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A NEW CLASS OF MILD AND STRONG SOLUTIONS OF INTEGRO-DIFFERENTIAL EQUATION OF ARBITRARY ORDER IN BANACH SPACE

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Abstract In the present paper, we investigate the existence and uniqueness of a new class of mild and strong solution to fractional integro-differential equations in the Hilfer fractional derivative sense in Banach space, by means of the continuously C_0-semigroup, Banach fixed point theorem and Gronwall inequality.

1 Introduction and motivation

Over the decades, differential equations have been a very fruitful field both in theory and applications. Numerous important and relevant results on the existence and uniqueness of mild and strong solutions of differential and integro-differential equations were investigated and published by many researchers [6, 19, 28]. On the other hand, with the expansion of fractional calculus, in particular, with new definitions of generalized fractional derivatives and fractional integrals [25, 37, 38], many researchers started to use such tools obtained within the fractional calculus, to obtain results only presented in the traditional sense in the field of differential equations, i.e., integer order [3, 4, 13, 16, 33, 41, 44]. Thus, since from the junction of the fractional calculus the different types of differential and integro-differential equations: impulsive, functional, evolution with instantaneous and non-instantaneous impulses, new and applicable results would arise and each time would consolidate the relation of these areas. For a brief reading on the existence, uniqueness and Ulam-Hyers stabilities of fractional differential and integro-differential equations, see [5, 9, 12, 26, 27, 35, 42, 39].

In 2010 Diagana et al. [15], using the Arzelà-Ascoli theorem, Schauder fixed point theorem and Lebesgue dominated convergence theorem, and investigated the existence and uniqueness of mild solutions for some nonlocal semilinear fractional integro-differential equations given by

$$\left\{ \begin{array}{rcl} D^{\beta}x\,(t) & = & -Ax(t) + f\,(t, x\,(t)) + \displaystyle\int_0^t a(t-s)h(s, x(s))ds \\ x(0) + g(x) & = & x_0 \end{array} \right. \tag{1.1}$$

where D^{β} $(0 < \beta < 1)$ is the Caputo fractional derivative, the linear operator $-A$ is the infinitesimal of an analytic semigroup $R(t)_{t\geq 0}$ that is uniformly bounded on X (Banach space) and compact for $t > 0$, the function $a(\cdot)$ is real-valued such that

$$a_T = \int_0^T a(s)ds < \infty \tag{1.2}$$

the functions f, g and h are continuous, and the nonlocal condition

$$g(x) = \sum_{k=1}^{p} c_k x(t_k), \tag{1.3}$$

with c_k, $k = 1, 2, ..., p$, are given constants and $0 < t_1 < t_2 < \cdots < t_p \leq T$.

In this sense, in 2011 Rashid and Al-Omari [31] also dedicated to investigate the existence of local and global mild solutions for fractional semilinear impulsive integro-differential equations in the Caputo sense in Banach space. In 2011, Agarwal et al. [1], presented sufficient conditions to investigate the existence and uniqueness of mild solutions for a class of fractional integro-differential equations with time-dependent delay in the Riemann-Liouville sense over the Banach space. In order to validate and consolidate the obtained results, they presented a concrete application on the conduction of heat in materials. In addition to the investigation of the existence and uniqueness of fractional integro-differential equations, we can also mention some important works about functional and impulsive differential equations [11, 17, 18, 34, 43, 45, 46].

On the other hand, in 2012 Debbouche et al. [14], by means of the Schauder's fixed point theorem, Gelfand-Shilov principles and the semigroup theory, dedicated to investigate the existence of mild and strong solutions of nonlinear fractional integro-differential equations Sobolev type given by

$$\frac{d^\alpha (Bu(t))}{dt^\alpha} + Au(t) = f(t, W(t)) + \int_0^t g(t, s, W(s))ds \tag{1.4}$$

$$u(0) = \sum_{k=1}^{p} c_k u(t_k) = u_0 \tag{1.5}$$

where $\dfrac{d^\alpha}{dt^\alpha}$ $(0 < \alpha \leq 1)$ is the Riemann-Liouville fractional derivative $0 \leq t_1 < \cdots < t_p \leq a$, $c_1, ..., c_p$ are real numbers, B and A are linear closed operators with domains contained in a Banach space X and ranges contained in a Banach space Y, $W(t) = (B_1(t)u(t), ..., B_r(t)u(t))$ $\{B_i(t) : i = 1, ..., r, t \in I = [0, a]\}$ is a family of linear closed operators defined on dense sets $S_1, ..., S_r \supset D(A) \supset D(B)$ respectively in X into X, $f : I \times X^r \to Y$ and $g : \Delta \times X^r \to Y$ are given abstract functions, with $\Delta = \{(s, t) : 0 \leq s \leq t \leq a\}$.

We can also highlight the work of the authors Qasen et al. [30] in 2015, in which they dedicated to investigate the existence and regularity of mild and strong solutions for a class of abstract integro-differential equations in the Caputo sense in Banach space Ω with fractional resolvent operator. We can mention important works that have been published in the context of stochastic fractional integro-differential equations [10, 8].

In 2021 Bedi et al. [23] investigated the existence of mild solutions of a coupled hybrid fractional order system with Caputo-Hadamard fractional derivatives using Dhage fixed point theorem in Banach algebras. The result discussed by the authors contribute significantly to the area. In this sense, an example in order to elucidate the result is presented. In 2021, Khan et al. [7], presented an important work on the existence and uniqueness of integro-differential equations with Mittag-Leffler kernel. Other interesting work on Ulam-Hyers stability and existence of fractional differential equation solutions with Mittag-Leffler kernel, see Khan et al. [24].

Motivated by the above works and by open problems, in this paper, we consider the fractional integro-differential equation (FIE) with initial local conditions given by

$$\left\{\begin{array}{rcl} {}^{\mathbf{H}}\mathbb{D}_{0+}^{\mu,\nu} u\left(t\right) + \mathcal{A}u(t) & = & f\left(t, u\left(t\right)\right) + \dfrac{1}{\Gamma(\mu)} \displaystyle\int_{t_0}^{t} \mathbb{H}^{\mu}(t,s) \mathbf{K}(t, s, u(s))ds \\ I_{t_0+}^{1-\gamma} u\left(t_{0+}\right) + g(t_1, ..., t_p, u(\cdot)) & = & u_0 \end{array}\right. \tag{1.6}$$

where ${}^{\mathbf{H}}\mathbb{D}_{0+}^{\mu,\nu}(\cdot)$ is the Hilfer fractional derivative, $I_{t_0}^{1-\gamma}(\cdot)$ is the Riemann-Liouville fractional integral with $0 < \mu \leq 1$, $0 \leq \nu \leq 1$, $\gamma = \mu + \nu(1 - \mu)$, $0 \leq t_0 < t_1 < ... < t_p \leq t_0 + a$ $(t \in (t_0, t_0 + a])$, $-\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $(\mathbb{S}(t))_{t \geq 0}$ on a Banach space Ω and $f : I \times \Omega \to \Omega$, $g(t_1, ..., t_p, \cdot) : \Omega \to \Omega$, $k : \Delta \times \Omega \to \Omega$, $\Delta = |(t, s) : t_0 \leq s \leq t \leq t_0 + a|$ and $\mathbb{H}^\mu(t, s) := \psi'(s)(\psi(t) - \psi(s))^{\mu - 1}$. We can substitute only elements of the set $[t_1, ..., t_p]$.

