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APPROXIMATION OF FUNCTIONS AND CONJUGATE OF FUNCTIONS USING PRODUCT MEAN $(E, q)(E, q)(E, q)$

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Abstract In this present work, we introduce the triple E^q summability method. We approximate functions and their conjugate with the help of triple E^q summability method. Further, few corollaries, resulting from theorems, have been presented.

1 Introduction

In recent times, approximation of functions by generalized Fourier series based on trigonometric polynomial plays a vital role in developing the various fields of mathematics and engineering. For example, using the properties of approximation of functions, a new L^2-based method was developed by Psarakis and Moustakiedes [8] for designing Finite Impulse Response digital filters to get an optimum approximation. Also, L^p-space, L^2-space, and L^∞-space are of much importance in developing digital filters. Since the last few years, error of approximation of periodic functions belonging to various classes through different summability methods has become an area of interest for many investigators. Many researchers like Değer [1], Gölbol and Değer [2], Jafarov [3], Krasniqi [4], Lal and Bhan [5], Nigam and Sharma [6, 7], Rathore and Singh [9], Sezgek and Dağadur [11], Singh [12], Sonkar [16], and Srivastava and Singh [18] have worked over the years in the direction of approximating functions by using different types of summability methods. Saxena and Prabhakar [10], Singh et al. [13], and Singh and Srivastava [14, 15] have worked on product summability of double Euler summability, $C^1.N_p$ and $C^1.T$, respectively. Recently, Sonkar and Sangwan [17] generalized the results of Saxena and Prabhakar [10] using triple E^1 Euler summability.

2 Definitions and Notations

The $L^p[0, 2\pi]$-space can be defined as

$$L^p[0, 2\pi] = \left\{ g : [0, 2\pi] \to \mathbb{R} : \int_0^{2\pi} |g(x)|^p dx < \infty \right\},\ p \geq 1.$$

For a periodic function g with period 2π and $g \in L^p[0, 2\pi]$ with $p \geq 1$, the trigonometric Fourier series and conjugate Fourier series of g can be written as

$$g(x) \sim \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx) = \sum_{m=0}^{\infty} A_m, \tag{2.1}$$

and

$$\widetilde{g}(x) \sim \sum_{m=1}^{\infty} (b_m \cos mx - a_m \sin mx) = \sum_{m=0}^{\infty} B_m, \tag{2.2}$$

respectively and the corresponding sequence of partial sums defined by

$$s_r(g; x) := \frac{a_0}{2} + \sum_{m=1}^{r} (a_m \cos mx + b_m \sin mx), r \in \mathbb{N} \text{ with } s_0(g; x) = \frac{a_0}{2},$$

and

$$\widetilde{s_r}(g;x) := \sum_{m=1}^{r}(b_m \cos mx - a_m \sin mx), r \in \mathbb{N} \text{ with } \widetilde{s_0}(g;x) = 0.$$

Here a_m and b_m are Fourier coefficients of g and $s_r(g;x)$ and $\widetilde{s_r}(g;x)$, represent the $(r+1)^{th}$ partial sums of Fourier series with respect to g and its conjugate series, respectively.

Let $\sum_{r=0}^{\infty} u_r$ be an infinite series and sequence $\{s_r\}$ is $(r+1)^{th}$ partial sum of given series, then

(i) The series $\sum_{r=0}^{\infty} u_r$ is said to be (E,q)-summable to s (throughout s denotes a definite number), if

$$E_r^q = t_r^{E^q} = \frac{1}{(1+q)^r}\sum_{j=0}^{r}\binom{r}{j}q^{r-j}s_j \to s, \text{ as } r \to \infty,$$

(ii) The series $\sum_{r=0}^{\infty} u_r$ is said to be $(E,q)(E,q)$-summable to s, if

$$E_r^q E_j^q = t_r^{E^q.E^q} = \frac{1}{(1+q)^r}\sum_{j=0}^{r}\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}q^{j-h}s_h \to s, \text{ as } r \to \infty,$$

(iii) The series $\sum_{r=0}^{\infty} u_r$ is said to be $(E,q)(E,q)(E,q)$-summable to s, if

$$E_r^q E_j^q E_h^q = t_r^{E^q.E^q.E^q} = \frac{1}{(1+q)^r}\sum_{j=0}^{r}\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\sum_{l=0}^{h}\binom{h}{l}q^{h-l}s_l \to s,$$

as $r \to \infty$.

