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A NOTE ON LEFT DUO SEMINEARRINGS

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Abstract A left ideal of a seminearring $\mathcal{R}$ need not be a right ideal in general. We concentrate on those seminearrings which exhibit this property. A seminearring $\mathcal{R}$ is a left duo if every left ideal is two sided ideal. It is quite natural for us to explore the distinct properties of a left duo seminearring. In this article, we continue the study of left duo seminearrings. We prove some of the salient features of a left duo seminearrings when admitting mate functions. We also characterise such a seminearring.

1 Introduction

Following Van Hoorn and Rootselaar [1] a seminearring is an algebraic structure more general than a nearring or semiring. Seminearrings came from monoids. By a seminearring we say that an algebraic system $(\mathcal{R}, +, .)$, where $(\mathcal{R}, +)$ and $(\mathcal{R}, .)$ are semigroups and it is connected by a right distributive law. The role of seminearring structure applied in many places of theoretical computer science, viz. algebra communicating processes, theory of automata and also seen in semigroup mapping, reversible computation models and cryptography [2, 3]. Let S be an additive semigroup. Let $M(S) := \{\alpha : S \to S\}$ be the set of all mappings of S into itself, with the operations of pointwise addition: $s(\alpha + \beta) = s\alpha + s\beta \; \forall \; s \in S$, and multiplication given by composition of maps: $s(\alpha\beta) = (s\alpha)\beta \; \forall \; s \in S$. Then $(M(S), +, .)$ is a seminearring.

In an earlier article [4, 5], the author proposed the left duo property and explored the characteristics of a $\mathcal{R}$ that each ideal (left) is two sided. A left ideal of $\mathcal{R}$ does not always have to be a right ideal. For instance in the seminearring $(\mathcal{R}, +, .)$ constructed on the semigroups $(\mathcal{R}, +)$ and $(\mathcal{R}, .)$ with $\mathcal{R} = \{0, t, r, s\}$ is defined as follows.

+	0	t	r	s
0	0	t	r	s
t	t	0	s	r
r	r	s	0	t
s	s	r	t	0

.	0	t	r	s
0	0	0	0	0
t	0	t	r	t
r	0	0	0	0
s	0	t	r	t

From the above table, the ideal $\{0, t\}$ is a left ideal but not a right. The left duo structure is present in some specific seminearring, like direct product of seminearfields. It is only necessary for us to continue our research and explore the many features of a seminearring $\mathcal{R}$ that is endowed with the structure of left duo. Enabled by the above, we aim in this article to reconcile and integrate the significant properties of some special structures mostly with structure of left duo in order to achieve a complete description of a left duo seminearring. We also obtain a structure theorem for left duo seminearring admitting mate functions.

2 Preliminaries

We consider some basic definitions related to seminearrings used in subsequent sections. A ring $\mathcal{R}$ is called (Von Neumann) regular if for all $a \in \mathcal{R}$ there exists $r \in \mathcal{R}$ such that $a = ara$ [6]. Regular rings have been extensively studied in the literature. The concept has been naturally extended to nearrings and seminearrings. The $r \in \mathcal{R}$ such that $a = ara$ need not be unique. This leads to the concept of mate functions. A mate function is a self-map $\phi : \mathcal{R} \to \mathcal{R}$ such that $a = a\phi(a)a$ for all $a \in \mathcal{R}$ [7]. It is clear that $\mathcal{R}$ admits mate function if and only if it is regular.