It is remarkable the excellence and importance of the results that have been obtained and published by many researchers, and that the range of options over the decades has been expanding due to the interdisciplinary between the fractional calculus and differential and integro-differential equations [21, 22, 29, 32, 40]. However, there are still many outstanding issues that need to be clarified and investigated. In this sense, in order to provide new results on the existence and uniqueness of mild solutions of fractional integro-differential equations in a Banach space, one of the motivations for the achievement of this paper is the contribution of such scientific growth.

Some points deserve to be highlighted in relation to the main results obtained in this paper:

(i) We present a new class of mild and strong solutions for the fractional integro-differential equation. This class can be obtained from the free choice of the parameters α and β. Note that from the choice of the limits $\beta \to 1$ and $\beta \to 0$ both in Eq.(1.6) as in their respective solution Eq.(2.15), we obtain the Caputo and Riemann-Liouville fractional derivatives;

(ii) We investigate the existence and uniqueness of mild solutions of fractional integro-differential equation in Banach space Ω;

(iii) We investigate the existence and uniqueness of strong solutions of fractional integro-differential equation in Banach Ω space.

This paper is organized as follows: in Section 2, we present some fundamental concepts and results for the development of this paper. In section 3, our main result we investigated the existence and uniqueness of mild and strong solutions of fractional integro-differential equation in the Hilfer sense in Banach space. In addition, we present two corollaries that are a direct consequence of the main results presented. Concluding remarks close this paper.

2 Mathematical background – auxiliary results

In this section, we present some definitions and theorem essential in the investigation of the main results of the paper.

First, we being with the introduction of the weighted space of functions $u \in J' := (t_0, t_0 + a]$ is given by [40]

$$C_{1-\gamma}(J,\Omega) = \left\{ u \in C(J',\Omega), t^{1-\gamma}u(t) \in C(J,\Omega) \right\}, 0 \leq \gamma \leq 1 \tag{2.1}$$

with norm

$$||u||_{C_{1-\gamma}} := \sup_{t \in J'} ||t^{1-\gamma}u(t)||. \tag{2.2}$$

Obviously, the space $C_{1-\gamma}(J,\Omega)$ is a Banach space.

Let $J = [t_0, t_0 + a]$ be a finite or infinite interval of the line $\mathbb{R}_+$ and $0 < \mu \leq 1$. Also, let $\psi(t)$ be an increasing and positive monotone function on $J' = (t_0, t_0 + a]$ having a continuous derivative $\psi'(t)$ on $J'' = (t_0, t_0 + a)$. The left-sided fractional integral of a function u with respect to the function ψ on $J = [t_0, t_0 + a]$ is defined by [38]

$$I^{\mu;\psi}_{t_{0+}} u(t) = \frac{1}{\Gamma(\mu)} \int_{t_0}^{t_0+a} \mathbb{H}^{\mu}(t,s)u(s)ds \tag{2.3}$$

where $\mathbb{H}^{\mu}(t,s) := \psi'(s)(\psi(t) - \psi(s))^{\mu-1}$.

Choosing $\psi(t) = t$, we have the Riemann-Liouville fractional integral given by

$$I^{\mu}_{t_{0+}} u(t) = \frac{1}{\Gamma(\mu)} \int_{t_0}^{t_0+a} (t-s)^{\mu-1}u(s)ds. \tag{2.4}$$

On the other hand, let $n - 1 < \mu \leq n$ with $n \in \mathbb{N}$, J the interval and $u, \psi \in C^n(J,\mathbb{R})$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$ for all $t \in J$. The left-sided ψ-Hilfer fractional derivative ${}^{\mathbf{H}}\mathbb{D}^{\mu,\nu;\psi}_{0+}(\cdot)$ of a function f of order μ and type $0 \leq \nu \leq 1$ is defined by [38]

$${}^{\mathbf{H}}\mathbb{D}^{\mu,\nu;\psi}_{0+} u(t) = I^{\nu(n-\mu);\psi}_{t_{0+}} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n I^{(1-\nu)(n-\mu)}_{t_{0+}} u(t). \tag{2.5}$$

Choosing $\psi(t) = t$, we have the Hilfer fractional derivative, given by

$$^{\mathbf{H}}\mathbb{D}_{0+}^{\mu,\nu}u(t) = I_{t_{0+}}^{\nu(n-\mu);\psi}\left(\frac{d}{dt}\right)^{n} I_{t_{0+}}^{(1-\nu)(n-\mu)}u(t). \tag{2.6}$$

In this paper, we use Eq.(2.4) and the so-called Hilfer fractional derivative Eq.(2.6). Consider the fractional initial value problem

$$\left\{\begin{array}{rcl} ^{\mathbf{H}}\mathbb{D}_{t_{0+}}^{\mu,\nu}u(t) & = & \mathcal{A}u(t) + f(t) \\ I_{t_0+}^{1-\gamma}u(t_0) & = & u_0 \end{array}\right. \tag{2.7}$$

where $^{\mathbf{H}}\mathbb{D}_{0+}^{\mu,\nu}(\cdot)$ is the Hilfer fractional derivative, $I_{t_0}^{1-\gamma}(\cdot)$ is the Riemann-Liouville fractional integral, with $0 < \mu \leq 1$, $0 \leq \nu \leq 1$, $\gamma = \mu + \nu(1-\mu)$, $f : (t_0, t_0+a] \to \Omega$, $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $(\mathbb{S}(t))_{t\geq 0}$, $u_0 \in \Omega$ and $t_0 \geq 0$.

Moreover a mild solution for Hilfer fractional evolution equations

$$\left\{\begin{array}{rcl} D_{0+}^{\gamma,\beta}x(t) & = & Ax(t) + f(t, x(t)),\ t \in J' = (0, b] \\ I_{t_0+}^{(1-\beta)(1-\gamma)}x(0) & = & x_0 \end{array}\right. \tag{2.8}$$

is given by a fractional version

$$x(t) = \mathbb{S}_{\gamma,\beta}(t)x_0 + \int_0^t \mathcal{K}_\beta(t-s)f(s, x(s))ds \tag{2.9}$$

where $\mathcal{K}_\beta(t) = t^{\beta-1}G_\beta(t)$, $G_\beta(t) = \displaystyle\int_0^\infty \beta\theta M_\beta(\theta)\mathbb{S}_{\gamma,\beta}(t^\beta\theta)\,d\theta$, $\mathbb{S}_{\gamma,\beta}(t) = I_\theta^{\beta(1-\gamma)}\mathcal{K}_\beta(t)$, $0 < \gamma \leq 1$ and $0 \leq \beta \leq 1$. For more details see [18] and references therein.

