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Some results on almost η-Ricci solitons

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Abstract. In this paper, we study almost η-Ricci soliton (g, V, λ, μ) and gradient almost η-Ricci soliton $(g, \text{grad}\,(\text{f}), \lambda, \mu)$ on β-paraKenmotsu 3-manifold. The conditions for this class of manifold to be η-Einstein are obtained. Some examples are also constructed.

1 Introduction

Paracontact geometry is an interesting area of research due to its applications in many branches of Physics. Kaneyuki and Williams in [1] initiated the study of almost paracontact manifolds, later on S. Zamkovoy in [2] systematically studied paracontact metric manifolds and their subclasses (paraSasakian manifolds, paraKenmotsu manifolds, paraCosymplectic manifolds etc.). Since then, many authors are making an effort to find out the complete geometry of these type of manifolds [3–6].

R. S. Hamilton in [7] introduced Ricci soliton as a natural generalization of an Einstein metric. Because of the important applications in physics, interest of researchers increases towards the geometry of almost Ricci solitons and gradient almost Ricci solitons [8–18, 20]. Ricci solitons in paracontact geometry were firstly studied by D. Perrone and G. Calvaruso in [21]. M. Kimura and J. T. Cho in [22] studied the more general notion known as η-Ricci soliton (briefly, η-RS) and gradient η-Ricci soliton (briefly, gradient η-RS).

Recently, U. C. De and K. Mandal in [23] studied almost Ricci solitons and gradient almost Ricci solitons in (k, μ)-paracontact geometry. D. M. Naik and V. Venkatesha [24], A. M. Blaga [25,26] also studied an almost η-RS.

Sectional study of this paper is given as: In Sect.2, we give basic definition of almost paracontact metric manifold $\mathbb{M}^3$ and its some subclasses: paraCosymplectic manifold $\mathbb{C}^3$, paraSasakian manifold $\mathbb{Q}^3$ and paraKenmotsu manifold $\mathbb{K}^3$. We also give the definition of a η-Einstein manifold. In Sect.3, we define and study almost η-RS in $\mathbb{M}^3$. In Sect.4-5, we prove the following main results:

Theorem 1.1. *Let a β-paraKenmotsu 3-manifold $\mathbb{K}^3$ admitting an almost η-RS having ξ as a potential vector field, then $\mathbb{K}^3$ is a η-Einstein manifold. Also if $\mu = \beta$ then $\mathbb{K}^3$ is an Einstein manifold.*

Theorem 1.2. *Let a β-paraKenmotsu 3-manifold $\mathbb{K}^3$ admitting an almost η-RS and having potential vector field orthogonal to ξ, then the scalar curvature of $\mathbb{K}^3$ depends on the value of μ and β.*

Theorem 1.3. *Let a β-para Kenmotsu 3-manifold $\mathbb{K}^3$ admitting a gradient almost η-RS corresponding to the potential function Df, then it is a η-Einstein.*

Finally in Sect.6 we construct some examples of almost η-RS (g, ξ, λ, μ) and gradient almost η-RS $(g, \text{grad}\,(\text{f}), \lambda, \mu)$ on $\mathbb{K}^3$.

2 Priliminaries

Definition 2.1 (Almost paracontact manifold). Let $\mathbb{M}^{2m+1}$ be a differentiable manifold, then $\mathbb{M}^{2m+1}$ is said to be an *almost paracontact manifold* if it is equipped with (ϕ, ξ, η)-structure and satisfies:

$$\left.\begin{aligned} \eta(\xi) &= 1, \quad \phi^2 = \mathcal{I} - \eta \otimes \xi, \\ \phi\xi &= 0, \quad \eta \circ \phi = 0. \end{aligned}\right\} \tag{2.1}$$

Also, an endomorphism ϕ induces an almost paracomplex structure on each fiber of $\mathcal{D} = ker\eta$ (horizontal distribution) *i.e.* the eigendistribution corresponding to eigenvalues $+1$ and -1, the eigensubbundles $\mathcal{D}^+$ and $\mathcal{D}^-$ have equal dimension m.

