alpha, G_\beta] = -[G_\beta, G_\alpha],\ [G_\alpha, [G_\beta, G_\gamma]] + [G_\beta, [G_\gamma, G_\alpha]] + [G_\gamma, [G_\alpha, G_\beta]] = 0,\ \forall\ G_\alpha, G_\beta, G_\gamma \in L.$$

Then L is called a Lie algebra over $\mathbb{F}$.

As we require the concept of an ideal to define a solvable Lie algebra, we define an ideal first.

Definition 2.5. A Lie subalgebra M of a Lie algebra L is an ideal in L, if $[G_1, G_2] \in M$, $\forall\ G_1 \in M,\ G_2 \in L.$

We conclude this section by defining a solvable Lie algebra.

Definition 2.6. A $r-$dimensional Lie algebra L_r is said to be solvable if we can find a sequence of Lie subalgebras such that

(i) $\{0\} = L \subset L_1 \subset L_2 \subset \cdots \subset L_r$, where $L_i,\ 1 \leq i \leq r$ denotes a Lie subalgebra of dimension i.

(ii) L_i is an ideal of $L_{i+1},\ \forall\ 1 \leq i \leq r-1.$

Remark 2.7. (i) A Lie algebra of dimension two is always solvable.

(ii) A three or higher dimensional Lie algebra need not be solvable.

More details on Lie algebras can be found in [9].

3 Invariance Condition for Partial Differential Equations of Third Order

On differentiating equation (2.1) with respect to t and x, we can calculate the prolongation of a given transformation. We define the following *total derivatives*:

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t} + \cdots, \tag{3.1}$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + \cdots. \tag{3.2}$$

Assuming that the Jacobian is non-zero, the first two equations of (2.1) can be locally inverted to give t and x in terms of $\bar{t}$ and $\bar{x}$, that is,

$$\mathcal{J} = \begin{vmatrix} D_t\bar{t} & D_t\bar{x} \\ D_x\bar{t} & D_x\bar{x} \end{vmatrix} \neq 0, \quad \text{where} \quad u = u(x,t). \tag{3.3}$$

Since the last equation of (2.1) contains $\bar{u}$ as a function of some variables, one of which is $u(t,x)$, we see that if equation (3.3) is satisfied, then by local inverse function theorem, the last equation of (2.1) can be rewritten as

$$\bar{u} = \bar{u}(\bar{t}, \bar{x}). \tag{3.4}$$

Application of the chain rule to equation (3.4), gives

$$\begin{bmatrix} D_t\bar{u} \\ D_x\bar{u} \end{bmatrix} = \begin{bmatrix} D_t\bar{t} & D_t\bar{x} \\ D_x\bar{t} & D_x\bar{x} \end{bmatrix} \begin{bmatrix} \bar{u}_{\bar{t}} \\ \bar{u}_{\bar{x}} \end{bmatrix},$$

and therefore by (Cramer's rule)

$$\bar{u}_{\bar{t}} = \frac{1}{\mathcal{J}} \begin{vmatrix} D_t\bar{u} & D_t\bar{x} \\ D_x\bar{u} & D_x\bar{x} \end{vmatrix}, \quad \bar{u}_{\bar{x}} = \frac{1}{\mathcal{J}} \begin{vmatrix} D_t\bar{t} & D_t\bar{u} \\ D_x\bar{t} & D_x\bar{u} \end{vmatrix} \tag{3.5}$$

Equation (3.5) can be simplified to get the extended infinitesimal representation,

$$\bar{u}_{\bar{t}} = u_t + \delta U_{[t]} + O(\delta^2), \quad \bar{u}_{\bar{x}} = u_x + \delta U_{[x]} + O(\delta^2), \tag{3.6}$$

where

$$\begin{aligned} U_{[t]} &= D_t(U) - u_x D_t(X) - u_t D_t(T), \\ U_{[x]} &= D_x(U) - u_x D_x(X) - u_t D_x(T). \end{aligned} \tag{3.7}$$

The explicit expression for equation (3.7) is

$$\begin{aligned} U_{[t]} &= U_t - X_t u_x + (U_u - T_t)u_t - X_u u_x u_t - T_u u_t^2, \\ U_{[x]} &= U_x + (U_u - X_x)u_x - T_x u_t - X_u u_x^2 - T_u u_x u_t. \end{aligned} \tag{3.8}$$