Definition 2.1. A function u is said to be a strong solution of problem Eq.(2.7) on I, if u is differentiable almost everywhere (a.e) on I

$$^{\mathbf{H}}\mathbb{D}_{0+}^{\mu,\nu} \in L'((t_0, t_0 + a], \Omega),\ I_{t_0+}^{1-\gamma}u(t_0) = u_0 \tag{2.10}$$

and

$$^{\mathbf{H}}\mathbb{D}_{0+}^{\mu,\nu}u(t) = \mathcal{A}(t) + f(t), \tag{2.11}$$

a.e on I.

We introduce the definition of the Wright function, fundamental in mild solution of Eq.(1.6). Then, the Wright function $\mathbf{M}_\mu(\mathbf{Q})$ is defined by [18]

$$\mathbf{M}_\mu(\mathbf{Q}) = \sum_{n=1}^\infty \frac{(-\mathbf{Q})^{n-1}}{(n-1)!\Gamma(1-\mu n)},\ 0 < \mu < 1,\ \mathbf{Q} \in \mathbb{C} \tag{2.12}$$

satisfying the equation

$$\int_0^\infty \theta^{\overline{\delta}}\mathbf{M}_\mu(\theta)\,d\theta = \frac{\Gamma(1+\overline{\delta})}{\Gamma(1+\mu\overline{\delta})},\ \text{for } \overline{\delta}, \theta \geq 0. \tag{2.13}$$

Theorem 2.2. [2] *If Ω is a reflexive Banach space, $u_0 \in \mathcal{D}(\mathcal{A})$ and f is Lipschitz continuous on I* Eq.(2.7) *has a unique strong solution u on I given by the expression*

$$u(t) = \mathbb{S}_{\mu,\nu}(t)u_0 + \int_{t_0}^t \mathcal{K}_\mu(t-s)f(s)ds,\ t \in I. \tag{2.14}$$

Definition 2.3. A continuous solutions u of integral equation

$$\begin{aligned} u(t) &= \mathbb{S}_{\mu,\nu}(t)[u_0 - g(t_1, ..., t_p, u(\cdot))] + \int_{t_0}^t \mathcal{K}_\nu(t-s)f(s, u(s))ds \\ &+ \frac{1}{\Gamma(\mu)}\int_{t_0}^t \mathcal{K}_\nu(t-s)\int_{t_0}^s \mathbb{H}^\mu(s,\tau)\mathbf{K}(s,\tau,u(\tau))d\tau ds \end{aligned} \tag{2.15}$$

is said to be a mild solution of problem Eq.(1.6), on I, where $\mathcal{K}_\nu(t) = t^{\gamma-1}\mathbf{G}_\nu(t)$, $\mathbf{G}_\nu(t) = \displaystyle\int_0^\infty \nu\theta\mathbf{M}_\nu(\theta)\mathbb{S}_{\mu,\nu}(t^\nu\theta)\,d\theta$ and $\mathbb{S}_{\mu,\nu}(t) = I_\theta^{\nu(1-\mu)}\mathcal{K}_\nu(t)$.

The following theorems and Corollary, are very important when we want to investigation existence, uniqueness and another fundamental properties of the fractional differential equations.

Theorem 2.4. [35] (Banach fixed point theorem) *Let (Ω, d) be a generalized complete metric space. Assume that $\hat{\Omega} : \Omega \to \Omega$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d(\hat{\Omega}^{k+1}, \hat{\Omega}^k) < \infty$ for some $x \in \Omega$, then the following are true:*

(i) The sequence $\hat{\Omega}^k x$ converges to a fixed point x^* of $\hat{\Omega}$;

(ii) x^* is the unique fixed point of $\hat{\Omega}$ in $\hat{\Omega}^* = \left\{ y \in \Omega / d(\hat{\Omega}^k x, y) < \infty \right\}$;

(iii) If $y \in \hat{\Omega}^*$, then $d(y, x^*) \leq \dfrac{1}{1-L} d(\hat{\Omega} y, y)$.

Theorem 2.5. (Gronwall inequality) *Let u, v be two integrable functions and g continuous, with domain $[a, b]$. Let $\psi \in C^1[a, b]$ an increasing function such that $\psi'(t) \neq 0$, $\forall t \in [a, b]$. Assume that*
(1) *u and v are nonnegative;*
(2) *g is nonnegative and non-decreasing.*
If

$$u(t) \leq v(t) + g(t) \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} u(\tau) d\tau \tag{2.16}$$

then

$$u(t) \leq v(t) + \int_a^t \sum_{k=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^k}{\Gamma(\alpha k)} \psi'(\tau)(\psi(t) - \psi(\tau))^{k\alpha-1} v(\tau) d\tau, \tag{2.17}$$

$\forall t \in [a, b]$.

Proof. See [36]. □

Corollary 2.6. *Under the hypotheses of* Theorem 2.5*, let v be a non-decreasing function on $[a, b]$. Then, we have*

$$u(t) \leq v(t) \mathbb{E}_\alpha(g(t)\Gamma(\alpha)(\psi(t) - \psi(a))^\alpha), \ \forall t \in [a, b], \tag{2.18}$$

where $\mathbb{E}_\alpha(\cdot)$ is a Mittag-Leffler function with one parameter.

Proof. See [36]. □

3 Existence and uniqueness of a mild and strong solution to FIE

In this section, we present the main results of this paper, i.e., the existence and uniqueness of mild and strong solutions of fractional integro-differential equations.