Here, $\mathcal{I}$ is the identity transformation, ϕ is a $(1,1)$-*tensor field*, ξ is a *characteristic vector field* and η is a *paracontact form* on $\mathbb{M}^{2m+1}$.
Let g be a pseudo-Riemannian metric such that

$$g(\phi P_1, \phi P_2) = -g(P_1, P_2) + \eta(P_1)\eta(P_2), \quad P_1, P_2 \in \Gamma\left(\mathcal{T}\mathbb{M}^{2m+1}\right) \tag{2.2}$$

then g is compatible with (ϕ, ξ, η)-structure. Here signature of g is $(m+1, m)$ and $\eta(P_1) = g(P_1, \xi)$, for any vector field $P_1 \in \Gamma\left(\mathcal{T}\mathbb{M}^{2m+1}\right)$.

Definition 2.2. The manifold $\mathbb{M}^{2m+1}$ is called $(2m+1)$-dimensional *almost paracontact metric (briefly, a.p.c.m.) manifold* if it is furnished with (ϕ, ξ, η, g)-structure. Further, $\mathbb{M}^{2m+1}$ is known as *paracontact metric (briefly, PCM) manifold* if it satisfies $\Psi = d\eta$, where Ψ is the *fundamental* 2-*form* given by $\Psi(P_1, P_2) = g(P_1, \phi P_2)$, $P_1, P_2 \in \Gamma\left(\mathcal{T}\mathbb{M}^{2m+1}\right)$.

Also, for manifold $\mathbb{M}^{2m+1}$ we can find ϕ-basis which is a local orthonormal basis $\{X_i, \phi X_i, \xi\}$ such that $g(X_i, X_i) = 1$ and $g(\phi X_i, \phi X_i) = -1$, $i = 1, \cdots, m$.

2.1 Normality

On the product manifold $\mathbb{M}^{2m+1} \times \mathbb{R}$, **J** (almost paracomplex structure) is given as:

$$\mathbf{J}\left(P_1, \rho\frac{d}{dt}\right) = \left(\phi P_1 + \rho\xi; \eta(P_1)\frac{d}{dt}\right), \tag{2.3}$$

where $\left(P_1, \rho\frac{d}{dt}\right)$ is a tangent vector field, t is the standard Cartesian coordinate on $\mathbb{R}$ and ρ is a smooth function on $\mathbb{M}^{2m+1} \times \mathbb{R}$.
The almost paracontact structure is normal if paracomplex structure is integrable and for the integrability of paracomplex structure Nijenhuis tensor has to be vanish. So, the condition for normality of paracontact structure is given as:

$$N_\phi(P_1, P_2) - 2d\eta(P_1, P_2)\xi = 0, \tag{2.4}$$

where $N_\phi(P_1, P_2)$ is Nijenhuis tensor of ϕ which is given as follows:

$$N_\phi(P_1, P_2) = [\phi P_1, \phi P_2] - \phi[\phi P_1, P_2] - \phi[P_1, \phi P_2] + \phi^2[P_1, P_2], \tag{2.5}$$

for any vector fields $P_1, P_2 \in \Gamma\left(\mathcal{T}\mathbb{M}^{2m+1}\right)$.
Following [5], we present some results related to this case. For a.p.c.m. 3-manifold $\mathbb{M}^3$, we find that

$$(\nabla_{P_1}\phi)P_2 = g(\phi\nabla_{P_1}\xi, P_2)\,\xi - \eta(P_2)\,\phi\nabla_{P_1}\xi, \quad P_1, P_2 \in \Gamma(\mathcal{T}\mathbb{M}^3). \tag{2.6}$$

Proposition 2.3. *For* 3*-dimensional a.p.c.m. manifold* $\mathbb{M}^3$*, following conditions are equivalent:*

(i) $\mathbb{M}^3$ *is normal,*

(ii) $\exists$ *smooth functions* α *and* β *on* $\mathbb{M}^3$ *such that*

$$(\nabla_{P_1}\phi)P_2 = \beta\left(g(\phi P_1, P_2)\xi - \eta(P_2)\phi P_1\right) + \alpha\left(g(P_1, P_2)\xi - \eta(P_2)P_1\right), \tag{2.7}$$