Continuing the procedure, we can obtain the second-order prolongations as follows:

$$\bar{u}_{\overline{tt}} = \frac{1}{\mathcal{J}} \begin{vmatrix} D_t\bar{x} & D_t\bar{u}_{\bar{t}} \\ D_x\bar{x} & D_x\bar{u}_{\bar{t}} \end{vmatrix}, \quad \bar{u}_{\overline{xx}} = \frac{1}{\mathcal{J}} \begin{vmatrix} D_t\bar{u}_{\bar{x}} & D_t\bar{t} \\ D_x\bar{u}_{\bar{x}} & D_x\bar{t} \end{vmatrix} \tag{3.9}$$

$$\bar{u}_{\overline{xt}} = \frac{1}{\mathcal{J}} \begin{vmatrix} D_t\bar{x} & D_t\bar{u}_{\bar{x}} \\ D_x\bar{x} & D_x\bar{u}_{\bar{x}} \end{vmatrix} = \frac{1}{\mathcal{J}} \begin{vmatrix} D_t\bar{u}_{\bar{t}} & D_t\bar{t} \\ D_x\bar{u}_{\bar{t}} & D_x\bar{t} \end{vmatrix} \tag{3.10}$$

On simplifying (3.9) and (3.10) we get the extended infinitesimal representations, namely

$$\bar{u}_{\overline{tt}} = u_{tt} + \delta U_{[tt]} + O(\delta^2), \quad \bar{u}_{\overline{xx}} = u_{xx} + \delta U_{[xx]} + O(\delta^2), \quad \bar{u}_{\overline{tx}} = u_{tx} + \delta U_{[tx]} + O(\delta^2), \tag{3.11}$$

where

$$U_{[tt]} = D_t(U_{[t]}) - u_{tx} D_t(X) - u_{tt} D_t(T), \quad U_{[xx]} = D_x(U_{[x]}) - u_{xx} D_x(X) - u_{tx} D_x(T), \tag{3.12}$$

and,

$$\begin{aligned} U_{[tx]} &= D_t(U_{[x]}) - u_{xx}D_t(X) - u_{tx}D_t(T) \\ &= D_x(U_{[t]}) - u_{tx}D_x(X) - u_{tt}D_x(T) \end{aligned} \tag{3.13}$$

The explicit expressions for $U_{[tt]}, U_{[xx]}, U_{[tx]}$ given by equations (3.12) and (3.13) are

$$\begin{aligned} U_{[tt]} = U_{tt} - X_{tt}u_x + (2U_{tu} - T_{tt})u_t - 2X_{tu}u_xu_t + (U_{uu} - 2T_{tu})u_t^2 - X_{uu}u_xu_t^2 \\ - T_{uu}u_t^3 - 2X_tu_{xt} - 2X_uu_tu_{xt} + (U_u - 2T_t)u_{tt} - X_uu_xu_{tt} - 3T_uu_tu_{tt}, \end{aligned} \tag{3.14}$$

$$\begin{aligned} U_{[xx]} = U_{xx} + (2U_{xu} - X_{xx})u_x - T_{xx}u_t + (U_{uu} - 2X_{xu})u_x^2 - 2T_{xu}u_xu_t - X_{uu}u_x^3 \\ - T_{uu}u_x^2u_t + (U_u - 2X_x)u_{xx} - 2T_xu_{xt} - 3X_uu_xu_{xx} - T_uu_tu_{xx} - 2T_uu_xu_{xt}, \end{aligned} \tag{3.15}$$

$$\begin{aligned} U_{[xt]} = U_{xt} + (U_{tu} - X_{xt})u_x + (U_{xu} - T_{xt})u_t - X_{tu}u_x^2 + (U_{uu} - X_{xu} - T_{tu})u_xu_t \\ - T_{xu}u_t^2 - X_{uu}u_x^2u_t - T_{uu}u_xu_t^2 - X_tu_{xx} - X_uu_tu_{xx} + (U_u - X_x - T_t)u_{xt} \\ - 2X_uu_xu_{xt} - 2T_uu_tu_{xt} - T_xu_{tt} - T_uu_xu_{tt}. \end{aligned}$$