The discussion of the main results will be investigated in two steps. Firstly, for the investigation of the main results that will be presented through theorems, some conditions are necessary and sufficient to obtain them. In this sense, we have the following conditions **(CI)**:

(i) Ω is a Banach space with $||(\cdot)||_{C_{1-\gamma}}$ and $u_0 \in \Omega$;

(ii) $0 \leq t_0 < t_1 < ... < t_p \leq t_0 + a$ and $B_r = \left\{ v : ||v||_{C_{1-\gamma}} \leq r \right\} \subset \Omega$;

(iii) $f : I \times \Omega \to \Omega$ is continuous in t on I and there exists a constant $\mathbf{L} \geq 0$ such that

$$||f(s, v_1) - f(s, v_2)||_{C_{1-\gamma}} \leq \mathbf{L} ||v_1 - v_2||_{C_{1-\gamma}}, \text{for s} \in I, v_1, v_2 \in B_r; \tag{3.1}$$

(iv) $\mathbb{H} : \Delta \times \Omega \to \Omega$ is continuous and $\exists \mathbf{K}_0 > 0$ ($\mathbf{K}_0$ is constants) such that

$$||g(t, s, x) - g(t, s, y)||_{C_{1-\gamma}} \leq \mathbf{K}_0 ||x - y||_{C_{1-\gamma}}; \tag{3.2}$$

(v) $g : I^p \times \Omega \to \Omega$ and there exists a constant $\mathbf{Q}_0 > 0$ such that

$$||g(t_1, ..., t_p, u_1(\cdot)) - g(t_1, ..., t_p, u_2(\cdot))||_{C_{1-\gamma}} \leq \mathbf{Q}_0 ||t^{1-\gamma} x - y||_{C_{1-\gamma}}; \tag{3.3}$$

for $u_1, u_2 \in C_{1-\gamma}(I, B_r)$;

(vi) $-\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $(\mathbb{S}(t))_{t\geq 0}$ on Ω;

(vii) Consider the following

$$\mathbf{M} = \max_{t\in[0,a]} \|\mathbb{S}_{\mu,\nu}(t)\|; \mathbf{H} = \max_{s\in I} \left\|s^{1-\gamma} f(s,0)\right\|;$$

and

$$\mathbf{K}_1 = \max_{t_0\leq s\leq t\leq t_0+a} \left\|t^{1-\gamma} k(t,s,0)\right\|; \widetilde{\mathbf{G}}_1 = \max_{u\in C_{1-\gamma}(I,B_r)} \left\|t^{1-\gamma} g(t_1,...,t_p,u(\cdot))\right\|;$$

(viii) Admit that

$$\mathbf{M}\left(\|u_0\|_{C_{1-\gamma}} + \widetilde{\mathbf{G}_1} + (\mathbf{L}_r + \mathbf{H})\,a + \frac{(\mathbf{K}_0 r + \mathbf{K}_1)}{\Gamma(\mu)^2} (\psi(t_0+a) - \psi(t_0))^{\mu}\, a\right) \leq r \quad (3.4)$$

and

$$\mathbf{M}\mathbf{Q}_0 + \mathbf{M}\mathbf{L}a + \frac{\mathbf{M}\mathbf{K}_0 a}{\Gamma(\mu)^2} (\psi(t_0+a) - \psi(t_0))^{\mu} < 1. \quad (3.5)$$

On the other hand, we will impose some necessary and sufficient conditions for the discussion of the existence and uniqueness of strong solutions to FIE as discussed according to Theorem 3.2. In this sense, we have the following conditions **(CII)**:

(i) Ω is a reflexive Banach space with norm $\|\cdot\|_{C_{1-\gamma}}$ and $u_0 \in \Omega$;

(ii) $0 \leq t_0 \leq t_1 < ... < t_p \leq t_0 + a$ and $B_r := \left\{v : \|v\|_{C_{1-\gamma}} \leq r\right\} \subset \Omega$;

(iii) $f : I \times \Omega \to \Omega$ is continuous in t on I and there exists a constant $\mathbf{L} > 0$ such that

$$\|f(s_1,v_1) - f(s_2,v_2)\|_{C_{1-\gamma}} \leq \mathbf{L}\left(\|s_1 - s_2\|_{C_{1-\gamma}} + \|v_1 - v_2\|_{C_{1-\gamma}}\right) \quad (3.6)$$

for $s_1, s_2 \in I$ and $v_1, v_2 \in B_r$;

(iv) $\mathbf{K} : \Delta \times \Omega \to \Omega$ is continuous and there exists a constant $\mathbf{K}_0 > 0$ such that

$$\|\mathbf{K}(t_1,s,x) - \mathbf{K}(t_2,s,y)\|_{C_{1-\gamma}} \leq \mathbf{K}_0\left(|t_1 - t_2| + \|x_1 - x_2\|_{C_{1-\gamma}}\right); \quad (3.7)$$

(v) $g : I^p \times \Omega \to \Omega$ and there exists a constant $\mathbf{G}_0 > 0$ such that

$$\|g(t_1,...,t_p,u(\cdot)) - g(t_1,...,t_p,u(\cdot))\|_{C_{1-\gamma}} \leq \mathbf{G}_0 \sup_{t\in I} \left\|t^{1-\gamma}(u_1(t) - u_2(t))\right\|; \quad (3.8)$$

for $u_1, u_2 \in C_{1-\gamma}(I, B_r)$ and $g(t_1,...,t_p) \in \mathcal{D}(\mathcal{A})$;

(vi) $-\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $(\mathbb{S}(t))_{t\geq 0} \in \mathcal{D}(\mathcal{A})$;

(vii) Consider $u_0 \in \mathcal{D}(\mathcal{A})$;

(viii) Admit that $\mathbf{M}\left(\mathbf{Q}_0 + \mathbf{L}a + \frac{\mathbf{K}_0 a}{\Gamma(\mu)^2} (\psi(t_0+a) - \psi(t_0))^{\mu}\right) < 1.$

Definition 3.1. A function u is said to be a strong solution of problem Eq.(1.6) on I if u is a.e differentiable on I

$$\begin{array}{l} {}^{\mathbf{H}}\mathbb{D}^{\mu,\nu}_{t_0+} u(t) \in L'((t_{0+}, t_{0+}+a], \Omega) \\ I^{1-\gamma}_{t_{0+}} u(t) + g(t_1,...,t_p,u(\cdot)) = u_0 \end{array} \quad (3.9)$$

and

$${}^{\mathbf{H}}\mathbb{D}^{\mu,\nu}_{t_0+} u(t) + \mathcal{A}(t) = f(t,u(t)) + \frac{1}{\Gamma(\mu)} \int_{t_0}^{t} \mathbb{H}^{\mu}(t,s)\, \mathbf{K}(t,s,v(s))\, ds, \quad (3.10)$$

$t \in (t_0, t_0 + a]$.

