

CERTAIN NEW RESULTS FOR SOME HORNS HYPERGEOMETRIC FUNCTIONS IN TWO VARIABLES

S. Jain, J. Younis, P. Agarwal and Mohamed A. Abd El Salam

Communicated by Martin Bohner

MSC 2010 Classifications: Primary 33C20; Secondary 33C65.

Keywords and phrases: Beta function, Horn double functions, transformation formulas, Eulerian integrals.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Praveen Agarwal was very thankful to the NBHM (project 02011/12/ 2020NBHM(R.P)/R&D II/7867) for their necessary support and facility.

Abstract In this paper, we introduce new transformations for Horn’s double hypergeometric functions G_1, G_2 and G_3 . Also deduce new integral representations of Euler-type involving these functions in terms of hypergeometric functions of two variables.

1 Introduction

The theory of hypergeometric functions has evolved in many directions and has been useful in the study of a variety of useful properties in diverse areas of mathematics, physics, and engineering. Numerous works have been published on this topic in the literature (for example [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]). The enormous success of the theory of hypergeometric functions in a single variable motivated the development of the theory of hypergeometric functions in multiple variables, as the solutions of partial differential equations arising in various problems of mathematical physics are expressed in terms of such hypergeometric functions [11, 12, 13]. The study of multivariable hypergeometric functions has grown in popularity as a result of advances in number theory, Lie algebras, group theory, representation theory, algebraic geometry, and combinatorics, etc. have led to increasing interest in the study of multivariable hypergeometric functions [14, 15, 16].

First of all, we recall Horn’s hypergeometric functions of two variables $G_1, G_2, G_3, H_1, H_2, H_3, H_4, H_5, H_6$ and H_7 defined by (see[17, 18]):

$$G_1(a, b, c; x, y) = \sum_{m,n=0}^{\infty} (a)_{m+n} (b)_{n-m} (c)_{m-n} \frac{x^m y^n}{m! n!}, \tag{1.1}$$

with $|x| + |y| < 1$;

$$G_2(a, b, c, d; x, y) = \sum_{m,n=0}^{\infty} (a)_m (b)_n (c)_{n-m} (d)_{m-n} \frac{x^m y^n}{m! n!}, \tag{1.2}$$

with $|x| < 1$ and $|y| < 1$;

$$G_3(a, b; x, y) = \sum_{m,n=0}^{\infty} (a)_{2n-m} (b)_{2m-n} \frac{x^m y^n}{m! n!}, \tag{1.3}$$

with $27|x|^2|y|^2 + 18|x||y| \pm 4(|x| - |y|) < 1$.

$$H_1(a, b, c; d; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m-n} (b)_{m+n} (c)_n}{(d)_m} \frac{x^m y^n}{m! n!}, \tag{1.4}$$

with $4|x||y| + 2|y| - |y|^2 < 1$.

$$H_2(a, b, c, d; e; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m-n}(b)_m(c)_n(d)_n}{(e)_m} \frac{x^m y^n}{m! n!}, \quad (1.5)$$

with $\frac{1}{|y|-|x|} < 1$.

$$H_3(a, b, c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}(b)_n}{(c)_{m+n}} \frac{x^m y^n}{m! n!}, \quad (1.6)$$

with $|x| + |y|^2 - |y| < 0$.

$$H_4(a, b, c, d; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}(b)_n}{(c)_m(c)_n} \frac{x^m y^n}{m! n!}, \quad (1.7)$$

with $4|x| + 2|y|^2 - |y| < 1$.

$$H_5(a, b, c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}(b)_{n-m}}{(c)_n} \frac{x^m y^n}{m! n!}, \quad (1.8)$$

with $16|x|^2 - 36|x||y| \pm (8|x| - |y| + 27|x||y|^2) < 1$.

$$H_6(a, b, c; x, y) = \sum_{m,n=0}^{\infty} (a)_{2m-n}(b)_{n-m}(c)_n \frac{x^m y^n}{m! n!}, \quad (1.9)$$

with $|x||y|^2 + |y| < 1$.

$$H_7(a, b, c, d; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{2m-n}(b)_n(c)_n}{(d)_m} \frac{x^m y^n}{m! n!}, \quad (1.10)$$

with $4|x| + \frac{2}{|s|} - \frac{1}{|s|^2} < 1$.