The conjugate of function g is denoted by $\widetilde{g}$ and defined as below:

$$\widetilde{g}(x) = -\frac{1}{2\pi}\lim_{\epsilon \to 0}\int_{\epsilon}^{\pi}\psi(t)\cot(t/2)\,dt.$$

We also write

$$J_r(t) = \frac{1}{2\pi\,(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\left\{\sum_{l=0}^{h}\binom{h}{l}q^{h-l}\frac{\sin(l+1/2)t}{\sin(t/2)}\right\}\right],$$

$$\widetilde{J}_r(t) = \frac{1}{2\pi\,(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\left\{\sum_{l=0}^{h}\binom{h}{l}q^{h-l}\frac{\cos(l+1/2)t}{\sin(t/2)}\right\}\right],$$

$$\phi(t) = g(x+t) - 2g(t) + g(x-t) \text{ and } \psi(t) = \frac{g(x+t) - g(x-t)}{2}.$$

3 Main Results

Saxena and Prabhakar [10] derived new results in the field of approximation of functions. Recently, Sonkar and Sangwan [17] introduced the triple E^1 summability method, and with the help of this method, they generalized the results of Saxena and Prabhakar [10]. In this paper, we have generalized the results of Saxena and Prabhakar [10] and Sonkar and Sangwan [17] and prove the following theorems in a more general setting.

Theorem 3.1. *Let $\{p_r\}$ be a non-increasing positive sequence of real constant such that*

$$P_r = \sum_{z=0}^{r} p_z \to \infty, \text{ as } r \to \infty, \tag{3.1}$$

and given function $\phi(t)$ satisfies,

$$\Phi(t) = \int_0^t |\phi(u)|du = o\left[\frac{t}{\alpha(1/t).P_t}\right], \text{ as } t \to +0, \tag{3.2}$$

where $\alpha(t)$ is positive, monotonic and non-increasing function of t.

$$\log r = O[\{\alpha(r)\}.P_r], \text{ as } r \to \infty. \tag{3.3}$$

Then approximation of function g at $x = t$ by triple Euler product means of its Fourier series is given by

$$|t_r^{E^q.E^q.E^q} - g(x)| = O(1), \text{ as } r \to \infty.$$

Theorem 3.2. *Let $\{p_r\}$ be a non-increasing positive sequence of real constant and satisfies condition (3.1) and given function $\psi(t)$ satisfies,*

$$\Psi(t) = \int_0^t |\psi(u)|du = o\left[\frac{t}{\alpha(1/t).P_t}\right], \text{ as } t \to +0, \tag{3.4}$$

where $\alpha(t)$ is positive, monotonic and non-increasing function of t.

$$\log r = O[\{\alpha(r)\}.P_r], \text{ as } r \to \infty, \tag{3.5}$$

then approximation of function $\widetilde{g}$ at $x = t$ by triple Euler product means of its conjugate Fourier series is given by

$$|\widetilde{t_r}^{E^q.E^q.E^q} - \widetilde{g}(x)| = O(1), \text{ as } r \to \infty.$$

4 Lemmas

Here few lemmas are given, which are useful to prove our theorems:

Lemma 4.1. $|J_r(t)| = O(r), \text{ for } 0 \le t \le 1/r.$

Proof. Using $\sin(rt) \le rt$ and $\sin(t/2) \ge t/\pi$, we have

$$\begin{aligned}
|J_r(t)| &= \frac{1}{2\pi(1+q)^r}\left|\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\times\right.\right. \\
&\qquad\left.\left.\left\{\sum_{l=0}^{h}\binom{h}{l}q^{h-l}\frac{\sin(l+1/2)t}{\sin(t/2)}\right\}\right]\right| \\
&\le \frac{1}{2\pi(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\times\right. \\
&\qquad\left.\left\{\sum_{l=0}^{h}\binom{h}{l}q^{h-l}\frac{(l+1/2)t}{t/\pi}\right\}\right] \\
&\le \frac{1}{2(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\left\{\sum_{l=0}^{h}\binom{h}{l}q^{h-l}\frac{(2l+1)}{2}\right\}\right] \\
&\le \frac{1}{4(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}(2h+1)\left\{\sum_{l=0}^{h}\binom{h}{l}q^{h-l}\right\}\right] \\
&\le \frac{1}{4(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}q^{j-h}(2h+1)\right] \\
&\le \frac{1}{4(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}(2j+1)\sum_{h=0}^{j}\binom{j}{h}q^{j-h}\right] \\
&\le \frac{1}{4(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}q^{r-j}(2j+1)\right] \\
&\le \frac{1}{4}(2r+1) \\
&= O(r).
\end{aligned}$$

Lemma 4.2. $|J_r(t)| = O(1/t),\ \textit{for}\ \ 1/r \le t \le \pi.$

Proof. Applying $\sin(rt) \le 1$ and $\sin(t/2) \ge t/\pi$, we have

$$\begin{aligned}
|J_r(t)| &= \frac{1}{2\pi\,(1+q)^r}\left|\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\ \times\right.\right. \\
&\qquad\left.\left.\left\{\sum_{l=0}^{h}\binom{h}{l}q^{h-l}\frac{\sin(l+1/2)t}{\sin(t/2)}\right\}\right]\right| \\
&\le \frac{1}{2\pi\,(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\left\{\sum_{l=0}^{h}\binom{h}{l}q^{h-l}\frac{\pi}{t}\right\}\right] \\
&\le \frac{1}{2\,t\,(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\left\{\sum_{l=0}^{h}\binom{h}{l}q^{h-l}\right\}\right] \\
&\le \frac{1}{2\,t\,(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}q^{j-h}\right] \\
&\le \frac{1}{2\,t\,(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}q^{r-j}\right] \\
&\le \frac{1}{2\,t} \\
&= O(1/t).
\end{aligned}$$

Lemma 4.3. $|\widetilde{J}_r(t)| = O(1/t),\ \textit{for}\ \ 0 \le t \le 1/r.$

Proof. Applying $|\cos(rt) \le 1|$, we have

$$\begin{aligned}
|\widetilde{J}_r(t)| &= \frac{1}{2\pi\,(1+q)^r}\left|\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\ \times\right.\right. \\
&\qquad\left.\left.\left\{\sum_{l=0}^{h}\binom{h}{l}q^{h-l}\frac{\cos(l+1/2)t}{\sin(t/2)}\right\}\right]\right| \\
&\le \frac{1}{2\pi\,(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\left\{\sum_{l=0}^{h}\binom{h}{l}q^{h-l}\frac{\pi}{t}\right\}\right] \\
&\le \frac{1}{2\,t\,(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\left\{\sum_{l=0}^{h}\binom{h}{l}q^{h-l}\right\}\right] \\
&\le \frac{1}{2\,t\,(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}q^{j-h}\right] \\
&\le \frac{1}{2\,t\,(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}q^{r-j}\right] \\
&\le \frac{1}{2\,t} \\
&= O(1/t).
\end{aligned}$$