(iii) $\exists$ *smooth functions* α *and* β *on* $\mathbb{M}^3$ *such that*

$$\nabla_{P_1}\xi = \beta\left(P_1 - \eta\left(P_1\right)\xi\right) + \alpha\,\phi P_1, \tag{2.8}$$

for any vector fields $P_1, P_2 \in \Gamma(\mathcal{T}\mathbb{M}^3)$*. Also on* $\mathbb{M}^3$*,* α *and* β *are defined as:*

$$\alpha = \frac{1}{2}\text{trace}\left\{P_1 \to \phi\nabla_{P_1}\xi\right\}, \quad \beta = \frac{1}{2}\text{trace}\left\{P_1 \to \nabla_{P_1}\xi\right\}. \tag{2.9}$$

Next, we give subclasses of normal a.p.c.m. 3-manifold $\mathbb{M}^3$:

Definition 2.4 (β-paraKenmotsu Manifold). The normal a.p.c.m. 3-manifold $\mathbb{M}^3$ is called β-*paraKenmotsu metric* 3*-manifold* (denoted by $\mathbb{K}^3$) if we take $\alpha = 0$ and $\beta = (non-zero)\ constant$ in Proposition 2.3. Also, if $\beta = 1$ then $\mathbb{M}^3$ is called *paraKenmotsu metric* 3*-manifold*.

Example 2.5. Let $\mathcal{K}^3 := \mathbb{R}^3_1$ with the structure (ϕ, ξ, η), where $\xi = U_3$, 1-form $\eta = dz$, and ϕ is given by: $\phi U_1 = U_2,\ \phi U_2 = U_1,\ \phi U_3 = 0$ ((x, y, z), being the standard Cartesian coordinates and $U_1 = \frac{\partial}{\partial x}, U_2 = \frac{\partial}{\partial y}, U_3 = \frac{\partial}{\partial z}$) and $g = -e^z dx^2 + e^z dy^2 + \eta \otimes \eta$. Then $\mathcal{K}^3$ with the structure (ϕ, ξ, η, g) becomes a normal a.p.c.m. 3-manifold $\mathbb{M}^3$.
Next, we find the coefficients of Levi-Civita connection as:

$$\begin{aligned}
&\nabla_{U_1}U_1 = \frac{1}{2}e^z U_3,\ \nabla_{U_2}U_2 = -\frac{1}{2}e^z U_3,\ \nabla_{U_3}U_3 = \nabla_{U_1}U_2 = \nabla_{U_2}U_1 = 0,\\
&\nabla_{U_1}U_3 = \nabla_{U_3}U_1 = \frac{1}{2}U_1,\ \nabla_{U_2}U_3 = \nabla_{U_3}U_2 = \frac{1}{2}U_2.
\end{aligned} \tag{2.10}$$

Using Eqs. (2.7) and (2.10), we have $\alpha = 0$ and $\beta = \frac{1}{2}$.

Similarly, we can also define *paracosymplectic metric* 3*-manifold* $\mathbb{C}^3$ and *paraSasakian metric* 3*-manifold* $\mathbb{Q}^3$ using Proposition 2.3.

2.2 Curvature Properties

The curvature tensor $\mathcal{R}$ on a manifold $\mathbb{M}^{2m+1}$ with pseudo-Riemannian metric g is defined as:

$$\mathcal{R}\left(P_1, P_2\right)P_3 = \nabla_{P_1}\nabla_{P_2}P_3 - \nabla_{P_2}\nabla_{P_1}P_3 - \nabla_{[P_1,P_2]}P_3, \tag{2.11}$$

where $P_1, P_2, P_3 \in \Gamma\left(\mathcal{T}\mathbb{M}^{2m+1}\right)$.

Next, we recall some results from [3] for later use.

Lemma 2.6. *For normal a.p.c.m.* 3*-manifold* $\mathbb{M}^3$*, we have*

$$\begin{aligned}
\mathcal{R}\left(P_1, P_2\right)\xi = &\left\{\left(P_1\,\beta\right) + \left(\alpha^2 + \beta^2\right)\eta\left(P_1\right)\right\}\phi^2 P_2 - \left\{\left(P_2\,\beta\right) + \left(\alpha^2 + \beta^2\right)\eta\left(P_2\right)\right\}\phi^2 P_1\\
&+ \left\{\left(P_1\alpha\right) + 2\alpha\,\beta\,\eta\left(P_1\right)\right\}\phi P_2 - \left\{\left(P_2\,\alpha\right) + 2\alpha\,\beta\,\eta\left(P_2\right)\right\}\phi P_1,
\end{aligned} \tag{2.12}$$