A seminearring $\mathcal{R}$ is a left(right) normal if for every element a in $\mathcal{R}$, we have $a \in \mathcal{R}a(a \in a\mathcal{R})$ and normal means both normal (left and right) [8, 9]. Let A and B be two subsets of $\mathcal{R}$. Then $(A : B) = \{x \in \mathcal{R}/xB \subset A\}$. There exists $a \in \mathcal{R}$ then (i) idempotent if $a^2 = a$ (set of all idempotents of $\mathcal{R}$ is E) (ii) if $a^n = 0$ for some integer $n \geq 1$ is nilpotent [10]. Boolean seminearring defined as for all $x \in \mathcal{R}$ implies $x^2 = x$ [17]. If $\mathcal{R}$ is a left bipotent seminearring then $\mathcal{R}a = \mathcal{R}a^2$ for all $a \in \mathcal{R}$ [11]. If $A \subseteq \mathcal{R}$ then $\sqrt{A}$ is a radical of A is $\{t \in \mathcal{R} : t^k \in A \text{ for some positive integer } k\}$. It is obvious that $A \subseteq \sqrt{A}$. If left(right) ideal A is closed with respect to addition and $\mathcal{R}A \subseteq A(A\mathcal{R} \subseteq A)$, where A is non empty subset. If A is a left as well as a right ideal then it is two sided ideal(or an ideal) [12]. $(\mathcal{R}, +, .)$ is known as k-ideal if for $g, h \in \mathcal{R}$ there is a $t, r \in I$ so it is $g+h+t = r+h+g$[14]. An ideal I of $\mathcal{R}$ is a prime if P and Q of $\mathcal{R}$, $PQ \subset I$ implies either $P \subset I$ or $Q \subset I$. An ideal I of $\mathcal{R}$ is a completely prime if for $t, r \in \mathcal{R}$, $tr \in I$ implies $t \in I$ or $r \in I$. I is known as a semiprime ideal, if $\beta^2 \subset I \implies \beta \subset I$. A seminearring $\mathcal{R}$ is known as semiprime seminearring if $\{0\}$ is a semiprime ideal. $\mathcal{R}$ is called simple if it does not have any non trivial ideals [15]. Suppose $\mathcal{R}$ is a right seminearfield then we can say $(\mathcal{R}, +)$ is semigroup under addition, $(\mathcal{R}^*, .)$ is a group (where $\mathcal{R}^* = \mathcal{R} - \{0\}$) and $\forall\, b, c, d \in \mathcal{R}$, $(b + c)d = bd + cd$ [16]. Suppose $\mathcal{R}$ has an ideal I then $\dfrac{\mathcal{R}}{I}$ represents the all set congruence classes i.e., $\dfrac{\mathcal{R}}{I}$=$\{a + I; a \in \mathcal{R}\}$. Let I be a k-ideal of a seminearring $\mathcal{R}$ then $\dfrac{\mathcal{R}}{I}$ is also a seminearring with $(a + I) + (b + I) = a + b + I$ and $(a + I).(b + I) = ab + I$ [14]. Let index set I and $\{\mathcal{R}_i : i \in I\}$ is a seminearrings family, $\underset{i\in I}{X}\, \mathcal{R}_i$ product interms of component operations under '+' then under '.' is the $\prod_{i\in I} \mathcal{R}_i$ direct product of the seminearrings $\mathcal{R}_i(i \in I)$ [20]. A direct product $\prod_{k\in K} \mathcal{R}_k$ of a family $\{\mathcal{R}_k/k \in K\}$ (index set K) of sub seminearring of the seminearrings is known as subdirect product of the $\mathcal{R}_k$'s if all mapping of projection π_k (restricted to $\mathcal{R}$) is onto[17]. The concepts of seminearring homomorphisms, epimorphisms, isomorphisms etc., are defined by extending the definition of a ring. We have the following: If an ideal I is a k-ideal of seminearring $\mathcal{R}$ then $g : \mathcal{R} \to \dfrac{\mathcal{R}}{I}$ is a epimorphism of R. Therefore $\dfrac{R}{I}$ is a homomorphic image of $\mathcal{R}$. If $g : \mathcal{R} \to \dfrac{\mathcal{R}}{I}$ is the canonical epimorphism we get k-ideals J containing I, $\dfrac{\frac{\mathcal{R}}{I}}{\frac{J}{I}} = \dfrac{\mathcal{R}}{J}$ [13, 15]. A subdirect product of seminearring $\mathcal{R}$, $\{\mathcal{R}_k/k \in K\}$ is called a trivial whenever there exists one such $k \in K$ such that π_k, a projection is isomorphic. When $\mathcal{R}$ is not a isomorphic to a seminearring subdirect product (non-trivial) then $\mathcal{R}$ is considered a subdirectly irreducible [17]. If $\mathcal{R}$ is a subdirect product of seminearrings $\mathcal{R}_k$'s $(k \in K)$ then $\mathcal{R}_k$'s are homomorphic representations of $\mathcal{R}$ (projection π_k mapping) [18].

3 Main Results

We start this part with left duo structure possess the special structure of seminearring.

Proposition 3.1. Every left duo seminearring with mate functions is left bipotent.

Proof. Let seminearring $\mathcal{R}$ is a left duo and has a f mate function. Then $a = af(a)a$ for every $a \in \mathcal{R}$. This gives $a = af(a)af(a)a = (af(a))^2a$. Since left duo $\mathcal{R}$ and a left idea $\mathcal{R}a$ as such $\mathcal{R}a\mathcal{R} \subseteq \mathcal{R}a$. Then we can write $(af(a))af(a) = ba$ for some element b in $\mathcal{R}$, in which with $a = (af(a))^2a$ makes $a = ba^2$. This gives $\mathcal{R}a = \mathcal{R}a^2$ and therefore $\mathcal{R}$ is a left bipotent. □

Proposition 3.2. Any homomorphic image of a left duo seminearring $\mathcal{R}$ is a left duo seminearring.

