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ON THE CONVERGENCE OF WAVELET FRAME SERIES

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Abstract The wavelet frames have been defined with illustrative examples. Walter [14] has discussed the pointwise convergence of wavelet expansions. Till now, wavelet frame series' pointwise and uniform convergence have not been studied in detail. Working in the same direction, the wavelet frame series and its pointwise as well as uniform convergence have been investigated in this study. The importance of the proof of the main results was increased by the use of Parseval's equality for the Fourier transform as well as the Fourier series.

1 Introduction

Wavelet frames and their constructions have been studied by several researchers like Ron and Shen [12], Benedetto [1], Benedetto and Treiber [2], Chui and Shi [3] and others. Recently Lal and Kumar [10] have studied the approximation of functions belonging to $L^2(\mathbb{R})$ space by their wavelet expansions. Now orthonormal wavelet series has become a very useful tool in several branches of Engineering and Technology like data analysis, signal and image processions. As a result, the study of convergence behaviour of this series becomes essential. In this direction, an attempt has been made by Walter [14] and Hernandez and Weiss [8]. Working in the same direction, Wavelet frame series have been studied pointwise and uniformly by Zhang [15]. Similarly a function $f \in L^2(\mathbb{R})$ is also expressible as a wavelet frame series. Wavelet frame series' pointwise and uniform convergence have not been studied in depth until now. The nature of pointwise and uniform convergence of certain wavelet frame series has been investigated in detail in this paper to make an advance study.

2 Definitions and Preliminaries

2.1 Wavelet Frames

Let's say $F \in L^2(\mathbb{R})$. F's Fourier transform is defined as follows:

$$\hat{F}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi\upsilon} F(\upsilon) d\upsilon,$$

and $\hat{F}$'s compact support is defined as

$$\text{supp}\hat{F} = \text{clos}\{\xi, \hat{F}(\xi) \neq 0\},$$

where $closS$ is the closure of set S, i.e. $closS$ is the set containing all the elements of S and its limit points. When supp$\hat{F}$ is bounded, F is referred to as a band-limited function.

A Hilbert space is denoted by the symbol $\mathbb{H}$. A sequence in $\mathbb{H}$ is denoted by $\{h_n\}_{n=-\infty}^{\infty}$. If there are two positive real constants $0 < \Gamma_1 \leq \Gamma_2 < \infty$ such that

$$\Gamma_1 \|F\|^2 \leq \sum_{n=-\infty}^{\infty} |\langle F, h_n \rangle|^2 \leq \Gamma_2 \|F\|^2 \qquad \text{for any} F \in \mathbb{H},$$

the frame for $\mathbb{H}$ is designated $\{h_n\}_{n=-\infty}^{\infty}$, and the bounds of the frame are Γ_1, Γ_2.

Let $\varphi \in L^2(\mathbb{R})$. Define

$$\varphi_{m,n}(\upsilon) = 2^{\frac{m}{2}}\varphi(2^m\upsilon - n),\ m,n \in \mathbb{Z}.$$

Throughout the work, we use the abbreviation ONB to refer to orthonormal basis. A wavelet frame (or a wavelet ONB) is defined as a system $\{\varphi_{m,n}, m,n \in \mathbb{Z}\}$ that is a frame (or an ONB) for $L^2(\mathbb{R})$.

Example 1. Define the Haar function ψ_H as follows:

$$\psi_H(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2}; \\ -1, & \frac{1}{2} \le t < 1; \\ 0, & \text{otherwise.} \end{cases}$$

and consider the sampling rate $1/3$. Then the linearly dependent family $S = \{\psi_{H,1/3;j,k} : j,k \in \mathbb{Z}\}$ is a wavelet frame of $L^2(\mathbb{R})$ (see Chui [4], p. 70).

Example 2. Legendre wavelet (see Lal and Bhan [11]) is defined over the interval $[0,1)$ as follows:

$$\psi_{\frac{1}{3};n,m}(t) = \begin{cases} \sqrt{(m+\frac{1}{2})}2^{\frac{k}{2}}L_m(2^kt - \frac{\hat{n}}{3}), & \frac{\frac{\hat{n}}{3}-1}{2^k} \le t < \frac{\frac{\hat{n}}{3}+1}{2^k}; \\ 0, & \text{otherwise,} \end{cases}$$

where $\hat{n} = 2n-1,\ n = 1,2,\cdots,2^{k-1}$, k is the natural number and m is the degree of Legendre polynomial. Then $\{\psi_{\frac{1}{3};j,k}\}$ is a wavelet frame. It can be shown following the proof of Example 1.

2.2 Wavelet Frame Series

Let $\varphi \in L^2(\mathbb{R})$ and supp$\hat{\varphi} \subset [-\pi,\pi]$. Then

$$\Gamma_1 \le \sum_{m=-\infty}^{\infty} |\hat{\varphi}(2^{-m}\xi)|^2 \le \Gamma_2,\ \text{a.e. } \xi \in \mathbb{R}, \quad \text{(Heil and Walnat [7])}$$

if and only if $\{\varphi_{m,n}, m,n \in \mathbb{Z}\}$ is a wavelet frame with bounds Γ_1, Γ_2. Let $\varphi,\ \tilde{\varphi} \in L^2(\mathbb{R})$. The wavelet frames $\{\varphi_{m,n}, m,n \in \mathbb{Z}\}$ and $\{\tilde{\varphi}_{m,n}, m,n \in \mathbb{Z}\}$ which satisfy the following expression

$$F = \sum_{m=-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty}\langle F,\varphi_{m,n}\rangle\tilde{\varphi}_{m,n}\right) = \sum_{m=-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty}\langle F,\tilde{\varphi}_{m,n}\rangle\varphi_{m,n}\right), \forall F \in L^2(\mathbb{R}) \tag{2.1}$$

in the sense of L^2-norm, are referred to as dual wavelet frames, and both the series in (2.1) are referred to as wavelet frame series.