$$\mathcal{S}\left(P_2, \xi\right) = -\left(P_2\,\beta\right) + \left(\phi P_2\right)\alpha - \left\{\left(\xi\,\beta\right) + 2\left(\beta^2 + \beta^2\right)\right\}\eta\left(P_2\right), \tag{2.13}$$

$$\begin{aligned}
\mathcal{S}\left(P_1, P_2\right) = &\left\{\frac{\tau}{2} + (\xi\beta) + \left(\alpha^2 + \beta^2\right)\right\}g\left(P_1, P_2\right) - \left\{\eta\left(P_2\right)\left(P_1\beta\right) + \eta\left(P_1\right)\left(P_2\beta\right)\right\}\\
&- \left\{\frac{\tau}{2} + (\xi\beta) + 3\left(\alpha^2 + \beta^2\right)\right\}\eta\left(P_1\right)\eta\left(P_2\right) + \left\{\eta\left(P_2\right)\left(\phi P_1\right)\alpha + \eta\left(P_1\right)\left(\phi P_2\right)\alpha\right\}
\end{aligned} \tag{2.14}$$

for any vector fields $P_1, P_2 \in \Gamma\left(\mathcal{T}\mathbb{M}^3\right)$*,* $\tau = trace(\mathcal{S})$ *(scalar curvature of* $\mathbb{M}^3$*) and the functions* α *and* β *are as given in* (2.9)*.*

Lemma 2.7. *For* $\mathbb{K}^3$*, we have*

$$\begin{aligned}
\mathcal{R}\left(P_1, P_2\right)P_3 = &\left(\frac{\tau}{2} + 2\beta^2\right)\left\{g\left(P_2, P_3\right)P_1 - g\left(P_1, P_3\right)P_2\right\}\\
&- \left(\frac{\tau}{2} + 3\beta^2\right)\left\{g\left(P_2, P_3\right)\eta\left(P_1\right) - g\left(P_1, P_3\right)\eta\left(P_2\right)\right\}\xi\\
&+ \left(\frac{\tau}{2} + 3\beta^2\right)\left\{\eta\left(P_1\right)P_2 - \eta\left(P_2\right)P_1\right\}\eta\left(P_3\right).
\end{aligned} \tag{2.15}$$

for any vector fields $P_1, P_2, P_3 \in \Gamma\left(\mathcal{T}\mathbb{K}^3\right)$*.*

Definition 2.8. A manifold $\mathbb{M}^{2m+1}$ with pseudo-Riemannian metric g is called η-Einstein manifold if

$$\mathcal{S}(P_1, P_2) = a_1\, g(P_1, P_2) + a_2\, \eta(P_1)\, \eta(P_2), \tag{2.16}$$

where a_1 and a_2 are non-zero functions and η is a 1-form. If $a_2 = 0$, $\mathbb{M}^{2m+1}$ is an Einstein manifold.

3 Almost η-RS

Consider a pseudo-Riemannian manifold $(\mathbb{M}^{2m+1}, g)$, then (g, V, λ, μ) is called an *almost η-RS* on $\mathbb{M}^{2m+1}$ if it satisfies:

$$(\pounds_V\, g + 2\mathcal{S} - 2\lambda\, g + 2\mu\, \eta \otimes \eta)(P_1, P_2) = 0, \tag{3.1}$$

where $P_1, P_2 \in \Gamma\left(\mathcal{T}\mathbb{M}^{2m+1}\right)$, $\pounds_V$-Lie derivative in the direction V and $\mathcal{S}$-Ricci curvature tensor field of g, η is a 1-form and λ, μ- smooth functions on $\mathbb{M}^{2m+1}$. An almost η-RS on $\left(\mathbb{M}^{2m+1}, g\right)$ is *expanding* if λ is negative, *steady* if λ is zero or *shrinking* if λ is positive.
In particular, (g, V, λ, μ) becomes an almost Ricci soliton if $\mu = 0$.

Next, we define potential vector field V of an almost η-RS as $V := \xi$.

