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Certain decomposition formulas for hypergeometric Gaussian functions in three variables

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Abstract In this paper, we present several operator identities by using certain inverse pairs of symbolic operators. Using the obtained operator identities, we establish many decomposition formulas for Gaussian hypergeometric functions in three variables. Furthermore, some transformation formulas are introduced.

1 Introduction

The use of many mathematical operations goes beyond the class of elementary functions. Calculation of integrals, summation of series, solution of algebraic, transcendental, difference and differential equations and their systems require expanding the class of functions studied. The development of the concept of a function, going in parallel with the development of the concepts of number and space, led to the emergence of new hypergeometric functions of many complex variables.

The theory of hypergeometric functions is one of the main branches of mathematical physics [9, 17, 18]. Over the past three centuries, the need to solve problems in hydrodynamics, control theory, classical and quantum mechanics, as well as numerous problems in probability theory and mathematical statistics has stimulated the development of the theory of special functions of one and several variables. Mathematical models of physical processes contain, as a rule, ordinary differential equations, partial differential equations or systems of such equations. However, only a few of the equations encountered in practice can be solved in the class of elementary functions. New functions were often defined as solutions to differential equations or their systems and were called hypergeometric functions. This is how the Bessel functions, Hermite functions and the Gaussian hypergeometric function arose. We choose to recall here the following second-order Gaussian hypergeometric functions in three variables, under consideration $r=|x|,\ s=|y|,\ t=|z|$, it defined by (see [20, pp. 74-75]):

$$F_{1c}\left(a_1,a_2,a_3,a_4,b_1,b_2;c;x,y,z\right) = \sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_n(a_3)_p(a_4)_p(b_1)_{m-n}(b_2)_{n-p}}{(c)_m m!n!p!}x^m y^n z^p, \tag{1.1}$$

where, $\{r<1, \quad s<\frac{1}{1+r}, \quad t<\frac{1}{1+s+rs}\}$,

$$F_{2c}\left(a_1,a_2,a_3,a_4,a_5,b;c;x,y,z\right) = \sum_{m,n,p=0}^{\infty}\frac{(a_1)_m(a_2)_m(a_3)_n(a_4)_n(a_5)_p(b)_{p-m-n}}{(c)_p m!n!p!}x^m y^n z^p, \tag{1.2}$$

where, $\{t<1, \quad max[r,s]<\frac{1}{1+t}\}$,

$$F_{4b}\left(a_1,a_2,a_3,a_4,b;c_1,c_2;x,y,z\right)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_m(a_2)_n(a_3)_p(a_4)_p(b)_{m+n-p}}{(c_1)_m(c_2)_n m!n!p!} x^m y^n z^p, \tag{1.3}$$

where, $\{r+s<1, \quad t<\frac{1}{1+s+r}\}$,

$$F_{4c}\,(a_1,a_2,a_3,a_4,b;c_1,c_2;x,y,z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_n(a_3)_p(a_4)_p(b)_{m-p}}{(c_1)_m(c_2)_n m!n!p!} x^m y^n z^p, \tag{1.4}$$

where, $\{r+s<1, \quad t<\frac{1-s}{1+r-s}\}$,

$$F_{4d}\,(a_1,a_2,a_3,a_4,b;c_1,c_2;x,y,z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_m(a_3)_n(a_4)_p(b)_{p-m}}{(c_1)_n(c_2)_p m!n!p!} x^m y^n z^p, \tag{1.5}$$

where, $\{s<1, \quad t<1, \quad r<\frac{1-s}{1+t}\}$,

$$F_{4f}\,(a_1,a_2,a_3,b_1,b_2;c;x,y,z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_m(a_2)_n(a_3)_p(b_1)_{m+n-p}(b_2)_{p-n}}{(c)_m m!n!p!} x^m y^n z^p, \tag{1.6}$$

where, $\{r+s<1, \quad t<\frac{1}{1+r}\}$,

$$F_{4h}\,(a_1,a_2,a_3,b_1,b_2;c;x,y,z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_n(a_3)_p(b_1)_{m-p}(b_2)_{p-m}}{(c)_n m!n!p!} x^m y^n z^p, \tag{1.7}$$