Extending this procedure to third-order prolongations, we get

$$\bar{u}_{\overline{ttt}} = \frac{1}{\mathcal{J}}\begin{vmatrix} D_t\bar{x} & D_t\bar{u}_{\overline{tt}} \\ D_x\bar{x} & D_x\bar{u}_{\overline{tt}} \end{vmatrix}, \quad \bar{u}_{\overline{xxx}} = \frac{1}{\mathcal{J}}\begin{vmatrix} D_t\bar{u}_{\overline{xx}} & D_t\bar{t} \\ D_x\bar{u}_{\overline{xx}} & D_x\bar{t} \end{vmatrix} \tag{3.16}$$

$$\bar{u}_{\overline{ttx}} = \frac{1}{\mathcal{J}}\begin{vmatrix} D_t\bar{x} & D_t\bar{u}_{\overline{tx}} \\ D_x\bar{x} & D_x\bar{u}_{\overline{tx}} \end{vmatrix} = \bar{u}_{\overline{xtt}} = \frac{1}{\mathcal{J}}\begin{vmatrix} D_t\bar{u}_{\overline{tt}} & D_t\bar{t} \\ D_x\bar{u}_{\overline{tt}} & D_x\bar{t} \end{vmatrix} = \bar{u}_{\overline{txt}} = \frac{1}{\mathcal{J}}\begin{vmatrix} D_t\bar{x} & D_t\bar{u}_{\overline{xt}} \\ D_x\bar{x} & D_x\bar{u}_{\overline{xt}} \end{vmatrix} \tag{3.17}$$

$$\bar{u}_{\overline{xxt}} = \frac{1}{\mathcal{J}}\begin{vmatrix} D_t\bar{x} & D_t\bar{u}_{\overline{xx}} \\ D_x\bar{x} & D_x\bar{u}_{\overline{xx}} \end{vmatrix} = \bar{u}_{\overline{xtx}} = \frac{1}{\mathcal{J}}\begin{vmatrix} D_t\bar{u}_{\overline{xt}} & D_t\bar{t} \\ D_x\bar{u}_{\overline{xt}} & D_x\bar{t} \end{vmatrix} = \bar{u}_{\overline{txx}} = \frac{1}{\mathcal{J}}\begin{vmatrix} D_t\bar{u}_{\overline{tx}} & D_t\bar{t} \\ D_x\bar{u}_{\overline{tx}} & D_x\bar{t} \end{vmatrix} \tag{3.18}$$

On simplifying (3.16), (3.17) and (3.18) we get the extended infinitesimal representations, namely

$$\begin{aligned} \bar{u}_{\overline{ttt}} &= u_{ttt} + \delta U_{[ttt]} + O(\delta^2), \quad \bar{u}_{\overline{xxx}} = u_{xxx} + \delta U_{[xxx]} + O(\delta^2), \\ \bar{u}_{\overline{ttx}} &= u_{ttx} + \delta U_{[ttx]} + O(\delta^2), \quad \bar{u}_{\overline{txx}} = u_{txx} + \delta U_{[txx]} + O(\delta^2), \end{aligned} \tag{3.19}$$

where

$$U_{[ttt]} = D_t(U_{[tt]}) - u_{ttx}D_t(X) - u_{ttt}D_t(T), \; U_{[xxx]} = D_x(U_{[xx]}) - u_{xxx}D_x(X) - u_{txx}D_x(T), \tag{3.20}$$

$$\begin{aligned} U_{[ttx]} &= D_t(U_{[tx]}) - u_{txx}D_t(X) - u_{ttx}D_t(T) \\ &= D_x(U_{[tt]}) - u_{ttx}D_x(X) - u_{ttt}D_x(T) \end{aligned} \tag{3.21}$$

and,

$$\begin{aligned} U_{[txx]} &= D_t(U_{[xx]}) - u_{xxx}D_t(X) - u_{txx}D_t(T) \\ &= D_x(U_{[tx]}) - u_{txx}D_x(X) - u_{ttt}D_x(T) \end{aligned} \tag{3.22}$$