Proposition 3.1. *Let $\mathbb{M}^3$ be a normal a.p.c.m. 3-manifold and admits an almost η-RS having ξ as a potential vector field, then its scalar curvature is given as:*

$$scal = 2\beta + 3\lambda - \mu. \tag{3.2}$$

Proof. Equation (3.1) can be rewritten as:

$$\begin{aligned}
\mathcal{S}(P_1, P_2) &= -\frac{1}{2}\pounds_\xi g(P_1, P_2) + \lambda g(P_1, P_2) - \mu\eta(P_1)\eta(P_2) &(3.3)\\
&= -\frac{1}{2}\{g(P_1, \nabla_{P_2}\xi) + g(\nabla_{P_1}\xi, P_2)\} + \lambda g(P_1, P_2) - \mu\eta(P_1)\eta(P_2) &(3.4)\\
&= -\frac{1}{2}\{g(\alpha\phi P_1 + \beta(P_1 - \eta(P_1)\xi), P_2) + g(P_1, \alpha\phi P_2 + \beta(P_2 - \eta(P_2)\xi))\}\\
&\quad + \lambda g(P_1, P_2) - \mu\eta(P_1)\eta(P_2)\\
&= -\frac{1}{2}\{\alpha g(\phi P_1, P_2) + \beta g(P_1, P_2) + \alpha g(P_1, \phi P_2) + \beta g(P_1, P_2)\}\\
&\quad + \beta\eta(P_1)\eta(P_2) + \lambda g(P_1, P_2) - \mu\eta(P_1)\eta(P_2)\\
\mathcal{S}(P_1, P_2) &= \beta g(\phi P_1, \phi P_2) + \lambda g(P_1, P_2) - \mu\eta(P_1)\eta(P_2). &(3.5)
\end{aligned}$$

Now, by using equation (3.5) and the definition of scalar curvature we get the required result. □

Corollary 3.2. *An almost η-RS on normal a.p.c.m. 3-manifold $\mathbb{M}^3$ is steady if and only if $\mu = 2\left(\alpha^2 + \beta^2\right)$.*

Proof. By using equation (3.5) and (2.13), we get

$$\lambda = \mu - 2\left(\alpha^2 + \beta^2\right). \tag{3.6}$$

Thus, (g, ξ, λ, μ) is steady if $\mu = 2\left(\alpha^2 + \beta^2\right)$. □

Corollary 3.3. *An almost Ricci soliton on normal a.p.c.m. 3-manifold $\mathbb{M}^3$ is expanding.*

Proof. Since $\mu = 0$ for an almost Ricci soliton on $\mathbb{M}^3$, then equation (3.6) becomes $\lambda = -2\left(\alpha^2 + \beta^2\right)$, which implies λ will always be negative and hence an almost Ricci soliton is expanding. □

Corollary 3.4. *If an almost η-RS on normal a.p.c.m. 3-manifold $\mathbb{M}^3$ is steady, then the scalar curvature of $\mathbb{M}^3$ is*

$$scal = 2\beta - 2\left(\alpha^2 + \beta^2\right). \tag{3.7}$$

Proof. Since $\lambda = 0$, for a steady almost η-RS. From equation (3.6), we have $\mu = 2\left(\alpha^2 + \beta^2\right)$ and use these values of λ and μ in equation (3.2) to get the required result. □

Consider an almost η-RS on $\mathbb{M}^{2m+1}(\phi, \xi, \eta, g)$ then for any vector fields $P_1, P_2, P_3 \in \Gamma\left(\mathcal{T}\mathbb{M}^{2m+1}\right)$, we have

$$(\nabla_{P_1}\mathcal{S})(P_2, P_3) = P_1(\mathcal{S}(P_2, P_3)) - \mathcal{S}(\nabla_{P_1}P_2, P_3) - \mathcal{S}(P_2, \nabla_{P_1}P_3). \tag{3.8}$$

By substituting the value of $\mathcal{S}(P_1, P_2)$ from equation (3.5), we have

$$\begin{aligned}(\nabla_{P_1}\mathcal{S})(P_2, P_3) =& P_1(\beta\, g(\phi P_2, \phi P_3) + \lambda\, g(P_2, P_3) - \mu\, \eta(P_2)\, \eta(P_3)) - \beta\, g(\phi\nabla_{P_1}P_2, \phi P_3) \\ &- \mu\, \eta(\nabla_{P_1}P_2)\, \eta(P_3) - \beta\, g(\phi P_2, \phi\nabla_{P_1}P_3) - \mu\, \eta(P_2)\, \eta(\nabla_{P_1}P_3).\end{aligned} \tag{3.9}$$

Definition 3.5 (Ricci symmetric). A manifold $\mathbb{M}^{2m+1}$ with pseudo-Riemannian metric g is called Ricci symmetric if

$$(\nabla_{P_1}\mathcal{S})(P_2, P_3) = 0, \tag{3.10}$$

for any vector field $P_1, P_2, P_3 \in \Gamma\left(\mathcal{T}\mathbb{M}^{2m+1}\right)$.