Proof. Let $g : \mathcal{R} \to \mathcal{R}'$ is a seminearring epimorphism and assume that $a, r' \in \mathcal{R}'$. The fact that g is onto then there exist $x, r \in \mathcal{R}$ for which $a = g(x)$, $r' = g(r)$. Now

$r'a = g(r)g(x)$

= $g(rx)$ (since $rx \in \mathcal{R}x = x\mathcal{R} \Rightarrow rx = xr_1$ $for\ some$ $r_1 \in \mathcal{R}$)
= $g(xr_1)$
$= g(x)g(r_1)$
$= ar_1'$
So $\mathcal{R}'a \subset a\mathcal{R}'$.
Similarly $a\mathcal{R}' \subset \mathcal{R}'a$. Thus $\mathcal{R}'a = a\mathcal{R}'$. □

We obtained the following as a consequence of Proposition 3.2.

Theorem 3.3. [19][17] Each seminearring is isomorphic to a subdirect product of subdirectly irreducible seminearrings.

Theorem 3.4. Every left duo seminearring $\mathcal{R}$ with mate functions is isomorphic to a subdirect product of subdirectly irreducible left duo seminearrings.

Proof. By Theorem 3.3, $\mathcal{R}$ is mapping (isomorphic) to a subdirectly irreducible seminearrings subdirect product say $\mathcal{R}_i'$s then $\mathcal{R}$ has homomorphic image of every $\mathcal{R}_i$. The required outcomes are now obtained from Proposition 3.2. □

Theorem 3.5. Suppose any ideal I of a left duo, then $\dfrac{\mathcal{R}}{I}$ is also a left duo seminearring.

Proof. We observe that $\dfrac{\mathcal{R}}{I}$ is a homomorphic representation of $\mathcal{R}$ under the canonical homomorphism then the required result follows directly from Proposition 3.2. □

Theorem 3.6. Let Boolean seminearring $\mathcal{R}$ be a left duo. Then $\mathcal{R}$ has no non-zero divisors iff $\mathcal{R}$ is simple.

Proof. Obviously, as $\mathcal{R}$ is assume to be Boolean, then the identity map gives a mate function of $\mathcal{R}$. For every $x \in \mathcal{R}$, $\mathcal{R}x = \mathcal{R}x^2$ and if $\mathcal{R}$ has no non-zero divisors, it follows that $\mathcal{R} = \mathcal{R}x$ $\forall$ $x \in \mathcal{R} - \{0\}$. So $\mathcal{R}$ does not have any non-trivial ideals and therefore $\mathcal{R}$ is simple.

For the converse, let $b \in \mathcal{R}$ such that $ba = 0$ for some $a(\neq 0) \in \mathcal{R}$. Since $\mathcal{R}$ has left duo and simple $\mathcal{R}a = \mathcal{R}$. Therefore some y exists in $\mathcal{R}$ so it is $b = ya$. This yields $0 = ba = ya^2 = ya = b$ (as $\mathcal{R}$ is Boolean) and this implies $b = 0$. Therefore, the seminearring $\mathcal{R}$ contains no non-zero divisors. □

Proposition 3.7. Suppose $\mathcal{R}$ is a seminearring which has f then seminearring $\mathcal{R}$ is to be a left normal (right normal) seminearring.

Proof. From our hypothesis, f is a mate of a seminearring $\mathcal{R}$, $\forall\ t \in \mathcal{R}$, $t = tf(t)t \in \mathcal{R}t$. Then it is obvious to say $\mathcal{R}$ is a left normal seminearring. Similarly $\mathcal{R}$ is right normal. Thus the result follows. □

Proposition 3.8. In $\mathcal{R}$ is a left duo , we get $efe = ef$, $ef \in E$ for e and f are idempotents.

Proof. Suppose $x \in \mathcal{R}$, then the left ideal of $\mathcal{R}$ is written as $\mathcal{R}x$. The fact that seminearring $\mathcal{R}$ is a left duo, then $\mathcal{R}x$ is a right ideal of a seminearring $\mathcal{R}$. Therefore, we have $\mathcal{R}x\mathcal{R} \subseteq \mathcal{R}x$.
It follows $ef = eef \in \mathcal{R}e\mathcal{R} \subseteq \mathcal{R}e$ then $ef = ae$, $a \in \mathcal{R}$.
Therefore $efe = ae = ef$. Finally $ef \in E$. □

Lemma 3.9. If $Y \subset \mathcal{R}$ and an ideal X in $\mathcal{R}$, then left ideal is $(X : Y)$.