Here $(\alpha)_\beta$ denotes the Pochhammer symbol defined (for $\alpha, \beta \in \mathbb{C}$) by

$$(\alpha)_\beta := \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)} = \begin{cases} 1 & (\beta = 0; \alpha \neq 0) \\ \alpha(\alpha + 1) \cdots (\alpha + n - 1) & (\beta = n \in \mathbb{N}), \end{cases} \quad (1.11)$$

Γ being the familiar Gamma function and it being read traditionally that $(0)_0 := 1$.

Appell [19] developed four hypergeometric functions of two variables and designated them by F_1, F_2, F_3 and F_4 . The following is a definition of one of these functions:

$$F_3(a, b, c, d; e; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_m(b)_n(c)_m(d)_n}{(e)_{m+n}} \frac{x^m y^n}{m! n!}. \quad (1.12)$$

for $|x| < 1$, $|y| < 1$, and $e \neq 0, -1, -2, \dots$.

2 Transformation formulas

In this section, we find some new transformations of Horn functions G_1, G_2 and G_3 in series of some generalized hypergeometric functions. We recall the generalized hypergeometric series ${}_pF_q$ ($p, q \in \mathbb{N}_0$) given by (see cite14):

$$\begin{aligned} {}_pF_q \left[\begin{matrix} \lambda_1, \dots, \lambda_p; \\ \mu_1, \dots, \mu_q; \end{matrix} z \right] &= \sum_{n=0}^{\infty} \frac{(\lambda_1)_n \cdots (\lambda_p)_n}{(\mu_1)_n \cdots (\mu_q)_n} \frac{z^n}{n!} \\ &= {}_pF_q(\lambda_1, \dots, \lambda_p; \mu_1, \dots, \mu_q; z). \end{aligned} \quad (2.1)$$

Theorem 2.1. *If $|x| \neq 0$ and $\left|\frac{x}{y}\right| < 1$, then the following transformations holds*

$$G_1(a, b, c; x, y) = \sum_{m=0}^{\infty} \frac{(a)_m(c)_m}{(b-1)_m m!} x^m \times {}_5F_4 \left[-m, \frac{b}{2}, \frac{b+1}{2}, \frac{b-1}{2}, \frac{b}{2}; \frac{b-1+m}{2}, \frac{b+m}{2}, \frac{1-c-m}{2}, \frac{2-c-m}{2}; \frac{y}{x} \right]. \quad (2.2)$$

$$G_2(a, b, c, d; x, y) = \sum_{m=0}^{\infty} \frac{(a)_m(d)_m}{(c-1)_m m!} x^m \times {}_5F_5 \left[-m, \frac{c}{2}, \frac{c+1}{2}, \frac{c-1}{2}, \frac{c}{2}; 1-a-m, \frac{c-1+m}{2}, \frac{c+m}{2}, \frac{1-d-m}{2}, \frac{2-d-m}{2}; \frac{y}{x} \right]. \quad (2.3)$$

$$G_3(a, b; x, y) = \sum_{m=0}^{\infty} \frac{(b)_m}{(a-1)_m m!} x^m \times {}_7F_6 \left[-m, \frac{a}{3}, \frac{a+1}{3}, \frac{a+3}{3}, \frac{a-1}{3}, \frac{a}{3}, \frac{a+1}{3}; \frac{a-1+m}{a}, \frac{a+m}{3}, \frac{a+m+1}{3}, \frac{1-b-2m}{3}, \frac{2-b-2m}{3}, \frac{3-b-2m}{3}; \frac{y}{x} \right]. \quad (2.4)$$

Proof. To prove the result in equality (2.2), let Ω denote the left side of the equality (2.2). And with the help of following identity

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A_{k,n} = \sum_{n=0}^{\infty} \sum_{k=0}^n A_{k,n-k} \quad (2.5)$$

we get

$$\Omega = \sum_{m=0}^{\infty} \sum_{n=0}^m (a)_m (b)_{2n-m} (c)_{m-2n} \frac{x^{m-n}}{(m-n)!} \frac{y^n}{n!}. \quad (2.6)$$

By using the following identities (see [19])

$$(m-n)! = \frac{(-1)^n m!}{(-m)_n}, \quad (2.7)$$

$$(b)_{m-n} = \frac{(-1)^n (b)_m}{(1-b-m)_n}, \quad \text{and} \quad (b)_{2n-m} = \frac{(b)_{2n} (b-1)_{2n}}{(b-1)_m (b-1+m)_{2n}}. \quad (2.8)$$

and

$$(c)_{m-2n} = \frac{(-1)^{2n} (c)_m}{(1-c-m)_{2n}}, \quad (2.9)$$

Ω becomes

$$\Omega = \sum_{m=0}^{\infty} \frac{(a)_m (c)_m}{(b-1)_m m!} x^m \sum_{n=0}^m \frac{(-m)_n (b)_{2n} (b-1)_{2n}}{(b-1-m)_{2n} (1-c-m)_{2n}}. \quad (2.10)$$