Proposition 3.6. *If a normal a.p.c.m.* 3*-manifold* $\mathbb{M}^3$ *be Ricci symmetric, then*

$$\xi(\mu) = \xi(\lambda).$$

Proof. Since the manifold $\mathbb{M}^3$ is Ricci symmetric, then by taking $P_1 = P_2 = P_3 = \xi$ in equation (3.10), we have

$$(\nabla_\xi\mathcal{S})(\xi, \xi) = 0.$$

Now, by using equations (2.1), (2.8) and (3.9) in above equation we get the required result. □

Definition 3.7 (η-recurrent Ricci tensor). A manifold $\mathbb{M}^{2m+1}$ with pseudo-Riemannian metric g is said to have η-recurrent Ricci tensor if

$$(\nabla_{P_1}\mathcal{S})(P_2, P_3) = \eta(P_1)\, \mathcal{S}(P_2, P_3), \tag{3.11}$$

for any vector field $P_1, P_2, P_3 \in \Gamma\left(\mathcal{T}\mathbb{M}^{2m+1}\right)$.

Proposition 3.8. *Let the normal a.p.c.m.* 3*-manifold* $\mathbb{M}^3$ *having* η*-recurrent Ricci tensor then*

$$\xi(\lambda - \mu) = -2\left(\alpha^2 + \beta^2\right). \tag{3.12}$$

Proof. Since the Ricci tensor is η-recurrent, then by taking $P_1 = P_2 = P_3 = \xi$ in equation (3.11), we have

$$(\nabla_\xi\mathcal{S})(\xi, \xi) = \mathcal{S}(\xi, \xi).$$

Now, by using equations (2.1), (2.8), (2.13) and (3.9) in above equation, we get the required result. □

Corollary 3.9. *For a* β*-paraKenmotsu manifold* $\mathbb{K}^3$ *having* η*-recurrent Ricci tensor, we have* $\xi(\lambda - \mu) = -2\beta^2$.

Definition 3.10 (Codazzi type Ricci tensor). A manifold $\mathbb{M}^{2m+1}$ with pseudo-Riemannian metric g is said to have Ricci tensor of Codazzi type if

$$(\nabla_{P_2}\mathcal{S})(P_1, P_3) = (\nabla_{P_1}\mathcal{S})(P_2, P_3), \tag{3.13}$$

for any vector field $P_1, P_2, P_3 \in \Gamma\left(\mathcal{T}\mathbb{M}^{2m+1}\right)$.

Proposition 3.11. *Let the normal a.p.c.m.* 3*-manifold* $\mathbb{M}^3$ *having Codazzi type Ricci tensor, then*

$$P_1(\lambda - \mu) = \xi(\lambda - \mu)\, \eta(P_1), \tag{3.14}$$

for any vector field $P_1 \in \Gamma\left(\mathcal{T}\mathbb{M}^3\right)$.

Proof. Taking $P_2 = P_3 = \xi$ in equation (3.13), we have

$$(\nabla_{\xi}\mathcal{S})(P_1,\xi) = (\nabla_{P_1}\mathcal{S})(\xi,\xi),$$

for any vector field $P_1 \in \Gamma(\mathcal{T}\mathbb{M}^3)$.
Now, by using equations (2.1), (2.8) and (3.9) in above equation we get the required result. □

Definition 3.12 (η-parallel Ricci tensor). A manifold $\mathbb{M}^{2m+1}$ with pseudo-Riemannian metric g is said to have η-parallel Ricci tensor if

$$(\nabla_{P_1}\mathcal{S})(\phi P_2,\phi P_3) = 0, \tag{3.15}$$

for any vector field $P_1, P_2, P_3 \in \Gamma(\mathcal{T}\mathbb{M}^{2m+1})$.

Proposition 3.13. *Let the normal a.p.c.m. 3-manifold $\mathbb{M}^3$ having η-parallel Ricci tensor, then*

$$P_1(\lambda) = P_1(\beta), \tag{3.16}$$

for any vector field $P_1 \in \Gamma(\mathcal{T}\mathbb{M}^3)$.