Proof. Let $I = (X : Y) = \{u \in \mathcal{R}/uY \subseteq X\}$. If for $b \in Y$ and $u, v \in I$, $(u+v)b = ub+vb \subset X$ then it follows that $u + v \in I$. If $u \in I$ and $v \in \mathcal{R}$ then we can have $vub \in vX \subseteq X$ (since X is an ideal) which shows that $vu \in I$. Hence, I is a left ideal. □

We are discussing some important results of a left duo seminearring in the theorem below.

Theorem 3.10. *Suppose that $\mathcal{R}$ is a left duo seminearring which admits f, a mate function. We have:*

(i) For every left ideal A of seminearring $\mathcal{R}$, $A = \sqrt{A}$.

(ii) Suppose that S is any subset of seminearring $\mathcal{R}$. Then an ideal $(I : S)$ of seminearring $\mathcal{R}$ for each left ideal I.

(iii) For all left ideal I, $v_1, v_2, \dots, v_n \in \mathcal{R}$, if $v_1 v_2 \dots v_n \in I$ then $\langle v_1\rangle\langle v_2\rangle \ \dots \ \langle v_n\rangle \subseteq I$.

(iv) If I is prime, then it is completely prime.

(v) $\mathcal{R}$ is also a semiprime seminearring.

Proof. The fact that $\mathcal{R}$ has a mate function, then Proposition 3.7 requires that $\mathcal{R}$ to be left normal seminearring.
(i) Suppose that $v \in \sqrt{A}$. Then we can find a positive integer say k for which $v^k \in A$. Then $\mathcal{R}$ is a left duo seminearring, which implies seminearring $\mathcal{R}$ is a left bipotent (Proposition 3.1). So $v \in \mathcal{R}v = \mathcal{R}v^2 \implies v = yv^2$ for some $y \in \mathcal{R}$.
Then $v = yvv = y(yv^2)v = y^2v^3 = \cdots = y^{k-1}v^k \in \mathcal{R}A \subset A$ i.e., $v \in A$. Therefore

$$\sqrt{A} \subset A. \tag{3.1}$$

But it is obvious that

$$A \subset \sqrt{A}. \tag{3.2}$$

Therefore $A = \sqrt{A}$.
(ii) Let us assume I is any left ideal of a seminearring $\mathcal{R}$. Hence $\mathcal{R}$ is a left duo then I is also a right ideal. By Lemma 3.9, $(I : S) = \{v \in \mathcal{R}/vS \subseteq I\}$ is a ideal(left) of a seminearring $\mathcal{R}$. Again, since left duo seminearring is $\mathcal{R}$, so we can have $(I : S)$ to be a right ideal as well. Consequently $(I : S)$ is an ideal.
(iii) Let $v_1v_2 \dots v_n \in I$.
$\Rightarrow v_1 \in (I : v_2 \dots v_n)$.
$\Rightarrow \langle v_1\rangle \subseteq (I : v_2 \dots v_n)$.
$\Rightarrow \langle\ v_1\ \rangle\ v_2\ \dots\ v_n \subseteq$ I.
$\Rightarrow v_2 \dots v_n\langle v_1\rangle \subseteq I$.
$\Rightarrow \langle v_2\rangle \subseteq (I : v_3 \dots v_n\langle v_1\rangle)$.
$\Rightarrow \langle v_2\rangle v_3 \dots v_n\langle v_1\rangle \subseteq I$.
$\Rightarrow v_3 \dots v_n\langle v_1\rangle\langle v_2\rangle \subseteq I$.
Continuing in the same vein, we get $\langle v_1\rangle\langle v_2\rangle \dots \langle v_n\rangle \subseteq I$.
(iv) Suppose $a, b \in \mathcal{R}$ and I is a prime ideal of $\mathcal{R}$. If $ab \in I$, then by **(3)**, $\langle a\rangle\langle b\rangle \subseteq I$. As I is prime, $\langle a\rangle \subseteq I$ or $\langle b\rangle \subseteq I$ i.e., $a \in I$ or $b \in I$. Hence I is completely prime.
(v) Assume P is a left ideal of $\mathcal{R}$. Then P is an ideal (since $\mathcal{R}$ is left duo). Suppose that I is any ideal of $\mathcal{R}$ then $I^2 \subseteq P$. If $a \in I$, then $a = af(a)a \in I(\mathcal{R}I) \subseteq I^2 \subseteq P$. Therefore any left ideal P of $\mathcal{R}$ is a semiprime ideal. Especially $\{0\}$ is a semiprime ideal and therefore $\mathcal{R}$ is a semiprime seminearring. □

Conclusion

Even in theory of nearring, a left ideal is not the same as a right. This adds to the appeal of studying such a seminearring. In this study, we discover several beneficial findings of left duo seminearrings that do not follow the aforementioned criterion.

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