Chui & Shi [3] and Frazier etal. [6] obtained a necessary and sufficient conditions for a pair of wavelet frames $\{\varphi_{m,n}, m,n \in \mathbb{Z}\}$ & $\{\tilde{\varphi}_{m,n}, m,n \in \mathbb{Z}\}$ to be dual frames in the following form:

The wavelet frames $\{\varphi_{m,n}, m,n \in \mathbb{Z}\}$ and $\{\tilde{\varphi}_{m,n}, m,n \in \mathbb{Z}\}$ become dual pairwise if and only if the following conditions

$$\sum_{m=-\infty}^{\infty}\hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)\overline{\hat{\varphi}}\left(\frac{\xi}{2^m}\right) = 1,\ \text{a.e. } \xi$$

and

$$\sum_{m=0}^{\infty}\hat{\tilde{\varphi}}\left(2^m(\xi + (2k+1)2\pi)\right)\overline{\hat{\varphi}}\left(2^m\xi\right) = 0,\ \text{a.e. } \xi,\ k \in \mathbb{Z},$$

hold.

Researchers like Walter [14], Kelly, Kon and Raphaet [9] and Hernandez and Weiss [8] have studied the converegence of orthonormal wavelet series as following:

Let a multi resolution analysis (MRA) originate a wavelet ONB $\{\varphi_{m,n}, m,n \in \mathbb{Z}\}$ with the scaling function ϕ which satisfies the following inequality

$$|\phi(\upsilon)| \le \frac{C}{(1+|\upsilon|)^p},\ \upsilon \in \mathbb{R},\ p = 1,2,3,\cdots.$$

(i) If

$$F_M(\upsilon) = \sum_{m=-\infty}^{M} \left(\sum_{n=-\infty}^{\infty} \langle F, \varphi_{m,n} \rangle \varphi_{m,n}(\upsilon) \right), F \in L^2(\mathbb{R})$$

then $F_M(\upsilon) \to F(\upsilon)$ as $M \to \infty$ for a.e. $\upsilon \in \mathbb{R}$.

(ii) If $F \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ is continuous in an open interval (a, b), then $F_M(\upsilon)$ converges to $F(\upsilon)$ uniformly as $M \to \infty$ on every compact subset of (a, b).

Let $\varphi \in L^2(\mathbb{R})$ and $\{\varphi_{m,n}, m, n \in \mathbb{Z}\}$ be a wavelet frame with supp$\hat{\varphi} \subset [-\pi, \pi]$ and $\tilde{\varphi}(\upsilon)$ satisfies

$$\hat{\tilde{\varphi}}(\xi) \quad = \quad \frac{\hat{\varphi}(\xi)}{\sum_{m=-\infty}^{\infty} |\hat{\varphi}\left(\frac{\xi}{2^m}\right)|^2}, \text{ a.e. } \xi \in \mathbb{R}. \tag{2.2}$$

Then

$$\Gamma_1 \leq \sum_{m=-\infty}^{\infty} |\hat{\varphi}(2^{-m}\xi)|^2 \leq \Gamma_2, \text{ a.e. } \xi \in \mathbb{R}, \tag{2.3}$$

being Γ_1, Γ_2 are bounds of the frame $\{\varphi_{m,n}, m, n \in \mathbb{Z}\}$.

Consequently, we have, if

$$\tilde{\varphi} \in L^2(\mathbb{R}), \text{ supp}\hat{\tilde{\varphi}} = \text{supp}\hat{\varphi} \subset [-\pi, \pi], \tag{2.4}$$

then

$$\frac{1}{\Gamma_1} \leq \sum_{m=-\infty}^{\infty} |\hat{\tilde{\varphi}}(2^{-m}\xi)|^2 \leq \frac{1}{\Gamma_2}, \text{ a.e. } \xi \in \mathbb{R} \quad (\text{see}[15]). \tag{2.5}$$

Thus $\{\tilde{\varphi}_{m,n}, m, n \in \mathbb{Z}\}$ is a wavelet frame. Under these circumstances

$$\sum_{m=-\infty}^{\infty} \hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right) \overline{\hat{\varphi}}\left(\frac{\xi}{2^m}\right) = 1, \text{ a.e. } \xi$$

and

$$\hat{\tilde{\varphi}}\left(2^m(\xi + (2k+1)2\pi)\right) \overline{\hat{\varphi}}\left(2^m\xi\right) = 0, \text{ a.e. } \xi \ (m \geq 0;\ k \in \mathbb{Z}).$$

Thus, the formula for reconstruction

$$F(\upsilon) = \sum_{m=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} \langle F, \varphi_{m,n} \rangle \tilde{\varphi}_{m,n}(\upsilon) \right) = \sum_{m=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} \langle F, \tilde{\varphi}_{m,n} \rangle \varphi_{m,n}(\upsilon) \right) \tag{2.6}$$

hold in the sense of L^2-norm.