Lemma 4.4. $|\widetilde{J}_r(t)| = O(1/t),\ \textit{for}\ \ 1/r \le t \le \pi.$

Proof. Applying $\sin(t/2) \geq t/\pi$, we have

$$
\begin{aligned}
|\widetilde{J}_r(t)| &= \frac{1}{2\pi\,(1+q)^r}\left|\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\times\right.\right.\\
&\qquad\left.\left.\left\{\sum_{l=0}^{h}\binom{h}{l}q^{h-l}\frac{\cos(l+1/2)t}{\sin(t/2)}\right\}\right]\right|\\
&\leq \frac{1}{2\pi\,(1+q)^r}\left|\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\times\right.\right.\\
&\qquad\left.\left.\left\{\sum_{l=0}^{h}\binom{h}{l}q^{h-l}\frac{\cos(l+1/2)t}{t/\pi}\right\}\right]\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{2\,t\,(1+q)^r}\left|\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}Re\left\{\sum_{l=0}^{h}\binom{h}{l}q^{h-l}e^{hlt}\right\}\right]\right|\\
&\leq \frac{1}{2\,t\,(1+q)^r}\left|\sum_{j=0}^{\tau-1}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}Re\left\{\sum_{l=0}^{h}\binom{h}{l}q^{h-l}e^{hlt}\right\}\right]\right|\\
&\quad+\frac{1}{2\,t\,(1+q)^r}\left|\sum_{j=\tau}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}Re\left\{\sum_{l=0}^{h}\binom{h}{l}q^{h-l}e^{hlt}\right\}\right]\right|\\
&\leq \frac{1}{2\,t\,(1+q)^r}\left|\sum_{j=0}^{\tau-1}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\left\{\sum_{l=0}^{h}\binom{h}{l}q^{h-l}\right\}\right]\right||e^{hlt}|\\
&\quad+\frac{1}{2\,t\,(1+q)^r}\sum_{j=\tau}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\max_{0\leq l\leq h}\left|\sum_{l=0}^{h}\binom{h}{l}q^{h-l}e^{hlt}\right|\right]\\
&\leq \frac{1}{2\,t\,(1+q)^r}\left|\sum_{j=0}^{\tau-1}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}q^{j-h}\right]\right|\\
&\quad+\frac{1}{2\,t\,(1+q)^r}\sum_{j=\tau}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}q^{j-h}\right]\\
&= \frac{1}{2\,t\,(1+q)^r}\sum_{j=0}^{\tau-1}\binom{r}{j}q^{r-j}+\frac{1}{2\,t\,(1+q)^r}\sum_{j=\tau}^{r}\binom{r}{j}q^{r-j}\\
&= \frac{1}{2\,t\,(1+q)^r}\sum_{j=0}^{r}\binom{r}{j}q^{r-j}\\
&= \frac{1}{2\,t}\\
&= O(1/t).
\end{aligned}
$$

5 Proof of Theorem 3.1

We have

$$
s_r(g;x)-g(x)=\frac{1}{2\pi}\int_0^{\pi}\frac{\phi(t)\,\sin\left(r+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\,dt,
$$

and

$$\begin{aligned} t_r^{E^q.E^q.E^q}(g;x) - g(x) &= \frac{1}{2\pi\,(1+q)^r} \sum_{j=0}^{r} \left[\binom{r}{j} \frac{q^{r-j}}{(1+q)^j} \sum_{h=0}^{j} \binom{j}{h} \frac{q^{j-h}}{(1+q)^h} \times \right. \\ &\qquad \left. \int_0^{\pi} \phi(t) \left\{ \sum_{l=0}^{h} \binom{h}{l} q^{h-l} \frac{\sin(l+1/2)t}{\sin(t/2)} \right\} dt \right] \\ &= \int_0^{\pi} \phi(t)\, J_r(t)\, dt \\ &= \left[\int_0^{1/r} \phi(t) + \int_{1/r}^{\gamma} \phi(t) + \int_{\gamma}^{\pi} \phi(t) \right] J_r(t)\, dt \ (\text{for } 0 < \gamma < \pi) \\ &= K_1 + K_2 + K_3, \text{ say.} \end{aligned} \tag{5.1}$$

Applying Lemma 4.1, condition 3.2 and 3.3 and second mean value theorem is applying for second term integral, we have

$$\begin{aligned} |K_1| &\leq \int_0^{1/r} |\phi(t)\, J_r(t)|\, dt \\ &= O(r) \left[\int_0^{1/r} |\phi(t)|\, dt \right] \\ &= O(r) \left[o\left(\frac{1/r}{\alpha(r).P_r} \right) \right] \\ &= O\left(\frac{1}{\log r} \right) = O(1), \text{ as } r \to \infty. \end{aligned} \tag{5.2}$$