Theorem 3.2. *The fractional integro-differential equation with nonlocal conditions given by* Eq.(1.6)*, has a unique solution on I.*

Proof. Let $\Omega := C_{1-\gamma}(I, B_r)$. First, we need define the following operator

$$\begin{aligned}(\mathfrak{F})\, v(t) &= \mathbb{S}_{\mu,\nu}(t)\, u_0 - \mathbb{S}_{\mu,\nu}(t)\, g(t_1, ..., t_p, v(\cdot)) + \int_{t_0}^{t} \mathcal{K}_\nu(t-s)\, f(s, v(s))\, ds \\ &\quad + \frac{1}{\Gamma(\mu)} \int_{t_0}^{t} \mathcal{K}_\mu(t-s) \int_{t_0}^{s} \mathbb{H}^\mu(s,\tau)\, \mathbf{K}(s,\tau,v(\tau))\, d\tau ds, \quad t \in I \qquad (3.11)\end{aligned}$$

where $\mathcal{K}_\nu(t) = t^{\nu-1} G_\nu(t)$, $G_\nu(t) = \displaystyle\int_0^\infty \nu\theta M_\nu(\theta)\, \mathbb{S}_{\mu,\nu}(t^\nu\theta)\, d\theta$, $\mathbb{S}_{\mu,\nu}(t) = I_\theta^{\nu(1-\mu)} \mathcal{K}_\nu(t)$, $0 < \mu \leq 1$ and $0 \leq \nu \leq 1$.

Using the conditions **C**(I.1)-(I.8) as presented previously, yields

$$\begin{aligned}
&\left\| t^{1-\gamma} (\mathfrak{F} v)(t) \right\| \\
\leq\ & \left\| t^{1-\gamma} \mathbb{S}_{\mu,\nu}(t)\, u_0 \right\| + \left\| t^{1-\gamma} \mathbb{S}_{\mu,\nu}(t)\, g(t_1, ..., t_p, v(\cdot)) \right\| \\
&+ \left\| t^{1-\gamma} \int_{t_0}^{t} \mathcal{K}_\mu(-s)\, f(s, v(s))\, ds \right\| \\
&+ \left\| t^{1-\gamma} \frac{1}{\Gamma(\mu)} \int_{t_0}^{t} \mathcal{K}_\mu(t-s) \int_{t_0}^{s} \mathbb{H}^\mu(s,\tau)\, \mathbf{K}(s,\tau,v(\tau))\, d\tau ds \right\| \\
\leq\ & \left\| \mathbb{S}_{\mu,\nu}(t) \right\| \left\| t^{1-\gamma} u_0 \right\| + \left\| \mathbb{S}_{\mu,\nu}(t) \right\| \left\| t^{1-\gamma} g(t_1, ..., t_p, v(\cdot)) \right\| \\
&+ t^{1-\gamma} \left\| \int_{t_0}^{t} \left\| \mathcal{K}_\mu(t-s) \right\| s^{\gamma-1} \left(\left\| s^{1-\gamma} (f(s, v(s)) - f(s, 0)) \right\| + \left\| s^{1-\gamma} f(s, 0) \right\| \right) ds \right\| \\
&+ \frac{t^{1-\gamma}}{\Gamma(\mu)} \int_{t_0}^{t} \left\| \mathcal{K}_\mu(t-s) \right\| \int_{t_0}^{s} \tau^{\gamma-1} \mathbb{H}^\mu(s,\tau) \times \\
&\left(\left\| \tau^{1-\gamma} (\mathbf{K}(s,\tau,v(\tau)) - \mathbf{K}(s,\tau,0)) \right\| + \left\| \tau^{1-\gamma} \mathbf{K}(s,\tau,0) \right\| \right) d\tau ds \\
\leq\ & \mathbf{M} \left\| u_0 \right\|_{C_{1-\gamma}} + \mathbf{M} \widetilde{\mathbf{G}_1} + \mathbf{M}(\mathbf{L}r + \mathbf{H}) \int_{t_0}^{t} ds + \frac{\mathbf{M}(\mathbf{K}_0 r + \mathbf{K}_1)}{\Gamma(\mu)} \int_{t_0}^{t} \int_{t_0}^{s} N^\mu(s,\tau)\, d\tau ds \\
=\ & \mathbf{M} \left\| u_0 \right\|_{C_{1-\gamma}} + \mathbf{M} \widetilde{\mathbf{G}_1} + \mathbf{M}(\mathbf{L}r + \mathbf{H})(t - t_0) + \frac{\mathbf{M}(\mathbf{K}_0 r + \mathbf{K}_1)}{\Gamma(\mu)} \frac{(\psi(s) - \psi(t_0))^\mu}{\Gamma(\mu)} \int_{t_0}^{t} ds \\
\leq\ & \mathbf{M} \left[\left\| u_0 \right\|_{C_{1-\gamma}} + \widetilde{\mathbf{G}_1} + (\mathbf{L}r + \mathbf{H})\, a + \frac{(\mathbf{K}_0 r + \mathbf{K}_1)(\psi(t_0 + a) - \psi(t_0))^\mu a}{\Gamma(\mu)^2} \right] \leq r,
\end{aligned} \qquad (3.12)$$

for $v \in \Omega$. Thus, we conclude that $\mathfrak{F}\Omega \subset \Omega$.

On the other hand, for every $v_1, v_2 \in \Omega$ and $t \in I$, yields

$$\begin{aligned}
& \left\| t^{1-\gamma} \left(\left(\mathfrak{F} v_1 \right)(t) - \left(\mathfrak{F} v_2 \right)(t) \right) \right\| \\
\leq \; & \left\| \mathbb{S}_{\mu,\nu}(t) \right\| \left\| t^{1-\gamma} \left(g\left(t_1, ..., t_1, v_1(\cdot)\right) - g\left(t_1, ..., t_1, v_2(\cdot)\right) \right) \right\| \\
& + t^{1-\gamma} \int_{t_0}^{t} \left\| \mathcal{K}_\mu (t-s) \right\| \left\| f\left(s, v_1(s)\right) - f\left(s, v_2(s)\right) \right\| ds \\
& + \frac{t^{1-\gamma}}{\Gamma(\mu)} \int_{t_0}^{t} \left\| \mathcal{K}_\mu (t-s) \right\| \int_{t_0}^{s} \mathbb{H}^\mu (s,\tau) \left\| \mathbf{K}\left(s,\tau,v_1(\tau)\right) - \mathbf{K}\left(s,\tau,v_2(\tau)\right) \right\| d\tau ds \\
\leq \; & \mathbf{M}\mathbf{Q}_0 \left\| v_1 - v_2 \right\|_{C_{1-\gamma}} + \mathbf{M}\mathbf{L} \left\| v_1 - v_2 \right\|_{C_{1-\gamma}} \int_{t_0}^{t} ds \\
& + \frac{\mathbf{M}\mathbf{K}_0}{\Gamma(\mu)} \left\| v_1 - v_2 \right\|_{C_{1-\gamma}} \int_{t_0}^{t} \int_{t_0}^{s} \mathbb{H}^\mu (s,\tau) \, d\tau ds \\
\leq \; & \left(\mathbf{M}\mathbf{Q}_0 + \mathbf{M}\mathbf{L}a + \frac{\mathbf{M}\mathbf{K}_0 a}{\Gamma(\mu)^2} \left(\psi(t_0+a) - \psi(t_0) \right)^\mu \right) \left\| v_1 - v_2 \right\|_{C_{1-\gamma}}.
\end{aligned} \tag{3.13}$$

If we take $q := \mathbf{M}\mathbf{Q}_0 + \mathbf{M}\mathbf{L}a + \dfrac{\mathbf{M}\mathbf{K}_0 a}{\Gamma(\mu)^2} \left(\psi(t_0+a) - \psi(t_0) \right)^\mu$, then

$$\left\| \mathfrak{F} v_1 - \mathfrak{F} v_2 \right\|_{C_{1-\gamma}} = \sup_{t \in I} \left\| t^{1-\gamma} \left(\left(\mathfrak{F} v_1 \right)(t) - \left(\mathfrak{F} v_2 \right)(t) \right) \right\| \leq q \left\| v_1 - v_2 \right\|_{C_{1-\gamma}} \tag{3.14}$$

with $0 < q < 1$.