From above equation (2.10) second summation over n is

$${}_5F_4 \left[-m, \frac{b}{2}, \frac{b+1}{2}, \frac{b-1}{2}, \frac{b}{2}; \frac{b-1+m}{2}, \frac{b+m}{2}, \frac{1-c-m}{2}, \frac{2-c-m}{2}; \frac{y}{x} \right].$$

in view of which, complete the proof of (2.2). Similarly, we can prove (2.3) and (2.4) by using identities (2.5), (2.7), (2.8) and (2.9). \square

3 Integral representations

In this section, we give certain integral representations for Horn’s double hypergeometric functions G_1, G_2 and G_3 .

Theorem 3.1. *The integral representations are satisfied for G_1 :*

$$G_1(a, b, c; x, y) = \frac{\Gamma(a + a')(V - T)^a(W - T)^{a'}}{\Gamma(a)\Gamma(a')(V - W)^{a+a'-1}} \int_W^V \frac{(\alpha - W)^{a-1}(V - \alpha)^{a'-1}}{(\alpha - T)^{a+a'}} \\ \times G_2\left(a + a', 1 - a', b, c; \frac{(V - T)(\alpha - W)x}{(V - W)(\alpha - T)}, \frac{-y}{(W - T)(V - \alpha)}\right) d\alpha, \\ (\Re(a) > 0, \Re(a') > 0, T < W < V), \quad (3.1)$$

$$G_1(a, b, c; x, y) = \frac{4M_1^{a'}M_2^b\Gamma(a + a')\Gamma(b + b')}{\Gamma(a)\Gamma(a')\Gamma(b)\Gamma(b')} \\ \times \int_0^\infty \int_0^\infty \frac{\cosh \alpha (\sinh^2 \alpha)^{a'-\frac{1}{2}} \cosh \beta (\sinh^2 \beta)^{b-\frac{1}{2}}}{(1 + M_1 \sinh^2 \alpha)^{a+a'} (1 + M_2 \sinh^2 \beta)^{b+b'}} \\ \times H_2\left(c, 1 - a', a + a', b + b'; b'; -\frac{x \operatorname{csch}^2 \alpha \operatorname{csch}^2 \beta}{M_1 M_2}, \frac{M_2 y}{(1 + M_1 \sinh^2 \alpha)(1 + M_2 \sinh^2 \beta)}\right) d\alpha d\beta, \\ (\Re(a) > 0, \Re(a') > 0, \Re(b) > 0, \Re(b') > 0, M_1 > 0, M_2 > 0), \quad (3.2)$$

$$G_1(a, b, c; x, y) = \frac{\Gamma(a + b)\Gamma(c + c')}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(c')} \int_0^1 \int_0^1 \alpha^{c-1} (1 - \alpha)^{c'-1} \beta^{a-1} (1 - \beta)^{b-1} \\ \times H_7\left(1 - c', \frac{a + b}{2}, \frac{a + b + 1}{2}; 1 - c - c'; \frac{\alpha\beta x}{(1 - \alpha)^2(\beta - 1)}, \frac{4(\alpha - 1)\beta(1 - \beta)y}{\alpha}\right) d\alpha d\beta, \\ (\Re(a) > 0, \Re(b) > 0, \Re(c) > 0, \Re(c') > 0), \quad (3.3)$$

$$G_1(a, b, c; x, y) = \frac{4\Gamma(a + b + c)}{\Gamma(a)\Gamma(b)\Gamma(c)} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (\sin^2 \alpha)^{a+b-\frac{1}{2}} (\cos^2 \alpha)^{c-\frac{1}{2}} (\sin^2 \beta)^{b-\frac{1}{2}} (\cos^2 \beta)^{a-\frac{1}{2}} \\ \times \left[1 - x \cos^2 \alpha \cot^2 \beta - \frac{1}{4}y \sin^2 \alpha \tan^2 \alpha \sin^2 2\beta\right]^{-(a+b+c)} d\alpha, \\ (\Re(a) > 0, \Re(b) > 0, \Re(c) > 0). \quad (3.4)$$