Proof. By using equations (2.1), (2.2) and (3.9) in equation (3.15) we get $P_1(\lambda) = P_1(\beta)$, for any vector field $P_1 \in \Gamma(\mathcal{T}\mathbb{M}^3)$ which gives the required result. □

Corollary 3.14. *For β-paraKenmotsu manifold $\mathbb{K}^3$, if the Ricci tensor is η-parallel then the scalar function λ is locally constant.*

Definition 3.15 (Cyclic Ricci tensor). A manifold $\mathbb{M}^{2m+1}$ with pseudo-Riemannian metric g is said to have cyclic Ricci tensor if

$$(\nabla_{P_3}\mathcal{S})(P_1,P_2) + (\nabla_{P_1}\mathcal{S})(P_2,P_3) + (\nabla_{P_2}\mathcal{S})(P_3,P_1) = 0, \tag{3.17}$$

for any vector fields $P_1, P_2, P_3 \in \Gamma(\mathcal{T}\mathbb{M}^{2m+1})$.

Proposition 3.16. *Let the normal a.p.c.m. 3-manifold $\mathbb{M}^3$ having cyclic Ricci tensor, then*

$$P_1(\lambda-\mu)\xi = 2\xi(\mu-\lambda)P_1, \tag{3.18}$$

for any vector field $P_1 \in \Gamma(\mathcal{T}\mathbb{M}^3)$.

Proof. Since the Ricci tensor is cyclic, then by taking $P_2 = P_3 = \xi$ in equation (3.17), we have

$$(\nabla_{\xi}\mathcal{S})(P_1,\xi) + (\nabla_{P_1}\mathcal{S})(\xi,\xi) + (\nabla_{\xi}\mathcal{S})(\xi,P_1) = 0.$$

By using equations (2.1), (2.2) and (3.9) in above equation, we get the required result. □

Definition 3.17 (Cyclic η-recurrent Ricci tensor). A manifold $\mathbb{M}^{2m+1}$ with pseudo-Riemannian metric g is said to have cyclic η-recurrent Ricci tensor if

$$\begin{aligned}(\nabla_{P_1}\mathcal{S})(P_2,P_3) + (\nabla_{P_2}\mathcal{S})(P_3,P_1) + (\nabla_{P_3}\mathcal{S})(P_1,P_2) =& \eta(P_1)\mathcal{S}(P_2,P_3) + \eta(P_2)\mathcal{S}(P_3,P_1)\\ &+ \eta(P_3)\mathcal{S}(P_1,P_2),\end{aligned} \tag{3.19}$$

for any vector fields $P_1, P_2, P_3 \in \Gamma(\mathcal{T}\mathbb{M}^{2m+1})$.

Proposition 3.18. *Let the normal a.p.c.m. 3-manifold $\mathbb{M}^3$ having cyclic η-recurrent Ricci tensor, then*

$$P_1(\lambda-\mu)\xi + 2\xi(\lambda-\mu)P_1 = -6\left(\alpha^2+\beta^2\right)P_1, \tag{3.20}$$

for any vector field $P_1 \in \Gamma(\mathcal{T}\mathbb{M}^3)$.

Proof. Taking $P_2 = P_3 = \xi$ in equation (3.19), we have

$$(\nabla_{\xi}\mathcal{S})(P_1,\xi) + (\nabla_{P_1}\mathcal{S})(\xi,\xi) + (\nabla_{\xi}\mathcal{S})(\xi,P_1) = \eta(P_1)\mathcal{S}(\xi,\xi) + 2\mathcal{S}(X,\xi).$$

By using equations (2.1), (2.8), (2.13) and (3.9), we get the required result. □

Corollary 3.19. *For a β-paraKenmotsu manifold $\mathbb{K}^3$ having cyclic η-recurrent Ricci tensor, we have $P_1(\lambda-\mu)\xi + 2\xi(\lambda-\mu)P_1 = -6\beta^2 P_1$.*

Theorem 3.20. *Let $\mathbb{M}^3$ be a normal a.p.c.m. 3-manifold and consider an almost η-RS on $\mathbb{M}^3$. If V is conformal Killing, then*

$$\mathcal{S}(P_1,P_2) = (-f+\lambda)\,g(P_1,P_2) - \left[2\left(\alpha^2+\beta^2\right) + (-f+\lambda)\right]\eta(P_1)\,\eta(P_2), \tag{3.21}$$

and the manifold $\mathbb{M}^3$ is Einstein if and only if $\lambda = f - 2\left(\alpha^2+\beta^2\right)$. In this case, we have an almost Ricci soliton.