Applying Lemma 4.2, condition 3.2 and 3.3 and second mean value theorem is applying for second term integral, we have

$$\begin{aligned} |K_2| &\leq \int_{1/r}^{\gamma} |\phi(t)\, J_r(t)|\, dt \\ &= O\left[\int_{1/r}^{\gamma} |\phi(t)| \left(\frac{1}{t} \right) dt \right] \\ &= O\left[\left\{ \frac{1}{t} \Phi(t) \right\}_{1/r}^{\gamma} + \int_{1/r}^{\gamma} o\left\{ \frac{\phi(t)}{t^2} \right\} dt \right] \\ &= O\left[o\left\{ \frac{1}{\alpha(1/t).P_t} \right\}_{1/r}^{\gamma} + \int_{1/r}^{\gamma} o\left(\frac{1/t}{\alpha(1/t).P_t} \right) dt \right] \\ &= O\left[o\left\{ \frac{1}{\alpha(r).P_r} \right\} + \int_{1/\gamma}^{r} o\left(\frac{1/u}{\alpha(u).P_u} \right) du \right] \\ &= O\left(\frac{1}{\alpha(r)\, P_r} \right) + O\left(\frac{1/r}{\alpha(r)\, P_r} \right) \int_{1/\gamma}^{r} 1 du \\ &= O\left(\frac{1}{\log r} \right) + O\left(\frac{1}{\log r} \right) = O(1), \text{ as } r \to \infty. \end{aligned} \tag{5.3}$$

Applying Riemann-Lebesgue theorem and regularity condition of summability, we have

$$|K_3| \leq \int_{\gamma}^{\pi} |\phi(t)\, J_r(t)|\, dt = O(1), \text{ as } r \to \infty. \tag{5.4}$$

Collecting (5.1) – (5.4), we get

$$|t_r^{E^q.E^q.E^q}(g;x) - g(x)| = O(1), \text{ as } r \to \infty.$$

Hence, complete the proof. □

6 Proof of Theorem 3.2

We have

$$\widetilde{s_r}(g;x) - \widetilde{g}(x) = \frac{1}{2\pi}\int_0^\pi \frac{\psi(t)\ \cos\left(r+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\,dt$$

and

$$\begin{aligned}
\widetilde{t_r}^{E^q.E^q.E^q}(g;x) - \widetilde{g}(x) &= \frac{1}{2\pi\,(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\times\right.\\
&\qquad\left.\int_0^\pi \psi(t)\left\{\sum_{l=0}^{h}\binom{h}{l}q^{h-l}\frac{\cos(l+1/2)t}{\sin(t/2)}\right\}dt\right]\\
&= \int_0^\pi \psi(t)\,\widetilde{J}_r(t)\,dt\\
&= \left[\int_0^{1/r}\psi(t) + \int_{1/r}^{\gamma}\psi(t) + \int_\gamma^\pi \psi(t)\right]\widetilde{J}_r(t)dt \ (\text{for } 0<\gamma<\pi)\\
&= \widetilde{K_1} + \widetilde{K_2} + \widetilde{K_3}, \text{ say.} \qquad (6.1)
\end{aligned}$$

Applying Lemma 4.3, condition 3.4 and 3.5 and second mean value theorem is applying for second term integral, we have

$$\begin{aligned}
|\widetilde{K_1}| &\leq \int_0^{1/r}|\psi(t)\,\widetilde{J}_r(t)|\,dt\\
&= \left[\int_0^{1/r}|\psi(t)|\,\frac{1}{t}dt\right]\\
&= O(r)\left[\int_0^{1/r}|\psi(t)|dt\right]\\
&= O(r)\left[o\left(\frac{1/r}{\alpha(r).P_r}\right)\right]\\
&= O\left(\frac{1}{\log r}\right) = O(1), \text{ as } r\to\infty. \qquad (6.2)
\end{aligned}$$