Proof. To prove the result in equality (3.1), let \mathcal{U} denote the right side of the equality (3.1). Then from the definition (2), we get

$$\mathcal{U} = \frac{\Gamma(a + a')}{\Gamma(a)\Gamma(a')} \sum_{m,n=0}^\infty \frac{(a + a')_m (b)_{n-m} (c)_{m-n}}{(a)_{-n}} \\ \times \frac{(V - T)^{a+m} (W - T)^{a'-n}}{(V - W)^{a+a'+m-1}} \int_W^V \frac{(\alpha - W)^{a+m-1} (V - \alpha)^{a'-n-1}}{(\alpha - T)^{a+a'+m}} \times \frac{x^m y^n}{m! n!} d\alpha. \quad (3.5)$$

Applying the following integral representation (see, e.g., [17, p. 10,(14)])

$$B(a, b) = \frac{(V - T)^a (W - T)^b}{(V - W)^{a+b-1}} \int_W^V \frac{(\alpha - W)^{a-1} (V - \alpha)^{b-1}}{(\alpha - T)^{a+b}} d\alpha,$$

$$(T < W < V, \Re(a) > 0, \Re(b) > 0).$$

in (27), we have

$$\mathfrak{U} = \frac{\Gamma(a+a')}{\Gamma(a)\Gamma(a')} \sum_{m,n=0}^{\infty} \frac{(a+a')_m (b)_{n-m} (c)_{m-n} B(a+m+n, a'-n)}{(a)_{-n}}. \tag{3.6}$$

Now applying well known beta function (see, e.g., [20, 21])

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

in (28), we get the required result. On the same way, we find the results (24)-(26). □

As in the proof of Theorem 3.1, one can easily prove the following Theorems. So, details are omitted.

Theorem 3.2. *The integral representations are satisfied for G_2 :*

$$\begin{aligned} G_2(a, b, c, d; x, y) &= \frac{\Gamma(b+d)\Gamma(c+c')}{\Gamma(b)\Gamma(c)\Gamma(c')\Gamma(d)} \int_0^\infty \int_0^\infty \frac{\alpha^{c-1}}{(1+\alpha)^{c+c'}} \frac{\beta^{d-1}}{(1+\beta)^{b+d}} \\ &\times F_3\left(a, \frac{c+c'}{2}, b+d, \frac{c+c'+1}{2}; c'; \frac{\beta x}{\alpha(1+\beta)}, \frac{4\alpha y}{(1+\alpha)^2 \beta}\right) d\alpha d\beta, \\ &(\Re(b) > 0, \Re(c) > 0, \Re(c') > 0, \Re(d) > 0), \end{aligned} \tag{3.7}$$

$$\begin{aligned} G_2(a, b, c, d; x, y) &= \frac{(1+M_1)^a (1+M_2)^b \Gamma(a+d)\Gamma(b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)} \\ &\times \int_0^1 \int_0^1 \frac{\alpha^{a-1} (1-\alpha)^{d-1}}{(1+M_1\alpha)^{a+d}} \frac{\beta^{b-1} (1-\beta)^{c-1}}{(1+M_2\beta)^{b+c}} \\ &\times G_3\left(b+c, a+d; \frac{(1+M_1)\alpha(1-\alpha)(1+M_2\beta)x}{(1+M_1\alpha)^2(1-\alpha)}, \right. \\ &\quad \left. \frac{(1+M_2)\beta(1-\beta)(1+M_1\alpha)y}{(1+M_2\beta)^2(1-\alpha)}\right) d\alpha d\beta, \\ &(\Re(a) > 0, \Re(b) > 0, \Re(c) > 0, \Re(d) > 0, M_1 > -1, M_2 > -1), \end{aligned} \tag{3.8}$$

$$\begin{aligned} G_2(a, b, c, d; x, y) &= \frac{\Gamma(a+a')\Gamma(c+d)}{\Gamma(a)\Gamma(a')\Gamma(c)\Gamma(d)} \\ &\times \int_0^\infty \int_0^\infty (e^{-\alpha})^a (1-e^{-\alpha})^{a'-1} (e^{-\beta})^c (1-e^{-\beta})^{d-1} \\ &\times H_6\left(1-a', a+a', b; \frac{e^{-(\alpha-\beta)}(1-e^{-\beta})x}{(1-e^{-\alpha})^2}, \frac{(e^{-\alpha}-1)e^{-\beta}y}{(1-e^{-\beta})}\right) d\alpha d\beta, \\ &(\Re(a) > 0, \Re(a') > 0, \Re(c) > 0, \Re(d) > 0), \end{aligned} \tag{3.9}$$