When we say $|F(\upsilon)| = O\{\phi(\upsilon)\}$, we imply that $|F(\upsilon)| < A\phi(\upsilon)$, as υ approaches either $a \in \mathbb{R}$, or infinity, and $A > 0$, i.e.,

$$\lim_{\upsilon \to a} \frac{|F(\upsilon)|}{\phi(\upsilon)} = A.$$

$O(1)$ denotes a bounded function in particular . Thus

$$\sin(\upsilon) = O(1),\ (\upsilon + 1)^2 = O(1)$$

as $\upsilon \to 0$; and

$$\sin(\upsilon) = O(1),\ (\upsilon + 1)^2 = O(\upsilon^2)$$

as $\upsilon \to \infty$.

Sometimes, however, $F(\upsilon) = O\{\phi(\upsilon)\}$ is used to mean

$$|F(\upsilon)| < K\phi(\upsilon),$$

but it's self-evident enough what parameters are involved.

When we say $F(\upsilon) = o\{\phi(\upsilon)\}$, we indicate that $\frac{F(\upsilon)}{\phi(\upsilon)} \to 0$ as υ approaches either $a \in \mathbb{R}$, or infinity, i.e.,

$$\lim_{\upsilon \to a} \frac{|F(\upsilon)|}{\phi(\upsilon)} = 0.$$

. Thus

$$\sin(\upsilon) = o(\upsilon^2), \ (\upsilon + 1)^2 = o(\upsilon^3)$$

as $\upsilon \to \infty$. Specifically, $o(1)$ denotes a function that approaches to zero, (Titchmarsh [13]).

3 Main theorems

Three new theorems on the convergence of the wavelet frame series are presented in this study, and they are as follows:

Theorem 3.1. *If a function $F \in L^2(\mathbb{R})$ such that its Fourier transform $\hat{F} \in L^1(\mathbb{R})$ and*

$$F(\upsilon) = \sum_{m=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} \langle F, \varphi_{m,n} \rangle \tilde{\varphi}_{m,n}(\upsilon) \right)$$

is a wavelet frame series having M^{th} partial sums

$$(S_M F)(\upsilon) = \sum_{m=-M}^{M} \left(\sum_{n=-\infty}^{\infty} \langle F, \varphi_{m,n} \rangle \tilde{\varphi}_{m,n}(\upsilon) \right), M = 0, 1, 2, \cdots,$$

then $((S_M F)(\upsilon))_{M=0}^{\infty}$ converges pointwise to $F(\upsilon)$ $\forall \upsilon \in \mathbb{R}$.

Theorem 3.2. *Let $F \in L^2(\mathbb{R})$ and*

$$F_M(\upsilon) = \sum_{m=-M}^{\infty} \left(\sum_{n=-\infty}^{\infty} \langle F, \varphi_{m,n} \rangle \tilde{\varphi}_{m,n}(\upsilon) \right).$$

Then $(F_M(\upsilon))_{M=0}^{\infty}$ converges pointwise to $F(\upsilon)$ everywhere on $\mathbb{R}$.

Theorem 3.3. *If*

$$\varphi(t) = O\left(\frac{1}{(1 + |t|)^{\frac{2}{\alpha} + \delta}} \right), \ 0 < \alpha \le \frac{1}{2}; \ \delta > 0,$$

$\hat{\varphi}(\xi) = 0, \ \xi \in (-\epsilon, \epsilon), \ 0 < \epsilon < \pi$, as well as F is continuous in (a, b), where $-\infty < a < b < \infty$ and it belongs to $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ then $(F_M(\upsilon))_{M=0}^{\infty}$ converges uniformly to $F(\upsilon)$ in every closed subinterval of (a, b).

4 Proof of Theorem 3.1

The Fourier transform of $\tilde{\varphi}_{m,n}$ is defined by

$$\begin{aligned}
\hat{\tilde{\varphi}}_{m,n}(\xi) &= \int_{-\infty}^{\infty} e^{-i\xi\upsilon} \tilde{\varphi}_{m,n}(\upsilon) d\upsilon \\
&= \int_{-\infty}^{\infty} e^{-i\xi\upsilon} 2^{\frac{m}{2}} \tilde{\varphi}(2^m \upsilon - n) d\upsilon, \quad \text{setting } 2^m \upsilon - n = u \\
&= \int_{-\infty}^{\infty} e^{-i\xi\left(\frac{u+n}{2^m}\right)} 2^{\frac{m}{2}} \tilde{\varphi}(u) \frac{1}{2^m} du \\
&= \frac{e^{-i\xi\frac{n}{2^m}}}{2^{\frac{m}{2}}} \int_{-\infty}^{\infty} e^{-i\frac{\xi}{2^m} u} \tilde{\varphi}(u) du \\
&= \frac{e^{-i\frac{n}{2^m}\xi}}{2^{\frac{m}{2}}} \hat{\tilde{\varphi}}\left(\frac{\xi}{2^m} \right). \qquad (4.1)
\end{aligned}$$

Using the inverse Fourier transform formula, we have

$$\begin{aligned}
\tilde{\varphi}_{m,n}(\upsilon) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\upsilon\xi}\hat{\tilde{\varphi}}_{m,n}(\xi)d\xi, \quad \text{for every } \upsilon\in\mathbb{R}\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\upsilon\xi}\frac{e^{-i\frac{n}{2^m}\xi}}{2^{\frac{m}{2}}}\hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)d\xi \qquad (\text{from}(4.1))\\
&= \frac{1}{\sqrt{2\pi}}\int_{-2^m\pi}^{2^m\pi} e^{i\upsilon\xi}\hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)\frac{e^{-i\frac{n}{2^m}\xi}}{\sqrt{2^{m+1}\pi}}d\xi \ (\because \text{supp}\hat{\tilde{\varphi}}\subset[-\pi,\pi])\\
&= \frac{1}{\sqrt{2\pi}}\int_{-2^m\pi}^{2^m\pi}\left(e^{i\upsilon\xi}\hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)\right)\overline{h_n^{(m)}(\xi)}d\xi, \qquad (4.2)
\end{aligned}$$