where, $\{r+s<1, \quad t<1\}$,

$$F_{4i}\,(a_1,a_2,a_3,b_1,b_2;c;x,y,z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_n(a_3)_p(b_1)_{p-m}(b_2)_{m-n}}{(c)_p m!n!p!} x^m y^n z^p, \tag{1.8}$$

where, $\{s<1, \quad t<\frac{1}{1+s}, \quad r<min[1, \frac{(1-s)^2}{4s}, \frac{1-t-st}{st^2}]\}$,

$$F_{5c}\,(a_1,a_2,a_3,a_4,b;c;x,y,z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_m(a_3)_p(a_4)_p(b)_{n-p}}{(c)_{m+n} m!n!p!} x^m y^n z^p, \tag{1.9}$$

where, $\{r<1, \quad s<1, \quad t<\frac{1}{1+s}\}$,

$$F_{5d}\,(a_1,a_2,a_3,a_4,b;c;x,y,z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_m(a_3)_n(a_4)_p(b)_{p-m-n}}{(c)_p m!n!p!} x^m y^n z^p, \tag{1.10}$$

where, $\{t<1, \quad max[r,s]<\frac{1}{1+t}\}$, and $(a)_n$ denotes the Pochhammer symbol defined by:

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & (n=0), \\ a(a+1)...(a+n-1) & (n \in \mathbb{N} := \{1,2,...\}). \end{cases}$$

Over seven decades ago, Burchnall and Chaundy [3, 4], and Chaundy [5], gave a number of expansions of double hypergeometric functions in series of simpler hypergeometric functions. Their method is based upon the inverse pair of symbolic operators

$$\nabla(h) = \frac{\Gamma(h)\Gamma(\delta_1+\delta_2+h)}{\Gamma(\delta_1+h)\Gamma(\delta_2+h)} \tag{1.11}$$

and

$$\Delta(h) = \frac{\Gamma(\delta_1+h)\Gamma(\delta_2+h)}{\Gamma(h)\Gamma(\delta_1+\delta_2+h)}, \tag{1.12}$$

where

$$\delta_1 := x\frac{\partial}{\partial x} \text{ and } \delta_2 := y\frac{\partial}{\partial y}.$$

For various multivariable hypergeometric functions including the Lauricella multivariable functions $F_A^{(n)}, F_B^{(n)}, F_C^{(n)}$ and $F_D^{(n)}$, Hasanov and Srivastava [10, 11] presented a number of decomposition formulas in terms of such simpler hypergeometric functions as the Gauss and Appell functions. Choi and Hasanov [6] showed how some rather elementary techniques based upon certain inverse pairs of symbolic operators

$$H_{x_1,\ldots,x_r}(\alpha,\beta) = \frac{\Gamma(\beta)\Gamma(\alpha+\delta_1+\cdots+\delta_r)}{\Gamma(\alpha)\Gamma(\beta+\delta_1+\cdots+\delta_r)}$$

$$= \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{(\beta-\alpha)_{k_1+\cdots+k_r}(-\delta_1)_{k_1}\cdots(-\delta_r)_{k_r}}{(\beta)_{k_1+\cdots+k_r}k_1!\cdots k_r!}, \tag{1.13}$$

$$\bar{H}_{x_1,\ldots,x_r}(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta+\delta_1+\cdots+\delta_r)}{\Gamma(\beta)\Gamma(\alpha+\delta_1+\cdots+\delta_r)}$$

$$= \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{(\beta-\alpha)_{k_1+\cdots+k_r}(-\delta_1)_{k_1}\cdots(-\delta_r)_{k_r}}{(1-\alpha-\delta_1-\cdots-\delta_r)_{k_1+\cdots+k_r}k_1!\cdots k_r!} \tag{1.14}$$

$$\left(\delta_j := x_j\frac{\partial}{\partial x_j}, j=1,...,r; r\in\mathbb{N} := \{1,2,3,...\}\right),$$

would lead us easily to several decomposition formulas associated with Humbert's hypergeometric functions $\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, \Xi_1$ and Ξ_2. Recently, the decomposition formulas for various hypergeometric functions have been studied by many authors (see [2, 7, 12, 13, 14, 16]).