For any interval I in $\mathbb{R}$ and for any open set D, if $G : I \times D^{11} \to \mathbb{R}$ is a differentiable function, then by the invariance condition we should have,

$$\begin{aligned}
0 &= H(\bar{t}, \bar{x}, \bar{u}, \bar{u}_{\bar{t}}, \bar{u}_{\bar{x}}, \bar{u}_{\overline{tt}}, \bar{u}_{\overline{xx}}, \bar{u}_{\overline{ttt}}, \bar{u}_{\overline{xxx}}, \bar{u}_{\overline{tx}}, \bar{u}_{\overline{ttx}}, \bar{u}_{\overline{txx}}) \\
&= H(t + \delta T + O(\delta^2), x + \delta X + O(\delta^2), u + \delta U + O(\delta^2), u_t + \delta U_{[t]} + O(\delta^2), \\
&\quad u_x + \delta U_{[x]} + O(\delta^2), u_{tt} + \delta U_{[tt]} + O(\delta^2), u_{xx} + \delta U_{[xx]} + O(\delta^2), \\
&\quad u_{ttt} + \delta U_{[ttt]} + O(\delta^2), u_{xxx} + \delta U_{[xxx]} + O(\delta^2), u_{tx} + \delta U_{[tx]} + O(\delta^2), \\
&\quad u_{ttx} + \delta U_{[ttx]} + O(\delta^2), u_{txx} + \delta U_{[txx]} + O(\delta^2)) \\
&= H(t, x, u, u_t, u_x, u_{tt}, u_{xx}, u_{ttt}, u_{xxx}, u_{tx}, u_{ttx}, u_{txx}) + \delta\Big(TH_t + XH_x + UH_u \\
&\quad + U_{[t]}H_{u_t} + U_{[x]}H_{u_x} + U_{[tt]}H_{u_{tt}} + U_{[xx]}H_{u_{xx}} + U_{[ttt]}H_{u_{ttt}} + U_{[xxx]}H_{u_{xxx}} \\
&\quad + U_{[tx]}H_{u_{tx}} + U_{[ttx]}H_{u_{ttx}} + U_{[txx]}H_{u_{txx}}\Big) + O(\delta^2)
\end{aligned}$$

On comparing the coefficient of δ, we see that

$$\begin{aligned}
&TH_t + XH_x + UH_u + U_{[t]}H_{u_t} + U_{[x]}H_{u_x} + U_{[tt]}H_{u_{tt}} + U_{[xx]}H_{u_{xx}} \\
&\qquad + U_{[ttt]}H_{u_{ttt}} + U_{[xxx]}H_{u_{xxx}} + U_{[tx]}H_{u_{tx}} + U_{[ttx]}H_{u_{ttx}} + U_{[txx]}H_{u_{txx}} = 0.
\end{aligned}$$

We have the extension of the prolongation given by,

$$\begin{aligned}
\zeta^{(1)} &= TH_t + XH_x + UH_u + U_{[t]}H_{u_t} + U_{[x]}H_{u_x} + U_{[tt]}H_{u_{tt}} + U_{[xx]}H_{u_{xx}} \\
&\quad + U_{[ttt]}H_{u_{ttt}} + U_{[xxx]}H_{u_{xxx}} + U_{[tx]}H_{u_{tx}} + U_{[ttx]}H_{u_{ttx}} + U_{[txx]}H_{u_{txx}}.
\end{aligned} \tag{3.23}$$

Summarizing the above, we get the following:

Theorem 3.1. *The invariance condition for nonlinear second-order partial differential equations is given by* $\zeta^{(1)}H \mid_{H=0} = 0$, *where*
$H(t, x, u, u_t, u_x, u_{tt}, u_{xx}, u_{ttt}, u_{xxx}, u_{tx}, u_{ttx}, u_{txx}) = 0$, *and the operator* $\zeta^{(1)}$ *is defined by equation (3.23) .*

Remark 3.2. The infinitesimal generator of the Lie group (or tangent vector field) is,

$$\zeta^* = T\frac{\partial}{\partial t} + X\frac{\partial}{\partial x} + U\frac{\partial}{\partial u}. \tag{3.24}$$