Theorem 3.3. *The integral representations are satisfied for G_3 :*

$$\begin{aligned}
G_3(a, b; x, y) &= \frac{\Gamma(a+a')\Gamma(b+b')}{\Gamma(a)\Gamma(a')\Gamma(b)\Gamma(b')(V_1-W_1)^{a+a'-1}(V_2-W_2)^{b+b'-1}} \int_{W_1}^{V_1} \int_{W_2}^{V_2} \\
&\quad \times (\alpha - W_1)^{a-1} (V_1 - \alpha)^{a'-1} (\beta - W_2)^{b-1} (V_2 - \beta)^{b'-1} \\
&\times G_2 \left(b + b', a + a', 1 - a', 1 - b'; \frac{(V_1 - \alpha)(\beta - W_2)^2 x}{(S_2 - W_2)(\alpha - W_1)(S_2 - \beta)}, \frac{(\alpha - W_1)^2 (V_2 - \beta) y}{(V_1 - W_1)(V_1 - \alpha)(\beta - W_2)} \right) \\
&\times d\alpha d\beta, \\
&(\Re(a) > 0, \Re(a') > 0, \Re(b) > 0, \Re(b') > 0, W_1 < V_1, W_2 < V_2), \quad (3.10)
\end{aligned}$$

$$\begin{aligned}
G_3(a, b; x, y) &= \frac{\Gamma(a+a')}{2^{a+a'-2}\Gamma(a)\Gamma(a')} \int_{-1}^1 \frac{[(1+\alpha)^2]^{a'-\frac{1}{2}} [(1-\alpha)^2]^{a-\frac{1}{2}}}{(1+\alpha^2)^{a+a'}} \\
&\quad \times G_3 \left(1 - a', b; - \left(\frac{1+\alpha}{1-\alpha} \right)^2 x, \left(\frac{1-\alpha}{1+\alpha} \right)^4 y \right) d\alpha, \\
&(\Re(a) > 0, \Re(a') > 0). \quad (3.11)
\end{aligned}$$

$$\begin{aligned}
G_3(a, b; x, y) &= \frac{4(1+M_1)^a(1+M_2)^b\Gamma(a+a')\Gamma(b+b')}{\Gamma(a)\Gamma(a')\Gamma(b)\Gamma(b')} \\
&\quad \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{(\sin^2\alpha)^{a-\frac{1}{2}} (\cos^2\alpha)^{a'-\frac{1}{2}} (\sin^2\beta)^{b-\frac{1}{2}} (\cos^2\beta)^{b'-\frac{1}{2}}}{(1+M_1\sin^2\alpha)^{a+a'} (1+M_2\sin^2\beta)^{b+b'}} \\
&\quad H_2 \left(a + a', 1 - a', \frac{b+b'}{2}, \frac{b+b'+1}{2}; b; - \frac{(1+M_1)^2 y \sin^2\alpha \tan^2\alpha \cot^2\beta}{(1+M_2)(1+M_1\sin^2\alpha)}, \right. \\
&\quad \left. \frac{(1+M_2)^2 x \csc^2\alpha \sin^4\beta (1+M_1\sin^2\alpha)}{(1+M_1)(1+M_2\sin^2\beta)^2} \right) d\alpha d\beta, \\
&(\Re(a) > 0, \Re(a') > 0, \Re(b) > 0, \Re(b') > 0, M_1 > -1, M_2 > -1). \quad (3.12)
\end{aligned}$$

$$\begin{aligned}
G_3(a, b; x, y) &= \frac{\Gamma(a+a')}{\Gamma(a)\Gamma(a')} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{2} + \alpha \right)^{a'-1} \left(\frac{1}{2} - \alpha \right)^{a-1} \\
&\quad \times H_6 \left(b, 1 - a', a + a'; - \left(\frac{1+2\alpha}{1-2\alpha} \right) x, - \frac{(1-2\alpha)^2 y}{2(1+2\alpha)} \right) d\alpha, \\
&(\Re(a) > 0, \Re(a') > 0), \quad (3.13)
\end{aligned}$$

4 Conclusion remarks

This paper aims is to obtain new transformations for Horn's double hypergeometric functions G_1 , G_2 and G_3 . Also, some new integral representations of Euler-type involving these functions in terms of hypergeometric functions of two variables have been discussed. Therefore, the results of this work are variant, significant and so it is interesting and capable to develop its study in the future.