Here, in this work, we aim at presenting certain decomposition formulas for the triple hypergeometric functions $F_{1c}, F_{2c}, F_{4b}, F_{4c}, F_{4d}, F_{4f}, F_{4h}, F_{4i}$ and F_{5c} with the help of the inverse pairs of operators (1.13) and (1.14). By means of our decompositions, we also give some transformation formulas involving these triple functions.

2 Operator identities

In this section, we follow the method given by Burchnall and Chaundy [3, 4], Chaundy [5], and Choi and Hasanov [6] to use the symbolic operators (1.13) and (1.14) to derive the following set of operator identities:

$$F_{1c}(a_1,a_2,a_3,a_4,b_1,b_2;c;x,y,z)$$
$$= H_x(a_1,c)(1-x)^{-b_1}H_2(b_2,a_2,a_3,a_4;1-b_1;-(1-x)y,z), \tag{2.1}$$

$$(1-x)^{-b_1}H_2(b_2,a_2,a_3,a_4;1-b_1;-(1-x)y,z)$$
$$= \bar{H}_x(a_1,c)F_{1c}(a_1,a_2,a_3,a_4,b_1,b_2;c;x,y,z), \tag{2.2}$$

$$F_{2c}(a_1,a_2,a_3,a_4,a_5,b;c;x,y,z)$$

$$= H_z(a_5, c)\,(1-z)^{-b}\, F_3\left(a_1, a_3, a_2, a_4; 1-b; -x(1-z), -y(1-z)\right), \tag{2.3}$$

$$(1-z)^{-b}\, F_3\left(a_1, a_3, a_2, a_4; 1-b; -x\left(1-z\right), -y\left(1-z\right)\right) \\ = \bar{H}_z(a_5, c) F_{2c}\left(a_1, a_2, a_3, a_4, a_5, b; c; x, y, z\right), \tag{2.4}$$

$$F_{4b}\left(a_1, a_2, a_3, a_4, b; c_1, c_2; x, y, z\right) \\ = H_x(a_1, c_1)\,(1-x)^{-b}\, H_2\left(b, a_2, a_3, a_4; c_2; \frac{y}{1-x}, (1-x)z\right), \tag{2.5}$$

$$(1-x)^{-b}\, H_2\left(b, a_2, a_3, a_4; c_2; \frac{y}{1-x}, (1-x)z\right) \\ = \bar{H}_x(a_1, c_1) F_{4b}\left(a_1, a_2, a_3, a_4, b; c_1, c_2; x, y, z\right), \tag{2.6}$$

$$F_{4c}\left(a_1, a_2, a_3, a_4, b; c_1, c_2; x, y, z\right) \\ = H_y(a_2, c_2)\,(1-y)^{-a_1}\, H_2\left(b, a_1, a_3, a_4; c_1; \frac{x}{1-y}, z\right), \tag{2.7}$$

$$(1-y)^{-a_1}\, H_2\left(b, a_1, a_3, a_4; c_1; \frac{x}{1-y}, z\right) \\ = \bar{H}_y(a_2, c_2) F_{4c}\left(a_1, a_2, a_3, a_4, b; c_1, c_2; x, y, z\right), \tag{2.8}$$

$$F_{4d}\left(a_1, a_2, a_3, a_4, b; c_1, c_2; x, y, z\right) \\ = H_y(a_3, c_1)\,(1-y)^{-a_1}\, H_2\left(b, a_4, a_1, a_2; c_2; z, \frac{x}{1-y}\right), \tag{2.9}$$

$$(1-y)^{-a_1}\, H_2\left(b, a_4, a_1, a_2; c_2; z, \frac{x}{1-y}\right) \\ = \bar{H}_y(a_3, c_1) F_{4d}\left(a_1, a_2, a_3, a_4, b; c_1, c_2; x, y, z\right), \tag{2.10}$$