So, we have $\mathfrak{F}$, is a contraction on the space Ω. In this sense, applying the Banach fixed point theorem, we concluded that, the operator $\mathfrak{F}$ admits unique fixed point in space Ω and this sense is the mild solution of Eq.(1.6) on I. □

Corollary 3.3. *Assume that the conditions* **CI** *are true and taking the limit $\nu \to 1$ on both sides of* Eq.(2.6)*, then the Caputo fractional integro-differential equation with nonlocal conditions given by* Eq.(1.6) *has a unique mild solution on I.*

Proof. Direct consequence of Theorem 3.2. □

Corollary 3.4. *Assume that the conditions* **CI** *are true and taking the limit $\nu \to 0$ on both sides of* Eq.(2.6) *then the Riemann-Liouville fractional integro-differential equation with nonlocal conditions given by* Eq.(1.6) *has a unique mild solution on I.*

Proof. Direct consequence of Theorem 3.2. □

Corollary 3.5. *Assume that the conditions* **CI** *are true and taking the limit $\mu \to 1$ on both sides of* Eq.(2.6) *then the integro-differential equation with nonlocal conditions given by* Eq.(1.6) *has a unique mild solution on I.*

Proof. Direct consequence of Theorem 3.2. □

Theorem 3.6. *The fractional integro-differential equation with nonlocal conditions given by* Eq.(1.6)*, has a strong solution on I.*

Proof. By satisfying all the conditions of Theorem 3.2, the problem Eq.(1.6) admits a unique mild solution in $C_\gamma(I, \Omega)$ which we denote by u.

Now, we shall show that this mild solution is a strong solution of problem Eq.(1.6) on I. For any $t \in I$, yields

$$\begin{aligned}
& u(t+h)-u(t) \\
= \;& \mathbb{S}_{\mu,\nu}(t+h)u_0-\mathbb{S}_{\mu,\nu}(t)u_0-\mathbb{S}_{\mu,\nu}(t+h)g(t_1,...,t_2,u(\cdot)) \\
& +\mathbb{S}_{\mu,\nu}(t)g(t_1,...,t_2,u(\cdot))+\int_{t_0}^{t_0+h}\mathcal{K}_\mu(t+h-s)f(s,u(s))ds \\
& +\int_{t_0+h}^{t+h}\mathcal{K}_\mu(t+h-s)f(s,u(s))ds-\int_{t_0}^{t}\mathcal{K}_\mu(t-s)f(s,u(s))ds \\
& +\frac{1}{\Gamma(\mu)}\int_{t_0}^{t_0+h}\mathcal{K}_\mu(t+h-s)\int_{t_0}^{s}\mathbb{H}^\mu(s,\tau)\mathbf{K}(s,\tau,u(\tau))d\tau ds \\
& +\frac{1}{\Gamma(\mu)}\int_{t_0+h}^{t+h}\mathcal{K}_\mu(t+h-s)\int_{t_0}^{s}\mathbb{H}^\mu(s,\tau)\mathbf{K}(s,\tau,u(\tau))d\tau ds \\
& -\frac{1}{\Gamma(\mu)}\int_{t_0}^{t}\mathcal{K}_\mu(t-s)\int_{t_0}^{s}\mathbb{H}^\mu(s,\tau)\mathbf{K}(s,\tau,u(s))ds \\
= \;& \mathbb{S}_{\mu,\nu}(t+h)u_0-\mathbb{S}_{\mu,\nu}(t)u_0-\mathbb{S}_{\mu,\nu}(t+h)g(t_1,...,t_2,u(\cdot)) \\
& +\mathbb{S}_{\mu,\nu}(t)g(t_1,...,t_2,u(\cdot))+\int_{t_0}^{t_0+h}\mathcal{K}_\mu(t+h-s)[f(s,u(s))-f(s,0)]ds \\
& +\int_{t_0}^{t_0+h}\mathcal{K}_\mu(t+h-s)f(s,0)ds+\int_{t_0}^{t}\mathcal{K}_\mu(t-s)[f(s+h,u(s+h))-f(s,u(s))]ds \\
& +\frac{1}{\Gamma(\mu)}\int_{t_0}^{t_0+h}\mathcal{K}_\mu(t+h-s)\int_{t_0}^{s}\mathbb{H}^\mu(s,\tau)[\mathbf{K}(s,\tau,u(\tau))-\mathbf{K}(s,\tau,0)]d\tau ds \\
& +\frac{1}{\Gamma(\mu)}\int_{t_0}^{t_0+h}\mathcal{K}_\mu(t+h-s)\int_{t_0}^{s}\mathbb{H}^\mu(s,\tau)\mathbf{K}(s,\tau,0)d\tau ds \\
& +\frac{1}{\Gamma(\mu)}\int_{t_0}^{s+h}\mathbb{H}^\mu(s+h,\tau)[\mathbf{K}(s+h,\tau,u(\tau))-\mathbf{K}(s+h,\tau,0)]d\tau ds \\
& +\frac{1}{\Gamma(\mu)}\int_{t_0}^{t}\mathcal{K}_\mu(t-s)\int_{t_0}^{s+h}\mathbb{H}^\mu(s+h,\tau)\mathbf{K}(s+h,\tau,0)d\tau ds \\
& +\frac{1}{\Gamma(\mu)}\int_{t_0}^{t}\mathcal{K}_\mu(t-s)\int_{t_0}^{s}\mathbb{H}^\mu(s,\tau)[\mathbf{K}(s+h,\tau,0)-\mathbf{K}(s,\tau,u(\tau))]d\tau ds \\
& +\frac{1}{\Gamma(\mu)}\int_{t_0}^{t}\mathcal{K}_\mu(t-s)\int_{t_0}^{s}\mathbb{H}^\mu(s,\tau)\mathbf{K}(s+h,\tau,0)d\tau ds. \qquad (3.15)
\end{aligned}$$