4 Symmetry Analysis of the Korteweg-de Vries Equation

Consider the KdV equation given by

$$u_t + uu_x + u_{xxx} = 0 \tag{4.1}$$

Applying the invariance condition given by equation (3.23) to equation (4.1), we get

$$U_{[t]} + uU_{[x]} + Uu_x + U_{[xxx]} = 0. \tag{4.2}$$

Using equations (3.19) and (3.8) in equation (4.2), then substituting
$u_{xxx} = -u_t - uu_x$, and splitting the resulting equation with respect to
$u_t, u_x, u_{tx}, u_{xx}, u_{ttt}, u_{ttx}, u_{txx}$ ans various products, we get

$$U_t + uU_x + U_{xxx} = 0, \tag{4.3}$$

$$T_t + uT_x - 3X_x + T_{xxx} = 0, \tag{4.4}$$

$$X_t - 2uX_x + X_{xxx} - 3U_{xxu} - U = 0, \tag{4.5}$$

$$X_u - T_{xxu} = 0, \tag{4.6}$$

$$uX_u - X_{xxu} + U_{xuu} = 0, \tag{4.7}$$

$$T_{xuu} = 0, \quad U_{uuu} - 3X_{xuu} = 0, \quad T_{uuu} = 0, \tag{4.8}$$

$$X_{uuu} = 0, \quad T_{xx} = 0, \quad U_{xu} - X_{xx} = 0, \tag{4.9}$$

$$T_{xu} = 0, \quad U_{uu} - X_{xu} = 0, \quad X_{uu} = 0, \tag{4.10}$$

$$T_x = 0, \quad T_u = 0. \tag{4.11}$$

Solving equations (4.3)-(4.11), we get

$$T_x = 0, \quad T_u = 0, \quad X_u = 0, \quad U_{uu} = 0,$$

solving which we get,

$$T(t,x,u) = \alpha(t), \quad X(t,x,u) = \beta(t,x), \quad U(t,x,u) = \gamma(t,x)u + \rho(t,x),$$

where $\alpha = \alpha(t), \beta = \beta(t,x), \gamma = \gamma(t,x)$ and $\rho = \rho(t,x)$ are arbitrary functions.
Substituting these values of α, β, γ and ρ in equations (4.3)-(4.11), we get

$$(\gamma_t u + \rho_t) + (\gamma_x u + \rho_x)u + \gamma_{xxx}u + \rho_{xxx} = 0, \tag{4.12}$$

$$\beta_t - 2u\beta_x + \beta_{xxx} - 3\gamma_{xx} - \gamma u - \rho = 0, \tag{4.13}$$

$$\alpha_t - 3\beta_x = 0, \quad \gamma_x - \beta_{xx} = 0. \tag{4.14}$$

Splitting equations (4.12) and (4.13) with respect to u, we get

$$\alpha_t - 3\beta_x = 0, \quad \gamma_x - \beta_{xx} = 0, \quad \gamma + 2\beta_x = 0, \quad \gamma_x = 0, \tag{4.15}$$

$$\rho_t + \rho_{xxx} = 0, \quad \gamma_t + \rho_x + \gamma_{xxx} = 0, \quad 3\gamma_{xx} - \beta_t - \beta_{xx} + \rho = 0. \tag{4.16}$$

In order to get the coefficients of the infinitesimal transformations, we solve equations (4.15)-(4.16), the solutions of which are

$$T(t,x,u) = 3c_2 t + c_1, \quad X(t,x,u) = c_4 t + c_2 x + c_3, \quad U(t,x,u) = -2c_2 u + c_4,$$

where $c_1 - c_4$ are arbitrary constants.
The general form of the infinitesimal generator given by

$$\zeta^* = (3c_2 t + c_1)\frac{\partial}{\partial t} + (c_4 t + c_2 x + c_3)\frac{\partial}{\partial x} + (-2c_2 u + c_4)\frac{\partial}{\partial u}.$$

We see that the infinitesimal generators are

$$G_1 = \frac{\partial}{\partial t}, \quad G_2 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}, \quad G_3 = \frac{\partial}{\partial x}, \quad G_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}.$$

The commutator table is given by

	G_1	G_2	G_3	G_4
G_1	0	$3G_1$	0	G_3
G_2	$-3G_1$	0	G_3	$2G_4$
G_3	0	$-G_3$	0	0
G_4	$-G_3$	$-2G_4$	0	0

Choosing,
$L_1 = \{0\}, \quad L_2 = span\{G_3\}, \quad L_3 = span\{G_3, G_4\}, \quad L_4 = span\{G_1, G_3, G_4\},$
$L_5 = span\{G_1, G_2, G_3, G_4\} = L,$
we see that

$$\{0\} = L_1 \subset L_2 \subset L_3 \subset L_4 \subset L_5 = L.$$

Consequently, the Lie algebra L is solvable.