$$F_{4f}\left(a_1, a_2, a_3, b_1, b_2; c; x, y, z\right) \\ = H_x(a_1, c)\,(1-x)^{-b_1}\, G_2\left(a_2, a_3, b_2, b_1; \frac{y}{1-x}, (1-x)z\right), \tag{2.11}$$

$$(1-x)^{-b_1}\, G_2\left(a_2, a_3, b_2, b_1; \frac{y}{1-x}, (1-x)z\right) \\ = \bar{H}_x(a_1, c) F_{4f}\left(a_1, a_2, a_3, b_1, b_2; c; x, y, z\right), \tag{2.12}$$

$$F_{4h}\left(a_1, a_2, a_3, b_1, b_2; c; x, y, z\right) \\ = H_y(a_2, c)\,(1-y)^{-a_1}\, G_2\left(a_1, a_3, b_2, b_1; \frac{x}{1-y}, z\right), \tag{2.13}$$

$$(1-y)^{-a_1}\, G_2\left(a_1, a_3, b_2, b_1; \frac{x}{1-y}, z\right) \\ = \bar{H}_y(a_2, c) F_{4h}\left(a_1, a_2, a_3, b_1, b_2; c; x, y, z\right), \tag{2.14}$$

$$F_{4i}\left(a_1, a_2, a_3, b_1, b_2; c; x, y, z\right) \\ = H_z(a_3, c)\,(1-z)^{-b_1}\, H_1\left(b_2, a_1, a_2; 1-b_1; -x(1-z), y\right), \tag{2.15}$$

$$(1-z)^{-b_1} H_1(b_2, a_1, a_2; 1-b_1; -x(1-z), y)$$
$$= \bar{H}_z(a_3, c) F_{4i}(a_1, a_2, a_3, b_1, b_2; c; x, y, z), \tag{2.16}$$

$$F_{5c}(a_1, a_2, a_3, a_4, b; c; x, y, z)$$
$$= H_{x,y}(a_1, c)(1-x)^{-a_2}(1-y)^{-b}\,{}_2F_1(a_3, a_4; 1-b; -(1-y)z), \tag{2.17}$$

$$(1-x)^{-a_2}(1-y)^{-b}\,{}_2F_1(a_3, a_4; 1-b; -(1-y)z)$$
$$= \bar{H}_{x,y}(a_1, c) F_{5c}(a_1, a_2, a_3, a_4, b; c; x, y, z), \tag{2.18}$$

$$F_{5d}(a_1, a_2, a_3, a_4, b; c; x, y, z)$$
$$= H_z(a_4, c)(1-z)^{-b} F_1(a_1, a_2, a_3; 1-b; -x(1-z), -y(1-z)), \tag{2.19}$$

$$(1-z)^{-b} F_1(a_1, a_2, a_3; 1-b; -x(1-z), -y(1-z))$$
$$= \bar{H}_z(a_4, c) F_{5d}(a_1, a_2, a_3, a_4, b; c; x, y, z), \tag{2.20}$$

where

$${}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{x^m}{m!},$$

is the Gaussian hypergeometric function in one variable [8]. Appell's functions F_1 and F_3 in two variables [1] and the Horn's functions H_1, H_2 and G_2 in two variables [15] are defined, respectively, by

$$F_1(a, b, c; d; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(c)_n}{(d)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \{r < 1,\ s < 1\},$$

$$F_3(a, b, c, d; e; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_m(b)_n(c)_m(d)_n}{(e)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \{r < 1,\ s < 1\},$$

$$H_1(a, b, c; d; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m-n}(b)_{m+n}(c)_n}{(d)_m} \frac{x^m}{m!} \frac{y^n}{n!}, \{4rs = (s-1)^2\},$$

$$H_2(a, b, c, d; e; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m-n}(b)_m(c)_n(d)_n}{(e)_m} \frac{x^m}{m!} \frac{y^n}{n!}, \{-r + \frac{1}{s} = 1\},$$

$$G_2(a, b, c, d; x, y) = \sum_{m,n=0}^{\infty} (a)_m(b)_n(c)_{n-m}(d)_{m-n} \frac{x^m}{m!} \frac{y^n}{n!}, \{r < 1, s < 1\}. \tag{2.21}$$