Using our assumptions **(CII)** we observe that

$$\begin{aligned}
& \left\|t^{1-\gamma}(u(t+h)-u(t))\right\| \\
\leq \;& \|\mathbb{S}_{\mu,\nu}(t+h)\|\left\|t^{1-\gamma}u_0\right\|+\|\mathbb{S}_{\mu,\nu}(t+h)\|\left\|t^{1-\gamma}g(t_1,...,t_2,u(\cdot))\right\| \\
& +\|\mathbb{S}_{\mu,\nu}(t)\|\left\|t^{1-\gamma}u_0\right\|+\|\mathbb{S}_{\mu,\nu}(t)\|\left\|t^{1-\gamma}g(t_1,...,t_2,u(\cdot))\right\| \\
& +t^{1-\gamma}\int_{t_0}^{t_0+h}\|\mathcal{K}_\mu(t+h-s)\|s^{\gamma-1}\left\|s^{1-\gamma}[f(s,u(s))-f(s,0)]\right\|ds
\end{aligned}$$

$$
\begin{aligned}
& +t^{1-\gamma}\int_{t_0}^{t_0+h}\left\|\mathcal{K}_\mu(t+h-s)\right\|s^{\gamma-1}\left\|s^{1-\gamma}f(s,0)\right\|ds \\
& +t^{1-\gamma}\int_{t_0}^{t}\left\|\mathcal{K}_\mu(t-s)\right\|t^{\gamma-1}\left\|s^{1-\gamma}\left[f(s+h,u(s+h))-f(s,u(s))\right]\right\|ds \\
& +\frac{t^{1-\gamma}}{\Gamma(\mu)}\int_{t_0}^{t_0+h}\left\|\mathcal{K}_\mu(t+h-s)\right\|\int_{t_0}^{s}\mathbb{H}^\mu(s,\tau)\tau^{\gamma-1}\times\left\|\tau^{1-\gamma}\left[\mathbf{K}(s,\tau,u(\tau))-\mathbf{K}(s,\tau,0)\right]\right\|d\tau ds \\
& +\frac{t^{1-\gamma}}{\Gamma(\mu)}\int_{t_0}^{t_0+h}\left\|\mathcal{K}_\mu(t+h-s)\right\|\int_{t_0}^{s}\mathbb{H}^\mu(s,\tau)\tau^{\gamma-1}\left\|\tau^{1-\gamma}\mathbf{K}(s,\tau,0)\right\|d\tau ds \\
& +\frac{t^{1-\gamma}}{\Gamma(\mu)}\int_{t_0}^{t}\left\|\mathcal{K}_\mu(t-s)\right\|\int_{t_0}^{s+h}\mathbb{H}^\mu(s+h,\tau)\tau^{\gamma-1}\left\|\tau^{1-\gamma}\mathbf{K}(s,\tau,0)\right\|d\tau ds \\
& +\frac{t^{1-\gamma}}{\Gamma(\mu)}\int_{t_0}^{t}\left\|\mathcal{K}_\mu(t-s)\right\|\int_{t_0}^{s+h}\mathbb{H}^\mu(s+h,\tau)\tau^{\gamma-1} \\
& \qquad\times\left\|\tau^{1-\gamma}\left[\mathbf{K}(s+h,\tau,u(\tau))-\mathbf{K}(s+h,\tau,0)\right]\right\|d\tau ds \\
& +\frac{t^{1-\gamma}}{\Gamma(\mu)}\int_{t_0}^{t}\left\|\mathcal{K}_\mu(t-s)\right\|\int_{t_0}^{s+h}\mathbb{H}^\mu(s,\tau)\tau^{\gamma-1}\left\|\tau^{1-\gamma}\mathbf{K}(s+h,\tau,0)\right\|d\tau ds \\
& +\frac{t^{1-\gamma}}{\Gamma(\mu)}\int_{t_0}^{t}\left\|\mathcal{K}_\mu(t-s)\right\|\int_{t_0}^{s}\mathbb{H}^\mu(s,\tau)\tau^{\gamma-1}\times\left\|\tau^{1-\gamma}\left[\mathbf{K}(s+h,\tau,0)-\mathbf{K}(s+h,\tau,u(\tau))\right]\right\|d\tau ds \\
& -\frac{t^{1-\gamma}}{\Gamma(\mu)}\int_{t_0}^{t}\left\|\mathcal{K}_\mu(t-s)\right\|\int_{t_0}^{s}\mathbb{H}^\mu(s,\tau)\tau^{\gamma-1}\left\|\tau^{1-\gamma}\mathbf{K}(s+h,\tau,0)\right\|d\tau ds \\
\leq\ & \mathbf{M}\left\|u_0\right\|_{C_{1-\gamma}}+\mathbf{M}\left\|u_0\right\|_{C_{1-\gamma}}+\mathbf{M}\widetilde{\mathbf{G}_1}+\mathbf{M}\widetilde{\mathbf{G}_1}+\mathbf{ML}rh+\mathbf{ML}h \\
& +\mathbf{ML}ha+\mathbf{ML}\int_{t_0}^{t}\mathbb{H}^\mu(t,s)\left\|s^{1-\gamma}(u(s+h)-u(s))\right\|ds \\
& +\frac{M}{\Gamma(\mu)}\mathbf{K}_0 r\int_{t_0}^{t_0+h}\int_{t_0}^{s}\mathbb{H}^\mu(s,\tau)d\tau ds+\frac{\mathbf{M}}{\Gamma(\mu)}\mathbf{K}_1\int_{t_0}^{t_0+h}\int_{t_0}^{s}d\tau ds \\
& +\frac{\mathbf{M}}{\Gamma(\mu)}\mathbf{K}_0 r\int_{t_0}^{t}\int_{t_0}^{s+h}\mathbb{H}^\mu(s+h,\tau)d\tau ds+\frac{\mathbf{M}}{\Gamma(\mu)}\mathbf{K}_1\int_{t_0}^{t}\int_{t_0}^{s+h}\mathbb{H}^\mu(s+h,\tau)d\tau ds \\
& +\frac{\mathbf{M}}{\Gamma(\mu)}\mathbf{K}_0 r\int_{t_0}^{t}\int_{t_0}^{s}\mathbb{H}^\mu(s,\tau)d\tau ds-\frac{\mathbf{M}}{\Gamma(\mu)}\mathbf{K}_1\int_{t_0}^{t}\int_{t_0}^{s}d\tau ds \\
\leq\ & 2\mathbf{M}\left\|u_0\right\|_{C_{1-\gamma}}+2\mathbf{M}\widetilde{\mathbf{G}_1}+\mathbf{ML}rh+\mathbf{MN}h+\mathbf{ML}ha \\
& +\mathbf{ML}\int_{t_0}^{t}\mathbb{H}^\mu(t,s)\left\|s^{1-\gamma}(u(s+h)-u(s))\right\|ds \\
& +\frac{\mathbf{MK}_0 r}{\Gamma(\mu+1)}(\psi(t_0+h)-\psi(t_0))^\mu+\frac{\mathbf{MK}_0 r}{\Gamma(\mu+1)}(\psi(t_0+h+a)-\psi(t_0))^\mu \\
& +\frac{\mathbf{MK}_0 r}{\Gamma(\mu+1)}(\psi(t_0+a)-\psi(t_0))^\mu+\frac{\mathbf{MK}_1}{\Gamma(\mu+1)}(\psi(t_0+h+a)-\psi(t_0))^\mu \\
& -\frac{\mathbf{MK}_1 a}{2\Gamma(\mu)}+\frac{\mathbf{MK}_1 a^2}{2\Gamma(\mu)} \\
\leq\ & \widetilde{\mathbf{P}}h+\mathbf{ML}\int_{t_0}^{t}\mathbb{H}^\mu(t,s)\left\|s^{1-\gamma}(u(s+h)-u(s))\right\|ds
\end{aligned}
\tag{3.16}
$$