References

- [1] P. Agarwal, J. Choi and J. Shilpi, *Extended hypergeometric functions of two and three variables*, Commun. Korean Math. Soc., **30** , 403–414, (2015).
- [2] P. Agarwal, M. Chand and J. Choi, *Some Integrals Involving ${}_2F_1$ -Functions and Laguerre Polynomials*, Ukr Math J., **71**, 1321—1340,(2020).
- [3] J. Choi, *Certain applications of generalized Kummer's summation formulas for ${}_2F_1$* . Symmetry, **13(8)**, 1538, (2021).
- [4] D. Kumar and J. Choi, *Certain generalized fractional differentiation of the product of two \mathfrak{N} -Functions Associated with the Appell Function F_3* , Applied Mathematical Sciences, **10** , 187–196, (2016).
- [5] D.L. Suthar, P. Agarwal and Hafte Amsalu, *Marichev-Saigo-Maeda fractional integral operators involving the product of generalized Bessel-Maitland functions*, Bol. Soc. Paran. Mat., **39** , 95–105,(2021).
- [6] J. A. Younis, *New Integrals for Horn Hypergeometric Functions in Two Variables*, Global Journal of Science Frontier Research: F Mathematics and Decision Sciences, **20** , 1–10,(2020).
- [7] J. Choi, M. I. Qureshi, A. H. Bhat, and Majid, J. *Reduction formulas for generalized hypergeometric series associated with new sequences and applications*. Fractal and Fractional, **5(4)**, 150, (2021).
- [8] H.M. Srivastava, P. Agarwal and J. Shilpi, *Generating functions for the generalized Gauss hypergeometric functions*, Appl. Math. Comput. , **247**, 348–352, (2014).
- [9] M.J. Luo, G.V. Milovanovic and P. Agarwal, *Some results on the extended beta and extended hypergeometric functions*, Appl. Math. Comput., **248**, 631–651 , (2014).
- [10] A. Etinkaya, İ.Ö Kıymaz, P. Agarwal and R. Agarwal, *A comparative study on generating function relations for generalized hypergeometric functions via generalized fractional operators*, Adv Differ Equ., **2018(1)**, 1–11 , (2018).
- [11] G. Lohöfer, *Theory of an electromagnetically deviated metal sphere. I: Absorbed power*, SIAM J. Appl. Math., **49** , 567–581,(1989).
- [12] A.M. Mathai and R.K. Saxena, *Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences*, Springer-Verlag, Berlin, Heidelberg and New York. (1973).
- [13] H.M. Srivastava and B.R.K. Kashyap, *Special Functions in Queuing Theory and Related Stochastic Processes*, Academic Press, New York, London and San Francisco, (1982).
- [14] Yu. A. Brychkov, *Handbook of Special Functions, Derivatives, Integrals, Series, and Other Formulas*, CRC Press, Boca Raton, etc., (2008).
- [15] B. Davies, *Integral Transforms And Their Applications*, 2nd Edition. New York, NY: Springer-Verlag, (1984).
- [16] A.P. Prudnikov, Yu.A. Brychkov and O.I. Marichev, *Integrals and Series*, Vol. III, Gordon and Breach Science Publishers, New York (1990).
- [17] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions*, Vol. I, McGraw-Hill Book Company, New York, Toronto and London, (1953).
- [18] J. Horn, *Ueber die Convergenz der Hypergeometrische Reihen Zweier und Dreier Veränderlichen*, Math. Ann.,**34**, 544–600, (1889).
- [19] H.M. Srivastava and P.W. Karlsson, *Multiple Gaussian Hypergeometric Series* , Ellis Horwood Lt1., Chichester, (1984).
- [20] E.D. Rainville, *Special Functions*, Macmillan Company, New York, (1960), Reprinted by Chelsea Publishing Company, Bronx, New York, (1971).
- [21] H.M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York, (2012).

Author information

S. Jain, Department of Mathematics, Poornima College of Engineering, India.
E-mail: shilpijain1310@gmail.com

J. Younis, Department of Mathematics, Aden University, Yemen.
E-mail: jihadalsaqqaf@gmail.com

P. Agarwal, Applied Nonlinear Science Lab, Department of Mathematics, Anand International College of Engineering
Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman, AE 346, United Arab Emirates, India.
E-mail: goyal.praveen2011@gmail.com

Mohamed A. Abd El Salam, Mathematics Department, Faculty of Science, Al-Azhar University, Nasr-City 11884, Cairo & October High Institute for Engineering and Technology, Giza, Egypt.
E-mail: mohamed_salam@azhar.edu.eg

Received: September 30, 2021.

Accepted: December 7, 2021.