Proof. The operator identities (2.1) to (2.20) can be easily derived by just following the method in Burchnall and Chaundy [3, 4], (see also [5], [6]). So the details of proofs are omitted. □

3 Decomposition formulas

Here, we use the mutually inverse (1.13) and (1.14) from the operator identities (2.1) to (2.20), we obtain the following expansion formulas for the Gaussian hypergeomeric functions in three variables (1.1) to (1.10):

$$F_{1c}\left(a_1,a_2,a_3,a_4,b_1,b_2;c;x,y,z\right)=(1-x)^{-b_1}\sum_{i=0}^{\infty}\frac{(-1)^i(c-a_1)_i(b_1)_i}{(c)_i i!}$$
$$\times\left(\frac{x}{1-x}\right)^i H_2\left(b_2,a_2,a_3,a_4;1-b_1-i;-(1-x)y,z\right), \tag{3.1}$$

$$F_{1c}\left(a_1,a_2,a_3,1-b_2,b_1,b_2;c;x,y,z\right)=(1-x)^{-b_1}\left[1-(1-x)y]\right)^{-a_3}$$
$$\times\sum_{i=0}^{\infty}\frac{(-1)^i(c-a_1)_i(b_1)_i}{(c)_i i!}\left(\frac{x}{1-x}\right)^i$$
$$\times F_1\left(a_2,b_2,a_3;1-b_1-i;-(1-x)y,\frac{-(1-x)yz}{1+z}\right), \tag{3.2}$$

$$(1-x)^{-b_1}H_2\left(b_2,a_2,a_3,a_4;1-b_1;-(1-x)y,z\right)$$
$$=\sum_{i=0}^{\infty}\frac{(c-a_1)_i(b_1)_i}{(c)_i i!}x^iF_{1c}\left(a_1,a_2,a_3,a_4,b_1+i,b_2;c+i;x,y,z\right), \tag{3.3}$$

$$F_{2c}\left(a_1,a_2,a_3,a_4,a_5,b;c;x,y,z\right)=(1-z)^{-b}\sum_{i=0}^{\infty}\frac{(-1)^i(c-a_5)_i(b)_i}{(c)_i i!}$$
$$\times\left(\frac{z}{1-z}\right)^i F_3\left(a_1,a_3,a_2,a_4;1-b-i;-x(1-z),-y(1-z)\right), \tag{3.4}$$

$$(1-z)^{-b}F_3\left(a_1,a_3,a_2,a_4;1-b;-x\left(1-z\right),-y\left(1-z\right)\right)$$
$$=\sum_{i=0}^{\infty}\frac{(c-a_5)_i(b)_i}{(c)_i i!}z^iF_{2c}\left(a_1,a_2,a_3,a_4,a_5,b+i;c+i;x,y,z\right), \tag{3.5}$$

$$F_{4b}\left(a_1,a_2,a_3,a_4,b;c_1,c_2;x,y,z\right)=(1-x)^{-b}\sum_{i=0}^{\infty}\frac{(-1)^i(c_1-a_1)_i(b)_i}{(c_1)_i i!}$$
$$\times\left(\frac{x}{1-x}\right)^i H_2\left(b+i,a_2,a_3,a_4;c_2;\frac{y}{1-x},(1-x)z\right), \tag{3.6}$$

$$(1-x)^{-b}H_2\left(b,a_2,a_3,a_4;c_2;\frac{y}{1-x},(1-x)z\right)$$
$$=\sum_{i=0}^{\infty}\frac{(c_1-a_1)_i(b)_i}{(c_1)_i i!}x^iF_{4b}\left(a_1,a_2,a_3,a_4,b+i;c_1+i,c_2;x,y,z\right), \tag{3.7}$$

$$F_{4c}\left(a_1,a_2,a_3,a_4,b;c_1,c_2;x,y,z\right)=(1-y)^{-a_1}\sum_{i=0}^{\infty}\frac{(-1)^i(a_1)_i(c_2-a_2)_i}{(c_2)_i i!}$$
$$\times\left(\frac{y}{1-y}\right)^i H_2\left(b,a_1+i,a_3,a_4;c_1;\frac{x}{1-y},z\right), \tag{3.8}$$