where

$$h_n^{(m)}(\xi) = \frac{e^{i\frac{n}{2^m}\xi}}{\sqrt{2^{m+1}\pi}}.$$

Using equation (2.4) and the Parseval equality of the Fourier transform, we have

$$\begin{aligned}
\overline{\langle F,\varphi_{m,n}\rangle} &= \frac{1}{2\pi}\overline{\langle\hat{F},\hat{\varphi}_{m,n}\rangle}\\
&= \frac{1}{2\pi}\int_{-2^m\pi}^{2^m\pi}\overline{\hat{F}}(\xi)\hat{\varphi}_{m,n}(\xi)d\xi
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi}\int_{-2^m\pi}^{2^m\pi}\overline{\hat{F}}(\xi)\frac{e^{-i\frac{n}{2^m}\xi}}{2^{\frac{m}{2}}}\hat{\varphi}\left(\frac{\xi}{2^m}\right)d\xi \ \ (\text{using}(4.1))\\
&= \frac{1}{\sqrt{2\pi}}\int_{-2^m\pi}^{2^m\pi}\left(\overline{\hat{F}}(\xi)\hat{\varphi}\left(\frac{\xi}{2^m}\right)\right)\overline{h_n^{(m)}(\xi)}d\xi. \qquad (4.3)
\end{aligned}$$

The system $\{h_n^{(m)}(\xi)\}_{n=-\infty}^{\infty}$ is an orthonormal basis for $L^2[-2^m\pi,2^m\pi]$. By using equations (2.2) and (2.5), It is simple to demonstrate that the functions $\overline{\hat{F}}(\xi)\hat{\varphi}\left(\frac{\xi}{2^m}\right)$ and $e^{i\upsilon\xi}\hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)$ are in $L^2[-2^m\pi,2^m\pi]$.

Using equations (4.2) and (4.3), for all $\upsilon\in\mathbb{R}$, we have

$$\begin{aligned}
\langle F,\varphi_{m,n}\rangle\tilde{\varphi}_{m,n}(\upsilon) &= \left(\frac{1}{\sqrt{2\pi}}\int_{-2^m\pi}^{2^m\pi}\left(\hat{F}(\xi)\overline{\hat{\varphi}}\left(\frac{\xi}{2^m}\right)\right)h_n^{(m)}(\xi)d\xi\right)\\
&\quad\times\left(\frac{1}{\sqrt{2\pi}}\int_{-2^m\pi}^{2^m\pi}\left(e^{i\upsilon\xi}\hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)\right)\overline{h_n^{(m)}(\xi)}d\xi\right)\\
&= \left(\frac{1}{2\pi}\int_{-2^m\pi}^{2^m\pi}\hat{F}(\xi)e^{i\upsilon\xi}\hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)\overline{\hat{\varphi}}\left(\frac{\xi}{2^m}\right)d\xi\right)\left(\int_{-2^m\pi}^{2^m\pi}|h_n^{(m)}(\xi)|^2d\xi\right).
\end{aligned}$$

Using the Parseval equality formula for the Fourier series, we have

$$\begin{aligned}
\sum_{n=-\infty}^{\infty}\langle F,\varphi_{m,n}\rangle\tilde{\varphi}_{m,n}(\upsilon) &= \frac{1}{2\pi}\int_{-2^m\pi}^{2^m\pi}\hat{F}(\xi)e^{i\upsilon\xi}\hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)\overline{\hat{\varphi}}\left(\frac{\xi}{2^m}\right)d\xi\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{F}(\xi)e^{i\upsilon\xi}\hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)\overline{\hat{\varphi}}\left(\frac{\xi}{2^m}\right)d\xi \ (\text{by}(2.4)).
\end{aligned} \qquad (4.4)$$

As a result, for every $\upsilon\in\mathbb{R}$, the series on the left hand side of equation (4.4) is convergent.

Now, using equation (2.2), we get

$$\sum_{m=-M}^{M}\left|\hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)\overline{\hat{\varphi}}\left(\frac{\xi}{2^m}\right)\right| = \sum_{m=-M}^{M}\hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)\overline{\hat{\varphi}}\left(\frac{\xi}{2^m}\right) \qquad (4.5)$$

Therefore

$$(S_M F)(\upsilon) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(\xi)e^{i\upsilon\xi}\left(\sum_{m=-M}^{M}\hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)\overline{\hat{\varphi}}\left(\frac{\xi}{2^m}\right)\right)d\xi \ \forall\, \upsilon \in R.$$

This implies that

$$\begin{aligned}\lim_{M\to\infty}(S_M F)(\upsilon) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(\xi)e^{i\upsilon\xi}\left(\sum_{m=-\infty}^{\infty}\hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)\overline{\hat{\varphi}}\left(\frac{\xi}{2^m}\right)\right)d\xi \ \forall\, \upsilon \in R\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(\xi)e^{i\upsilon\xi}d\xi\left(\because \sum_{m=-\infty}^{\infty}\hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)\overline{\hat{\varphi}}\left(\frac{\xi}{2^m}\right)=1\right)\\ &= F(\upsilon).\end{aligned}$$

As a result, Theorem 3.1 has been fully proved.