where $\widetilde{\mathbf{P}}:=2\mathbf{M}\left\|u_0\right\|_{C_{1-\gamma}}+2\mathbf{M}\widetilde{\mathbf{G}_1}+\mathbf{ML}r+\mathbf{MN}+\mathbf{ML}a-\frac{\mathbf{MK}_1 a}{2\Gamma(\mu)}+\frac{\mathbf{MK}_1 a^2}{2\Gamma(\mu)}+$

$$\frac{\mathbf{MK}_0 r}{\Gamma(\mu+1)}\left(\psi(t_0+h)-\psi(t_0)\right)^\mu+\frac{\mathbf{MK}_0 r}{\Gamma(\mu+1)}\left(\psi(t_0+h+a)-\psi(t_0)\right)^\mu+$$
$$\frac{\mathbf{MK}_0 r}{\Gamma(\mu+1)}\left(\psi(t_0+a)-\psi(t_0)\right)^\mu+\frac{\mathbf{MK}_1}{\Gamma(\mu+1)}\left(\psi(t_0+h+a)-\psi(t_0)\right)^\mu.$$

By means of the Gronwall inequality, follows that

$$\left\|t^{1-\gamma}\left(u(t+h)-u(t)\right)\right\| \leq \widetilde{\mathbf{P}} h \mathbb{E}_\mu\left[\mathbf{ML}\Gamma(\mu)\left(\psi(t)-\psi(a)\right)^\mu\right],\ t\in I, \tag{3.17}$$

where $\mathbb{E}_{\mu(\cdot)}$ is one-parameter Mittag-Leffler function.

Therefore, u is Lipschitz continuous on I. The Lipschitz continuity of u on I combined with **(CII)** (3) give that $t\to f(t,u(t))$ is Lipschitz continuous on I. Also by assumption **(CII)** (5), we have $t\to \frac{1}{\Gamma(\mu)}\int_{t_0}^t \mathbb{H}^\mu(t,s)\mathbf{K}(t,s,u(s))\,ds$ which is Lipschitz continuous on I.

Using Theorem 3.2, we observe that the equation

$$\begin{aligned} {}^H\mathbb{D}^{\mu,\nu}_{t_0+}v(t)+\mathcal{A}(t) &= f(t,u(t))+\frac{1}{\Gamma(\mu)}\int_{t_0}^t \mathbb{H}^\mu(t,s)\mathbf{K}(t,s,u(s))\,ds \\ I^{1-\gamma}_{t_0}v(t_0) &= u_0-g(t_1,...,t_p,u(\cdot)) \end{aligned} \tag{3.18}$$

has a unique strong solution v on I satisfying the equation

$$\begin{aligned} v(t) &= \mathbb{S}_{\mu,\nu}(t)\left[u_0-g(t_1,...,t_p,u(\cdot))\right]+\int_{t_0}^t \mathbf{K}_\mu(t-s)f(s,u(s))\,ds \\ &\quad+\int_{t_0}^t \mathbf{K}_\mu(t-s)\int_{t_0}^s \mathbb{H}^\mu(s,\tau)\mathbf{K}(s,\tau,u(\tau))\,d\tau ds \\ &= u(t),\ t\in I. \end{aligned} \tag{3.19}$$

Consequently, u is a strong solution of Eq.(1.6) on I. □

Corollary 3.7. *Assume that the* **CII** *are true and taking the limit $\nu\to 0$ on both sides of* Eq.(2.6) *then the Riemann-Liouville fractional integro-differential equation with nonlocal conditions given by* Eq.(1.6) *has a unique strong solution in I.*

Proof. Direct consequence of Theorem 3.6. □

Corollary 3.8. *Assume that the* **CII** *are true and taking the limit $\nu\to 1$ on both sides of* Eq.(2.6) *then the Caputo fractional integro-differential equation with nonlocal conditions given by* Eq.(1.6) *has a unique strong solution in I.*

Proof. Direct consequence of Theorem 3.6. □

Corollary 3.9. *Assume that the* **CII** *are true and taking the limit $\mu\to 1$ on both sides of* Eq.(2.6) *then the integro-differential equation with nonlocal conditions given by* Eq.(1.6) *has a unique strong solution in I.*

Proof. Direct consequence of Theorem 3.6. □

4 Concluding remarks and future works

We conclude the paper with the main purpose achieved, i.e., we investigate the existence and uniqueness of a new class of mild and strong solutions of Hilfer fractional integro-differential equations in Banach space through the Banach fixed point and the Gronwall inequality in the context of continuously C_0-semigroup.

However, some questions remain open and are motivations for a future work among them:

(i) A natural continuation of this work is to investigate the Ulam-Hyers stability and continuous dependence of mild solutions for Eq.(1.6).

(ii) Investigate the existence, uniqueness and Ulam-Hyers stability of mild and strong solutions of integro-differential and fractional differential equations involving the ψ-Hilfer fractional derivative. Here, we also consider A to be an infinitesimal generator. To obtain such a result, two points are needed. First point would be to obtain a Laplace transform with respect to another function. This point has already been obtained and discussed in the interesting work [20]. However, it is still necessary to obtain a mild solution to a problem of Eq.(1.6) type (closed formula). This question is still an open problem when it involves an infinitesimal generator.

(iii) Another open point is to discuss problems of fractional pseudo-differential equations with the Hilfer fractional derivative, especially involving the ψ-Hilfer fractional derivative.

We believe that the objectives discussed here certainly contribute to the construction and enrichment of the theory of fractional differential equations. In addition, we highlight some next and possible steps that we intend to continue in the research involving differential equations and fractional operators.

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