$$(1-y)^{-a_1}H_2\left(b,a_1,a_3,a_4;c_1;\frac{x}{1-y},z\right)$$
$$=\sum_{i=0}^{\infty}\frac{(a_1)_i(c_2-a_2)_i}{(c_2)_i i!}y^iF_{4c}\left(a_1+i,a_2,a_3,a_4,b;c_1,c_2+i;x,y,z\right), \tag{3.9}$$

$$F_{4d}\left(a_1,a_2,a_3,a_4,b;c_1,c_2;x,y,z\right)=(1-y)^{-a_1}\sum_{i=0}^{\infty}\frac{(-1)^i(a_1)_i(c_1-a_3)_i}{(c_1)_i i!}$$

$$\times\left(\frac{y}{1-y}\right)^i H_2\left(b,a_4,a_1+i,a_2;c_2;z,\frac{x}{1-y}\right), \tag{3.10}$$

$$(1-y)^{-a_1} H_2\left(b,a_4,a_1,a_2;c_2;z,\frac{x}{1-y}\right)$$

$$=\sum_{i=0}^{\infty}\frac{(a_1)_i(c_1-a_3)_i}{(c_1)_i i!}y^i F_{4d}\left(a_1+i,a_2,a_3,a_4,b;c_1+i,c_2;x,y,z\right), \tag{3.11}$$

$$F_{4f}\left(a_1,a_2,a_3,b_1,b_2;c;x,y,z\right)=(1-x)^{-b_1}\sum_{i=0}^{\infty}\frac{(-1)^i(c-a_1)_i(b_1)_i}{(c)_i i!}$$

$$\times\left(\frac{x}{1-x}\right)^i G_2\left(a_2,a_3,b_2,b_1+i;\frac{y}{1-x},(1-x)z\right), \tag{3.12}$$

$$(1-x)^{-b_1} G_2\left(a_2,a_3,b_2,b_1;\frac{y}{1-x},(1-x)z\right)$$

$$=\sum_{i=0}^{\infty}\frac{(c-a_1)_i(b_1)_i}{(c)_i i!}x^i F_{4f}\left(a_1,a_2,a_3,b_1+i,b_2;c+i;x,y,z\right), \tag{3.13}$$

$$F_{4h}\left(a_1,a_2,a_3,b_1,b_2;c;x,y,z\right)=(1-y)^{-a_1}\sum_{i=0}^{\infty}\frac{(-1)^i(a_1)_i(c-a_2)_i}{(c)_i i!}$$

$$\times\left(\frac{y}{1-y}\right)^i G_2\left(a_1+i,a_3,b_2,b_1;\frac{x}{1-y},z\right), \tag{3.14}$$

$$(1-y)^{-a_1} G_2\left(a_1,a_3,b_2,b_1;\frac{x}{1-y},z\right)$$

$$=\sum_{i=0}^{\infty}\frac{(a_1)_i(c-a_2)_i}{(c)_i i!}y^i F_{4h}\left(a_1+i,a_2,a_3,b_1,b_2;c+i;x,y,z\right), \tag{3.15}$$

$$F_{4i}\left(a_1,a_2,a_3,b_1,b_2;c;x,y,z\right)=(1-z)^{-b_1}\sum_{i=0}^{\infty}\frac{(-1)^i(c-a_3)_i(b_1)_i}{(c)_i i!}$$

$$\times\left(\frac{z}{1-z}\right)^i H_1\left(b_2,a_1,a_2;1-b_1-i;-x(1-z),y\right), \tag{3.16}$$

$$(1-z)^{-b_1} H_1\left(b_2,a_1,a_2;1-b_1;-x(1-z),y\right)$$

$$=\sum_{i=0}^{\infty}\frac{(c-a_3)_i(b_1)_i}{(c)_i i!}z^i F_{4i}\left(a_1,a_2,a_3,b_1+i,b_2;c+i;x,y,z\right), \tag{3.17}$$