Applying Lemma 4.4, condition 3.4 and 3.5 and second mean value theorem is applying for second term integral, we have

$$\begin{aligned}
|\widetilde{K_2}| &\leq \int_{1/r}^{\gamma} |\psi(t)\,\widetilde{J}_r(t)|\,dt \\
&= O\left[\int_{1/r}^{\gamma} |\psi(t)| \left(\frac{1}{t}\right) dt\right] \\
&= O\left[\left\{\frac{1}{t}\Psi(t)\right\}_{1/r}^{\gamma} + \int_{1/r}^{\gamma} o\left\{\frac{\psi(t)}{t^2}\right\} dt\right] \\
&= O\left[o\left\{\frac{1}{\alpha(1/t).P_t}\right\}_{1/r}^{\gamma} + \int_{1/r}^{\gamma} o\left(\frac{1/t}{\alpha(1/t).P_t}\right) dt\right] \\
&= O\left[o\left\{\frac{1}{\alpha(r).P_r}\right\} + \int_{1/\gamma}^{r} o\left(\frac{1/u}{\alpha(u).P_u}\right) du\right] \\
&= O\left(\frac{1}{\alpha(r)\,P_r}\right) + O\left(\frac{1/r}{\alpha(r)\,P_r}\right) \int_{1/\gamma}^{r} 1du \\
&= O\left(\frac{1}{\log r}\right) + O\left(\frac{1}{\log r}\right) = O(1), \text{ as } r \to \infty. \qquad (6.3)
\end{aligned}$$

Applying Riemann-Lebesgue theorem and regularity condition of summability, we have

$$|\widetilde{K_3}| \leq \int_{\gamma}^{\pi} |\psi(t)\,\widetilde{J}_r(t)|\,dt = O(1), \text{ as } r \to \infty. \qquad (6.4)$$

Collecting (6.1) – (6.4), we get

$$|\widetilde{t_r}^{E^q.E^q.E^q}(g;x) - \widetilde{g}(x)| = O(1), \text{ as } r \to \infty.$$

Hence, complete the proof. □

7 Corollaries

Here few corollaries are given, which are derived from our Theorems 3.1 and 3.2.

Corollary 7.1. *If we take $q = 1$ in Theorem 3.1, then triple summability $(E, q)(E, q)(E, q)$ reduce to $(E, 1)(E, 1)(E, 1)$, then*

$$|t_r^{E^1.E^1.E^1}(g;x) - g(x)| = O(1), \text{ as } r \to \infty.$$

Corollary 7.2. *If we take $q = 1$ in Theorem 3.2, then triple summability $(E, q)(E, q)(E, q)$ reduce to $(E, 1)(E, 1)(E, 1)$, then*

$$|\widetilde{t_r}^{E^1.E^1.E^1}(g;x) - \widetilde{g}(x)| = O(1), \text{ as } r \to \infty.$$

Remark 7.3. In view of Corollary 7.1 and 7.2, Theorem 3.1 and 3.2 [17] are particular cases of our Theorems 3.1 and 3.2.

Corollary 7.4. *If we take $q = 1$ and one $(E, q) = 1$ in Theorem 3.1, then triple summability $(E, q)(E, q)(E, q)$ reduce to $(E, 1)(E, 1)$, then*

$$|t_r^{E^1.E^1}(g;x) - g(x)| = O(1), \text{ as } r \to \infty.$$

Corollary 7.5. *If we take $q = 1$ and one $(E, q) = 1$ in Theorem 3.2, then triple summability $(E, q)(E, q)(E, q)$ reduce to $(E, 1)(E, 1)$, then*

$$|\widetilde{t_r}^{E^1.E^1}(g;x) - \widetilde{g}(x)| = O(1), \text{ as } r \to \infty.$$

Remark 7.6. In view of Corollary 7.4 and 7.5, Theorem 1 and 2 [10] are particular cases of our Theorems 3.1 and 3.2.

8 Conclusion

The results of the paper are aimed to formulate the problem of approximation of functions g and their conjugates $\widetilde{g}$ by triple E^q summability means of their Fourier series and conjugate Fourier series, respectively in a simpler manner.