5 Proof of Theorem 3.2

Using equations (4.4) and (2.4) for $m < M$, we have

$$\begin{aligned}\lim_{N\to\infty}\sum_{n=-N}^{N}\langle F,\varphi_{m,n}\rangle\tilde{\varphi}_{m,n}(\upsilon) &= \sum_{n=-\infty}^{\infty}\langle F,\varphi_{m,n}\rangle\tilde{\varphi}_{m,n}(\upsilon)\\ &= \frac{1}{2\pi}\int_{-2^m\pi}^{2^m\pi}\hat{F}(\xi)e^{i\upsilon\xi}\hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)\overline{\hat{\varphi}}\left(\frac{\xi}{2^m}\right)d\xi.\end{aligned} \tag{5.1}$$

Since the functions $F \in L^2(\mathbb{R})$ and $\hat{F}$ are both Lebesgue-integrable on $[-2^M\pi, 2^M\pi]$. Using Lebesgue's dominant convergence theorem and equation (4.5) in equation (5.1) for every $\upsilon \in \mathbb{R}$, we have

$$\begin{aligned}&\sum_{m=-M}^{\infty}\left(\lim_{N\to\infty}\sum_{n=-N}^{N}\langle F,\varphi_{m,n}\rangle\tilde{\varphi}_{m,n}(\upsilon)\right)\\ &= \frac{1}{2\pi}\int_{-2^m\pi}^{2^m\pi}\hat{F}(\xi)e^{i\upsilon\xi}\left(\sum_{m=-M}^{\infty}\hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)\overline{\hat{\varphi}}\left(\frac{\xi}{2^m}\right)\right)d\xi.\end{aligned}$$

Taking $M \to \infty$ in above expression, we get

$$\begin{aligned}\lim_{M\to\infty}\lim_{N\to\infty}F_{M,N}(\upsilon) &= \sum_{m=-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty}\langle F,\varphi_{m,n}\rangle\tilde{\varphi}_{m,n}(\upsilon)\right)\\ &= \frac{1}{2\pi}\int_{-2^m\pi}^{2^m\pi}\hat{F}(\xi)e^{i\upsilon\xi}\lim_{M\to\infty}\left(\sum_{m=-M}^{\infty}\hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)\overline{\hat{\varphi}}\left(\frac{\xi}{2^m}\right)\right)d\xi\\ &= \frac{1}{2\pi}\int_{-2^m\pi}^{2^m\pi}\hat{F}(\xi)e^{i\upsilon\xi}\left(\sum_{m=-\infty}^{\infty}\hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)\overline{\hat{\varphi}}\left(\frac{\xi}{2^m}\right)\right)d\xi\\ &= \frac{1}{2\pi}\int_{-2^m\pi}^{2^m\pi}\hat{F}(\xi)e^{i\upsilon\xi}d\xi\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{F}(\xi)e^{i\upsilon\xi}d\xi = F(\upsilon).\end{aligned}$$

As a result, the Theorem 3.2's proof is complete.

6 Proof of Theorem 3.3

Since

$$F_M(\upsilon) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(\xi)e^{i\upsilon\xi}\left(\sum_{m=-M}^{\infty}\hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)\overline{\hat{\varphi}}\left(\frac{\xi}{2^m}\right)\right)d\xi \text{ for every } \upsilon\in\mathbb{R}. \tag{6.1}$$

Take a look at its kernel functions:

$$\sum_{m=-M}^{\infty}\hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)\overline{\hat{\varphi}}\left(\frac{\xi}{2^m}\right).$$

Since

$$\text{supp}\hat{\tilde{\varphi}} = \text{supp}\hat{\varphi} \subset [-\pi,-\epsilon]\cup[\epsilon,\pi] \tag{6.2}$$

and $\varphi(t) = O\left(\frac{1}{(1+|t|)^{\frac{2}{\alpha}+\delta}}\right)$, $0<\alpha<\frac{1}{2}$ therefore it is obvious that both $\hat{\varphi}(\xi)$ and $|\hat{\varphi}\left(\frac{\xi}{2^m}\right)|^2$ are differentiable three times on $\mathbb{R}$. When $|m|$ is large enough, then $\text{supp}\hat{\varphi}\left(\frac{\xi}{2^m}\right)\cap(\xi-\eta,\xi+\eta)=\phi$ by equation (6.2), for $\xi\neq 0,\ 0<\eta<|\xi|$. As a result, there are only finitely many nonzero terms of the sum $\sum_{m=-\infty}^{\infty}\left|\hat{\varphi}\left(\frac{\xi}{2^m}\right)\right|^2$ in any neighbourhood $\xi\neq 0$ that does not contain $\xi=0$, and the sum $\sum_{m=-\infty}^{\infty}\left|\hat{\varphi}\left(\frac{\xi}{2^m}\right)\right|^2$ is three times differentiable on non zero real ξ. From equations (2.2) and (2.3), $\hat{\tilde{\varphi}}'''(\xi)$ exists on $\xi\in\mathbb{R}-\{0\}$.