$$F_{5c}\left(a_1,a_2,a_3,a_4,b;c;x,y,z\right)=(1-x)^{-a_2}(1-y)^{-b}$$

$$\times\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(a_2)_i(c-a_1)_{i+j}(b)_j}{(c)_{i+j}i!j!}\left(\frac{x}{1-x}\right)^i\left(\frac{y}{1-y}\right)^j$$

$$\times{}_2F_1\left(a_3,a_4;1-b-j;-(1-y)z\right), \tag{3.18}$$

$$(1-x)^{-a_2}(1-y)^{-b}\,{}_2F_1\left(a_3,a_4;1-b;-(1-y)z\right)$$

$$=\sum_{i,j=0}^{\infty}\frac{(a_2)_i(c-a_1)_{i+j}(b)_j}{(c)_{i+j}i!j!}x^i y^j F_{5c}\left(a_1,a_2+i,a_3,a_4,b+j;c+i+j;x,y,z\right), \tag{3.19}$$

$$F_{5d}\left(a_1,a_2,a_3,a_4,b;c;x,y,z\right) =) \left(1-z\right)^{-b}\sum_{i=0}^{\infty}\frac{(-1)^i(c-a_4)_i(b)_i}{(c)_i i!}$$

$$\times\left(\frac{z}{1-z}\right)^i F_1\left(a_1,a_2,a_3;1-b-i;-x(1-z),-y(1-z)\right), \tag{3.20}$$

$$\left(1-z\right)^{-b}F_1\left(a_1,a_2,a_3;1-b;-x(1-z),-y(1-z)\right)$$

$$=\sum_{i=0}^{\infty}\frac{(c-a_4)_i(b)_i}{(c)_i i!}z^i F_{5d}\left(a_1,a_2,a_3,a_4,b+i;c+i;x,y,z\right). \tag{3.21}$$

Proof. We first recall the following formulas [19, p.93]:

$$(-\delta)_n\{f(\xi)\} = (-1)^n\xi^n\frac{d^n}{d\xi^n}\{f(\xi)\}, \tag{3.22}$$

$$(\alpha+\delta)_n\{f(\xi)\} = \xi^{1-\alpha}\frac{d^n}{d\xi^n}\{\xi^{\alpha+n-1}f(\xi)\}, \tag{3.23}$$

$$\left(\delta := \xi\frac{d}{d\xi}; \alpha\in\mathbb{C}; n\in\mathbb{N}_0 := \{0,1,2,...\}\right),$$

where $f(\xi)$ is an analytic function.
Let us prove the expansion formula (3.1). It is not difficult to show that the equality

$$(1-x)^{-b_1}H_2\left(b_2,a_2,a_3,a_4;1-b_1;-(1-x)y,z\right)$$

$$=\sum_{m,n,p=0}^{\infty}\frac{(a_2)_n(a_3)_p(a_4)_p(b_1)_{m-n}(b_2)_{n-p}}{m!n!p!}x^m y^n z^p. \tag{3.24}$$

Now, taking into account operator (1.13) and equality (3.24), in expanded form, we write the operaton identity (2.1)

$$F_{1c}\left(a_1,a_2,a_3,a_4,b_1,b_2;c;x,y,z\right)$$

$$=\sum_{i=0}^{\infty}\frac{(c-a_1)_i(-\delta_x)_i}{(c)_i i!}\sum_{m,n,p=0}^{\infty}\frac{(a_2)_n(a_3)_p(a_4)_p(b_1)_{m-n}(b_2)_{n-p}}{m!n!p!}x^m y^n z^p. \tag{3.25}$$

Using the differentiation formula for the hypergeometric formula, from (3.25) we have

$$(-\delta_x)_i\sum_{m,n,p=0}^{\infty}\frac{(a_2)_n(a_3)_p(a_4)_p(b_1)_{m-n}(b_2)_{n-p}}{m!n!p!}x^m y^n z^p$$

$$=(-1)^i x^i (b_1)_i\sum_{m,n,p=0}^{\infty}\frac{(a_2)_n(a_3)_p(a_4)_p(b_1+i)_{m-n}(b_2)_{n-p}}{m!n!p!}x^m y^n z^p. \tag{3.26}$$