The following theorem summarizes our results:

Theorem 4.1. *The KdV equation given by equation (4.1) admits a four dimensional solvable Lie algebra generated by*

$$G_1 = \frac{\partial}{\partial t}, \quad G_2 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}, \quad G_3 = \frac{\partial}{\partial x}, \quad G_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}.$$

5 Invariant Surface Condition and Solutions of the Korteweg-de Vries Equation

To the characteristic equations given by

$$\frac{dt}{T} = \frac{dx}{X} = \frac{du}{U},$$

we can associate the first-order partial differential equation

$$Tu_t + Xu_x = U. \tag{5.1}$$

Equation (5.1) is referred to as the "invariant surface condition."
So for the KdV equation under study, equation (5.1), becomes

$$(3c_2t + c_1)u_t + (c_4t + c_2x + c_3)u_x = -2c_2u + c_4. \tag{5.2}$$

This is the invariant surface condition for the KdV equation.
We will now explain the reason for this name. Consider the solution

$$u = u(t, x).$$

If we consider

$$\bar{u} = u(\bar{t}, \bar{x}) \tag{5.3}$$

then under the infinitesimal transformations given by equation (2.1), equation (5.3) becomes

$$u + \delta U(t, x, u) + O(\delta^2) = u(t + \delta T(t, x, u) + O(\delta^2), x + \delta X(t, x, u) + O(\delta^2)),$$

then on expanding, we get

$$u + \delta U(t, x, u) + O(\delta^2) = u(t, x) + \delta(T(t, x, u)u_t + X(t, x, u)u_x) + O(\delta^2).$$

If $u = u(t, x)$, then comparing the coefficients of δ, we get equation (5.1).

Having obtained the invariant surface condition associated with the KdV equation, we will now this to find explicit solutions of equation (4.1). We will take particular cases of the invariant surface condition given in equation (5.2).

Let us first consider, $c_1 = 1, c_2 = 0, c_3 = c, c_4 = 0$, where c is an arbitrary constant. Equation (5.2) reduces to

$$u_t + cu_x = 0.$$

whose solution is

$$u = \theta(x - ct).$$

Substituting this into equation (4.1) yields the third order ordinary differential equation

$$\theta'''(r) + \theta(r)\theta'(r) - c\theta'(r) = 0,$$

where $r = x - ct$. Integrating this equation once and suppressing the constant of integration gives

$$\theta'' + \frac{1}{2}\theta^2 - c\theta = 0.$$

One particular solution of this equation is

$$u(t,x) = \theta(x - 4p^2 t) = 12p^2 sech(px), \tag{5.4}$$

where the constant c is chosen as $4p^2$.
As this solution does not depend on time, it is a "steady state solution".
This solution is referred to as the *"one solition"* solution and is graphically represented in Figure 1:

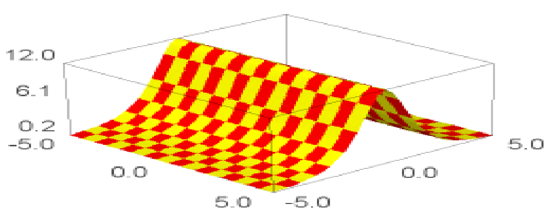

Figure 1. One Soliton Solution $u(t,x) = 12p^2 sech(px),\ t, x \in (-5,5),\ p = 1$