Define function $g\in L^2(\mathbb{R})$ as

$$\hat{g}(\xi) = \begin{cases}\sum_{m=0}^{\infty}\hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)\overline{\hat{\varphi}}\left(\frac{\xi}{2^m}\right), & \xi\neq 0;\\ 1, & \xi=0.\end{cases} \tag{6.3}$$

By $\hat{\varphi}(\xi)=0, \xi\in(-\epsilon,\epsilon)$, we have

$$\sum_{m=0}^{\infty}\hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)\overline{\hat{\varphi}}\left(\frac{\xi}{2^m}\right) = \sum_{m=-\infty}^{\infty}\hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)\overline{\hat{\varphi}}\left(\frac{\xi}{2^m}\right) = 1$$

for $\xi\in(-\epsilon,\epsilon)-\{0\}$. Using the above expression and equation (6.3), we have

$$\hat{g}(\xi)=1,\ \text{for } \xi\in(-\epsilon,\epsilon). \tag{6.4}$$

A similar argument demonstrates that in some neighborhood of $\xi\neq 0$, $\sum_{m=0}^{\infty}\hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)\overline{\hat{\varphi}}\left(\frac{\xi}{2^m}\right)$ has a finite number nonzero terms. Also by equations (6.2), (6.3) and (6.4), $\hat{g}'''(\xi)$ exists on $\mathbb{R}$ and

$$\text{supp}\hat{g}\subset[-\frac{\pi}{2},\frac{\pi}{2}]. \tag{6.5}$$

Allow $\tilde{g}$ to satisfy the following conditions once more:

$$\text{supp}\hat{\tilde{g}}\subset[-\pi,\pi],\ \hat{\tilde{g}}(\xi)=1,\ \xi\in[-\frac{\pi}{2},\frac{\pi}{2}] \tag{6.6}$$

and $\hat{\tilde{g}}'''(\xi)$ exists on $\mathbb{R}$.

Because both $\hat{g}$ and $\hat{\tilde{g}}$ are three-time differentiable,

$$g(y) = O\left((1+|y|)^{-3}\right) \text{ and } \tilde{g}(y) = O\left((1+|y|)^{-3}\right). \tag{6.7}$$

By equations (2.2), (6.5) and (6.6), we have $\hat{g}(\xi)\geq 0$ and

$$\overline{\hat{g}}(\xi)\hat{\tilde{g}}(\xi) = \overline{\hat{g}}(\xi) = \hat{g}(\xi) \text{ for } \xi\in\mathbb{R}. \tag{6.8}$$

Using equations (6.3) and (6.8), we have

$$\sum_{m=-M}^{\infty} \hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)\overline{\hat{\varphi}}\left(\frac{\xi}{2^m}\right) = \hat{g}\left(\frac{\xi}{2^M}\right) = \overline{\hat{g}}\left(\frac{\xi}{2^M}\right)\hat{\tilde{g}}\left(\frac{\xi}{2^M}\right), \quad \xi \neq 0. \tag{6.9}$$

By equations (6.1), (6.5) and (6.9), it is easily obtained that

$$F_M(\upsilon) = \frac{1}{2\pi}\int_{-2^M\pi}^{2^M\pi} \hat{F}(\xi)e^{i\upsilon\xi}\overline{\hat{g}}\left(\frac{\xi}{2^M}\right)\hat{\tilde{g}}\left(\frac{\xi}{2^M}\right)d\xi \text{ for every } \upsilon \in \mathbb{R}.$$

Since

$$\{h_n^{(M)}(\xi)\}_{n\in\mathbb{Z}} := \{\frac{e^{-i\frac{n}{2^M}\xi}}{\sqrt{2^{M+1}\pi}}\}_{n\in\mathbb{Z}}$$

is an ONB on $L^2[-2^M\pi, 2^M\pi]$, and the functions

$$\hat{F}(\xi)\overline{\hat{g}}\left(\frac{\xi}{2^M}\right), e^{-i\upsilon\xi}\overline{\hat{\tilde{g}}}\left(\frac{\xi}{2^M}\right) \in L^2[-2^M\pi, 2^M\pi].$$

Using the Fourier series' Parseval equality, we have

$$F_M(\upsilon) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left(\int_{-2^M\pi}^{2^M\pi}\left(\hat{F}(\xi)\overline{\hat{g}}\left(\frac{\xi}{2^M}\right)\right)\overline{h_n^{(M)}}(\xi)d\xi\right)\overline{\left(\int_{-2^M\pi}^{2^M\pi} e^{-i\upsilon\xi}\overline{\hat{\tilde{g}}}\left(\frac{\xi}{2^M}\right)\overline{h_n^{(M)}}(\xi)d\xi\right)}. \tag{6.10}$$

Consider

$$g_{M,n}(t) = 2^{\frac{M}{2}}g(2^Mt-n) \textit{ and } \tilde{g}_{M,n}(t) = 2^{\frac{M}{2}}\tilde{g}(2^Mt-n).$$

Then the Fourier transforms of $g_{M,n}$ and $\tilde{g}_{M,n}$ are given by

$$\hat{g}_{M,n}(\xi) = \frac{e^{-in\frac{\xi}{2^M}}}{2^{\frac{M}{2}}}\hat{g}\left(\frac{\xi}{2^M}\right) \textit{ and } \hat{\tilde{g}}_{M,n}(\xi) = \frac{e^{-in\frac{\xi}{2^M}}}{2^{\frac{M}{2}}}\hat{\tilde{g}}\left(\frac{\xi}{2^M}\right), \text{ (using equation (4.1))},$$

respectively. From the above expressions and equations (6.5), (6.6) and (6.10), we have

$$\begin{aligned} F_M(\upsilon) &= \sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{F}(\xi)\overline{\hat{g}}_{M,n}(\xi)d\xi\right)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{\hat{\tilde{g}}}_{M,n}(\xi)e^{i\upsilon\xi}d\xi\right) \\ &= \sum_{n=-\infty}^{\infty}\left(\int_{-\infty}^{\infty}F(t)\overline{g}_{M,n}(t)dt\right)\tilde{g}_{M,n}(\upsilon), \quad \upsilon \in \mathbb{R}. \end{aligned} \tag{6.11}$$