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References

[1] U. Değer, On approximation to functions in the $W(L_p, \xi(t))$ class by a new matrix mean, *Novi Sad J. Math.* **46 (1)**, 1-14 (2016).

[2] S. Y. Gölbol and U. Değer, Approximation in the weighted generalized Lipschitz class by deferred Cesàro-Matrix product submethods, *Palestine J. Math.* **10 (2)**, 740–750 (2021).

[3] S. Z. Jafarov, On approximation of a weighted Lipschitz class functions by means $t_n(f;x)$, $N_n^\beta(f;x)$ and $R_n^\beta(f;x)$ of Fourier series, *Trans. Natl. Acad. Sci. Azerb. Ser. Phys. -Tech. Math. Sci. Mathematics* **40 (4)**, 118-129 (2020).

[4] X. Z. Krasniqi, Approximation by sub-matrix means of multiple Fourier series in the Hölder metric, *Palestine J. Math.* **9 (2)**, 761-770 (2020).

[5] S. Lal and I. Bhan, Approximation of functions with first and second derivatives f', f'' belonging to Lipschitz class of order $0 < \alpha \leq 1$ by generalized Legendre wavelet expansion, *Palestine J. Math.* **9 (2)**, 711–729 (2020).

[6] H. K. Nigam and K. Sharma, Degree of approximation of a class of function by $(C,1)(E,q)$ means of Fourier series, *Int. J. Appl. Math.* **41 (2)**, 1-5 (2007).

[7] H. K. Nigam and K. Sharma, A study on degree of approximation by Karamata summability method, *J. Inequal. Appl.* **85 (1)**, 1-28 (2011).

[8] E. Z. Psarakis and G. V. Moustakides, An L_2-based method for the design of $1-D$ zero phase FIR digital filters, *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.* **44 (7)**, 551-601 (1997).

[9] A. Rathore and U. Singh, On the degree of approximation of functions in a weighted Lipschitz class by almost matrix summability method, *J. Anal.* **28 (1)**, 21-33 (2020).

[10] K. Saxena and M. Prabhakar, A study of double Euler Summability method of Fourier series and its conjugate series, *Int. J. Sci. Innov. Math. Res.* **4 (1)**, 46-52 (2016).

[11] Ş. Sezgek and İ. Dağadur, Approximation by double Cesàro submethods of double Fourier series for Lipschitz fuctions, *Palestine J. Math.* **8 (1)**, 71–85 (2019).

[12] U. Singh, On the trigonometric approximation of functions in a weighted Lipschitz class, *J. Anal.* 1-11 (2020).

[13] U. Singh, M. L. Mittal and S. Sonkar, Trigonometric approximation of signals (functions) belonging to $W(L_r, \xi(t))$-class by matrix $(C^1.N_p)$ operator, *Int. J. Math. Math. Sci.* **2012**, 1–11 (2012).

[14] U. Singh and S. K. Srivastava, Fourier approximation of functions conjugate to the functions belonging to weighted Lipschitz class, *In: Proc. WCE* **1**, 236-240 (2013).

[15] U. Singh and S. K. Srivastava, Trigonometric approximation of functions belonging to certain Lipschitz classes by $C^1.T$ operator, *Asian-European J. Math.* **7 (4)**, 1-13 (2014).

[16] S. Sonkar, Approximation of periodic functions belonging to $W(L^r, \xi(t), \beta \geq 0)$- class by $(C^1.T)$ means of Fourier series, *Math. Anal. Appl.* **143**, 73-83 (2015).

[17] S. Sonker and P. Sangwan, Approximation of Fourier and its conjugate series by triple Euler product summability, *J. Phys. Conf. Ser.* **1770 (1)**, 012003 (2021).

[18] S. K. Srivastava and U. Singh, Trigonometric approximation of periodic functions belonging to weighted Lipschitz class $W(L^p, \psi(t), \beta)$, *Contemp. Math.* **645**, 283-291 (2015).

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