Substitute (3.26) into (3.25) and in view of the definition (2.21), we obtain the expansion formula (3.1). Similarly, by means of the identities (3.22) and (3.23), one can easily obtain the other expansion formulas. □

4 Transformation formulas

In this section, we establish some transformation formulas for the Gaussian triple functions as below:

$$F_{1c}\left(a_1,a_2,a_3,a_4,b_1,b_2;c;x,y,z\right)$$

$$=(1-x)^{-b_1}F_{1c}\left(c-a_1,a_2,a_3,a_4,b_1,b_2;c;\frac{x}{x-1},(1-x)y,z\right), \tag{4.1}$$

$$F_{2c}\left(a_1,a_2,a_3,a_4,a_5,b;c;x,y,z\right)$$

$$= (1-z)^{-b} F_{2c}\left(a_1, a_2, a_3, a_4, c-a_5, b; c; x(1-z), y(1-z), \frac{z}{z-1}\right), \tag{4.2}$$

$$F_{4b}\left(a_1, a_2, a_3, a_4, b; c_1, c_2; x, y, z\right)$$
$$= (1-x)^{-b} F_{4b}\left(c_1-a_1, a_2, a_3, a_4, b; c_1, c_2; \frac{x}{x-1}, \frac{y}{1-x}, (1-x)z\right), \tag{4.3}$$

$$F_{4c}\left(a_1, a_2, a_3, a_4, b; c_1, c_2; x, y, z\right)$$
$$= (1-y)^{-a_1} F_{4c}\left(a_1, c_2-a_2, a_3, a_4, b; c_1, c_2; \frac{x}{1-y}, \frac{y}{y-1}, z\right), \tag{4.4}$$

$$F_{4d}\left(a_1, a_2, a_3, a_4, b; c_1, c_2; x, y, z\right)$$
$$= (1-y)^{-a_1} F_{4d}\left(a_1, a_2, c_1-a_3, a_4, b; c_1, c_2; \frac{x}{1-y}, \frac{y}{y-1}, z\right), \tag{4.5}$$

$$F_{4f}\left(a_1, a_2, a_3, b_1, b_2; c; x, y, z\right)$$
$$= (1-x)^{-b_1} F_{4f}\left(c-a_1, a_2, a_3, b_1, b_2; c; \frac{x}{x-1}, \frac{y}{1-x}, (1-x)z\right), \tag{4.6}$$

$$F_{4h}\left(a_1, a_2, a_3, b_1, b_2; c; x, y, z\right)$$
$$= (1-y)^{-a_1} F_{4h}\left(a_1, c-a_2, a_3, b_1, b_2; c; \frac{x}{1-y}, \frac{y}{y-1}, z\right), \tag{4.7}$$

$$F_{4i}\left(a_1, a_2, a_3, b_1, b_2; c; x, y, z\right)$$
$$= (1-z)^{-b_1} F_{4i}\left(a_1, a_2, c-a_3, b_1, b_2; c; x(1-z), y, \frac{z}{z-1}\right), \tag{4.8}$$

$$F_{5c}\left(a_1, a_2, a_3, a_4, b; c; x, y, z\right)$$
$$= (1-x)^{-a_2}(1-y)^{-b} F_{5c}\left(c-a_1, a_2, a_3, a_4, b; c; \frac{x}{x-1}, \frac{y}{y-1}, (1-y)z\right), \tag{4.9}$$

$$F_{5d}\left(a_1, a_2, a_3, a_4, b; c; x, y, z\right)$$
$$= (1-z)^{-b} F_{5d}\left(a_1, a_2, a_3, c-a_4, b; c; x(1-z), y(1-z), \frac{z}{z-1}\right). \tag{4.10}$$

Proof. The relations (4.1)-(4.10) follow easily from the expansion formulas (3.1) to (3.21). So details of proofs are omitted. □

5 Concluding remarks

In this paper, with the help of the inverse pairs of symbolic operators, we established a number of decomposition formulas for some Gaussian triple hypergeometric functions. Additionally, we investigated certain transformation formulas for these functions. We conclude that mutually inverse operators (1.13) and (1.14) can be applied to other multiple hypergeometric functions.

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