Define the kernel functions

$$K_M(t,\upsilon) := \sum_{n=-\infty}^{\infty}\overline{g}_{M,n}(t)\tilde{g}_{M,n}(\upsilon). \tag{6.12}$$

By equation (6.7), it observes that

$$\begin{aligned} \overline{g}_{M,n}(t)\tilde{g}_{M,n}(\upsilon) &= 2^{\frac{M}{2}}g(2^Mt-n)2^{\frac{M}{2}}\tilde{g}(2^M\upsilon-n) \\ &= O(2^M)(1+|2^Mt-n|)^{-3}(1+|2^M\upsilon-n|)^{-3} \\ &= O(2^M)(1+|2^Mt-n|)^{-3/2}(1+|2^M\upsilon-n|)^{-3/2}. \end{aligned}$$

As a result, given $t, \upsilon \in \mathbb{R}$, the series in equation (6.12) is convergent, and the kernel functions $K_M(t,\upsilon)$ are properly defined. Also, by using the inequality $(1+|p|)(1+|q|) \geq 1+|p-q|$, we get

$$\begin{aligned} K_M(t,\upsilon) &= \sum_{n=-\infty}^{\infty} O(2^M)(1+|2^Mt-n|)^{-3/2}(1+|2^M\upsilon-n|)^{-3/2} \\ &= \sum_{n=-\infty}^{\infty} O(2^M)(1+2^M|t-\upsilon|)^{-3/2} \\ &= O(2^M)(1+2^M|t-\upsilon|)^{-3/2}. \end{aligned} \tag{6.13}$$

Therefore, the integral

$$\begin{aligned}\int_{-\infty}^{\infty}|K_M(t,\upsilon)|dt &= \int_{-\infty}^{\infty} O(2^M)(1+2^M|t-\upsilon|)^{-3/2}dt \\ &= O(2^M)\left(\int_{-\infty}^{\upsilon}\left(1+2^M(\upsilon-t)\right)^{-3/2}dt+\int_{\upsilon}^{\infty}\left(1+2^M(t-\upsilon)\right)^{-3/2}dt\right) \\ &= O(2^M)\frac{1}{2^M}\int_1^{\infty}2u^{-3/2}du, \quad (\because 1+2^M(t-\upsilon)=u) \\ &= O(1) \end{aligned} \tag{6.14}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty}K_M(t,\upsilon)dt &= \int_{-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty}\overline{g}_{M,n}(t)\tilde{g}_{M,n}(\upsilon)\right)dt \\ &= \sum_{n=-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\overline{g}_{M,n}(t)dt\right)\tilde{g}_{M,n}(\upsilon) \\ &= \sum_{n=-\infty}^{\infty}\left(\int_{-\infty}^{\infty}2^M\overline{g}(2^Mt-n)dt\right)\tilde{g}(2^M\upsilon-n), \text{ setting } 2^Mt=u \end{aligned}$$

$$\begin{aligned} &= \sum_{n=-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\overline{g}(u-n)du\right)\tilde{g}(2^M\upsilon-n) \\ &= \sum_{n=-\infty}^{\infty}\tilde{g}(2^M\upsilon-n) \\ &\quad (\because \int_{-\infty}^{\infty}\overline{g}(u-n)du=\int_{-\infty}^{\infty}\overline{g}(t)dt=\hat{\overline{g}}(0)=1, \text{by equation (6.4)}). \end{aligned} \tag{6.15}$$

Further, the Poisson's summing formula and equation (6.6) ensure that

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\tilde{g}(2^M\upsilon-n) &= \sum_{k=-\infty}^{\infty}\hat{\tilde{g}}(2k\pi)e^{i2^{M+1}k\pi\upsilon} \\ &= \hat{\tilde{g}}(0)=1, \ \upsilon\in\mathbb{R}.\end{aligned}$$

Thus,

$$\int_{-\infty}^{\infty}K_M(t,\upsilon)dt=1, \quad \upsilon\in\mathbb{R}. \tag{6.16}$$

Using the dominant convergence theorem of Lebesgue in equation (6.11), we now obtain

$$\begin{aligned}F_M(\upsilon) &= \int_{-\infty}^{\infty}F(t)\left(\sum_{n=-\infty}^{\infty}\overline{g}_{M,n}(t)\tilde{g}_{M,n}(\upsilon)\right)dt \\ &= \int_{-\infty}^{\infty}F(t)K_M(t,\upsilon)dt \text{ for all } \upsilon\in\mathbb{R} \quad (\text{using equation (6.12)}).\end{aligned} \tag{6.17}$$

Let $[\beta,\gamma]\subset(a,b)$. Since F is continuous in (a,b), we may deduce that for each $\epsilon=\frac{1}{M}>0$, there exists a $\delta>0$ such that

$$|F(t)-F(\upsilon)|\leq\epsilon \text{ for } |t-\upsilon|\leq\delta.$$

Let us simplify equation (6.17) as following:

$$\begin{aligned} F_M(\upsilon) &= \int_{|t-\upsilon|\le\delta} (F(t)-F(\upsilon))\, K_M(t,\upsilon)dt + \int_{|t-\upsilon|\le\delta} F(\upsilon)K_M(t,\upsilon)dt \\ &\quad + \int_{|t-\upsilon|\ge\delta} F(t)K_M(t,\upsilon)dt \\ &= I_1 + I_2 + I_3,\ (\text{say}). \end{aligned} \tag{6.18}$$