We will now explicitly find, for this case, the Lie group under which the KdV equation is invariant. To do this, we need to solve the system (2.3) subjected to

$$\bar{t} = t,\ \bar{x} = x,\ \bar{u} = u,\ \text{ when }\ \delta = 0.$$

Therefore, with $T(t,x,u) = 1,\ X(t,x,u) = c,\ U(t,x,u) = 0$, the above system can be solved to give

$$\bar{t} = t + \delta, \quad \bar{x} = x + c\delta, \quad \bar{u} = u.$$

We illustrate another choice of constants in equation (5.2) to find solutions of equation (4.1).
Consider $c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 1$ in equation (5.2), which then gives

$$tu_x = 1,$$

which can be solved to yield

$$u = \frac{x}{t}\phi(t). \tag{5.5}$$

Substituting this in the original equation gives

$$t\phi'(t) + \phi(t) = 0,$$

whose solution with an arbitrary constant k is

$$\phi(t) = \frac{k}{t}. \tag{5.6}$$

Substituting equation (5.6) in equation (5.5), we get the exact solution of the KdV equation given by

$$u(t,x) = \frac{k+x}{t}. \tag{5.7}$$

If the space is bounded, then the solution tends to zero as time t tends to infinity while if the space is unbounded, then with the passage of time, the solution becomes bounded.
Equation (5.7) is graphically represented in Figure 2:

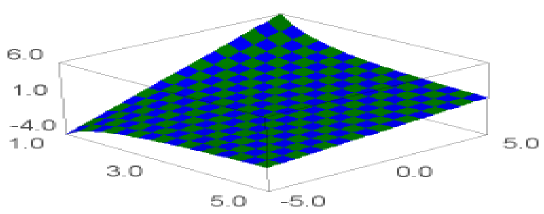

Figure 2. Exact Solution $u(t,x) = \dfrac{k+x}{t},\ t \in (1,5),\ x \in (-5,5),\ k = 1$

For this case, with $T(t,x,u) = 0,\ X(t,x,u) = t,\ U(t,x,u) = 1$, we see that the KdV equation is invariant under

$$\bar{t} = t, \quad \bar{x} = x + t\delta, \quad \bar{u} = u + \delta.$$

As a final illustration, we exhibit the similarity solutions admitted by the KdV equation. For this, consider $c_1 = 0, c_2 = 1, c_3 = 0, c_4 = 0$ in equation (5.2), which then gives

$$3tu_t + xu_x = -2u,$$

which can be solved to yield the similarity solution of the KdV equation given by

$$u(t,x) = t^{-2/3}\psi\left(\frac{x}{t^{1/3}}\right).$$

Substituting the similarity solution in the KdV equation (4.1) gives

$$\psi'''(r) = \left(\frac{2\psi(r) + r\psi'(r)}{3}\right) - \psi(r)\psi'(r), \tag{5.8}$$

where $r = \dfrac{x}{t^{1/3}}$.
Solutions of equation (5.8) will give exact solutions of the KdV equation.

As we now have $T(t, x, u) = 3t,\ X(t, x, u) = x,\ U(t, x, u) = -2u$, we see that the KdV equation is invariant under the stretching group

$$\bar{t} = e^{3\delta}t, \quad \bar{x} = e^{\delta}x, \quad \bar{u} = e^{-2\delta}u.$$

The graphical representation of the similarity solution is shown in Figure 3:

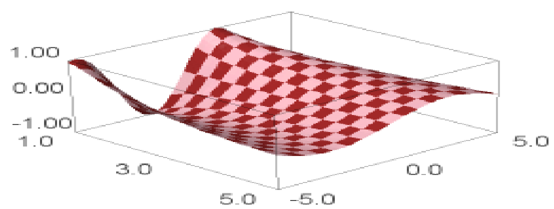

Figure 3. Similarity Solution $u(t, x) = t^{-2/3}\psi\left(\dfrac{x}{t^{1/3}}\right),\ t \in (1, 5), x \in (-5, 5),\ \psi(\cdot) = \sin(\cdot)$

6 Conclusion

We have established the Lie invariance condition for third-order nonlinear partial differential equations using local inverse theorem – a novel approach. Using this, we have computed the symmetry algebra of the KdV equation and found that it is 4-dimensional. We have seen that this algebra is solvable. We then obtained the invariant surface condition associated with this partial differential equation. Using this, we have computed the exact solutions of the KdV equation, commented on them and represented them graphically. We have also obtained a similarity solution and represented a particular case graphically.

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