Therefore,

$$\begin{aligned} I_1 &= O(\epsilon)\int_{-\infty}^{\infty} |K_M(t,\upsilon)|dt \\ &= O\left(\frac{1}{M}\right) \ \text{ for } \upsilon\in[\beta,\gamma] \text{ (by equation(6.14))}. \end{aligned} \tag{6.19}$$

Using equations (6.13) and (6.16), we have

$$\begin{aligned} I_2 &= F(\upsilon)\left(\int_{-\infty}^{\infty} K_M(t,\upsilon)dt - \int_{|t-\upsilon|\ge\delta} K_M(t,\upsilon)dt\right) \\ &= F(\upsilon)\left(1 + O(1)\int_{|u|\ge 2^M\delta} (1+|u|)^{-\frac{3}{2}}du\right) \\ &= F(\upsilon)(1+o(1)) \text{ for } M\to\infty \\ &= F(\upsilon)+o(1). \end{aligned} \tag{6.20}$$

For $|t-\upsilon|\ge\delta$, by equation (6.13), we have

$$\begin{aligned} K_M(t,\upsilon) &= O\left(\frac{1}{2^M(1+2^M|t-\upsilon|)}\right)^{\frac{3}{2}} \\ &= O\left(\frac{1}{2^M(1+2^M\delta)}\right)^{\frac{3}{2}} \\ &= O\left(\frac{1}{2^{2M}\delta)}\right)^{\frac{3}{2}} \\ &= O\left(\frac{1}{2^{3M}\delta^{\frac{3}{2}}}\right) \\ &= O\left(\frac{1}{2^{\frac{M}{2}}}\right),\ \text{taking } \delta = \frac{1}{2^{\frac{5M}{2}}}. \end{aligned} \tag{6.21}$$

Lastly,

$$\begin{aligned} |I_3| &\le \int_{|t-\upsilon|\ge\delta} |F(t)||K_M(t-\upsilon)|dt \\ &\le \max_{|t-\upsilon|\ge\delta} |K_M(t-\upsilon)| \int_{|t-\upsilon|\ge\delta} |F(t)|dt \\ &= O\left(\frac{1}{2^{\frac{M}{2}}}\right) O(1),\ \text{by equation (6.21) and} F\in L^1(\mathbb{R}) \\ &= O\left(\frac{1}{2^{\frac{M}{2}}}\right). \end{aligned} \tag{6.22}$$

By equations (6.19), (6.20) and (6.22), we have

$$\begin{aligned} |F_M(\upsilon)-F(\upsilon)| &= O\left(\frac{1}{M}\right) + o(1) + O\left(\frac{1}{2^{\frac{M}{2}}}\right) \\ &= o(1),\ \text{as } M\to\infty\ \forall\, \upsilon\in[\beta,\gamma]. \end{aligned}$$

As a result, $\{F_M\}_{M=0}^{\infty}$ on $[\beta,\gamma]$ converges uniformly to $F(\upsilon)$.

Thus the Theorem 3.3 has been completely established.

7 Corollaries

Following corollary can be derived from our theorem:

If a function $F \in L^2(\mathbb{R})$ such that its Fourier transform $\hat{F} \in L^1(\mathbb{R})$ and

$$F(\upsilon) = \sum_{m=-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty}\langle F,\varphi_{m,n}\rangle\tilde{\varphi}_{m,n}(\upsilon)\right)$$

is a wavelet frame series having M^{th} partial sums

$$(S_M F)(\upsilon) = \sum_{m=-M}^{M}\left(\sum_{n=-\infty}^{\infty}\langle F,\varphi_{m,n}\rangle\tilde{\varphi}_{m,n}(\upsilon)\right), M = 0,1,2,\cdots,$$

then

$$F(\upsilon) - (S_M F)(\upsilon) = o(1) \text{ as } M \to \infty \;\forall\, \upsilon \in \mathbb{R}.$$

Proof Following the proof of Theorem 3.1, we have

$$\begin{aligned}
& F(\upsilon) - (S_M F)(\upsilon) \\
= & \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{F}(\xi)e^{i\upsilon\xi}d\xi - \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{F}(\xi)e^{i\upsilon\xi}\left(\sum_{m=-M}^{M}\hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)\overline{\hat{\varphi}}\left(\frac{\xi}{2^m}\right)\right)d\xi \\
= & \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{F}(\xi)e^{i\upsilon\xi}\left(\sum_{m=-\infty}^{\infty}\hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)\overline{\hat{\varphi}}\left(\frac{\xi}{2^m}\right)\right)d\xi \\
& -\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{F}(\xi)e^{i\upsilon\xi}\left(\sum_{m=-M}^{M}\hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)\overline{\hat{\varphi}}\left(\frac{\xi}{2^m}\right)\right)d\xi \\
= & \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{F}(\xi)e^{i\upsilon\xi}\left(\sum_{m\le-(M+1)}\hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)\overline{\hat{\varphi}}\left(\frac{\xi}{2^m}\right)+\sum_{m\ge(M+1)}\hat{\tilde{\varphi}}\left(\frac{\xi}{2^m}\right)\overline{\hat{\varphi}}\left(\frac{\xi}{2^m}\right)\right)d\xi \\
\to & \; 0 \text{ as } M \to \infty.
\end{aligned}$$

8 Comparision and Justification

(1) Theorem 3.1-3.3 are verified and justified when the wavelet φ is replaced by Haar wavelet as well as Legendre wavelet.

(2) The results of this paper are verified by Haar wavelet which is discontinuous. These are also justified by Legendre wavelet which are continuous.

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