Proof. Since V is conformal Killing, then $\frac{1}{2}\pounds_\xi\, g = fg$. Now, using equation (2.13) and (3.3), we get the required result. □

Corollary 3.21. *If V is torse forming* i.e. $\nabla_{P_1}\xi = f\phi^2(P_1)$*, then we also get the same result as previous result.*

4 β-paraKenmotsu manifold and almost η-RS

Proposition 4.1. *For a β-paraKenmotsu 3-manifold $\mathbb{K}^3$, we have*

(i) $(\nabla_{P_1}\phi)P_2 = \beta\{g(\phi P_1,P_2)\xi - \eta(P_2)\phi P_1\}$,

(ii) $\nabla_{P_1}\xi = \beta\{P_1 - \eta(P_1)\xi\}$,

(iii) $\mathcal{R}(P_1,P_2)\xi = \beta^2\left\{\eta(P_1)\phi^2 P_2 - \eta(P_2)\phi^2 P_1\right\}$,

(iv) $\mathcal{S}(P_2,\xi) = -2\beta^2\eta(P_2)$,

(v) $\mathcal{Q}X_2 = -2\beta^2\eta(P_2)\xi$,

where $P_1,P_2\in\Gamma\left(\mathcal{T}\mathbb{K}^3\right)$.

Proof. For (i) and (ii) take $\beta\,(\neq 0)$ a constant and $\alpha = 0$ in Proposition 2.3.
For (iii) and (iv) take $\beta\,(\neq 0)$ a constant and $\alpha = 0$ in Lemma 2.6.
For (v) rewrite (iv) as: $g(\mathcal{Q}X_2,\xi) = -2\beta^2\eta(P_2)$. □

Next, we study an almost η-RS on $\mathbb{K}^3$ in two cases:

Case 1: When potential vector field $V=\xi$.

Proof of Theorem 1.1

For $\mathbb{K}^3$ with $V=\xi$, equation (3.1) reduces to

$$\left(\pounds_\xi g + 2\mathcal{S} - 2\lambda\, g + 2\mu\,\eta\otimes\eta\right)(P_1,P_2) = 0. \tag{4.1}$$

Now, using Proposition 4.1 we have

$$\mathcal{S}(P_1,P_2) = (\lambda-\beta)\,g(P_1,P_2) + (\beta-\mu)\,\eta(P_1)\,\eta(P_2). \tag{4.2}$$

Thus, $\mathbb{K}^3$ is a η-Einstein manifold. Take $P_1 = P_2 = \xi$ and together with Proposition 4.1, we get $\lambda = \mu - 2\,\beta^2$. Also from equation (4.2), we see that if $\mu=\beta$, then $\mathbb{K}^3$ is an Einstein manifold. Which is the complete proof of Theorem 1.1. □

Corollary 4.2. *On a β-paraKenmotsu 3-manifold $\mathbb{K}^3$ there does not exist Ricci soliton (g,V,λ,μ) with $V=\xi$.*

Proposition 4.3. *Let us consider a β-paraKenmotsu 3-manifold $\mathbb{K}^3$. If $\mathbb{K}^3$ be a η-Einstein manifold with functions a_1 and a_2, then $\mathbb{K}^3$ admits an almost η-RS having ξ as a potential vector field, $\lambda = \beta + a_1$ and $\mu = \beta - a_2$.*

Proof. Let $\mathbb{K}^3$ be a η-Einstein β-paraKenmotsu 3-manifold and $V=\xi$. Then

$$\mathcal{S}(P_1,P_2)=a_1 g(P_1,P_2)+a_2\eta(P_1)\eta(P_2), \tag{4.3}$$

where $P_1,\ P_2\in\Gamma\left(\mathcal{T}\mathbb{K}^3\right)$ and a_1, a_2 are functions on $\mathbb{K}^3$.
By using equation (2.6), we get

$$\begin{aligned}\pounds_\xi\, g(P_1,P_2)&+2S(P_1,P_2)-2\lambda\, g(P_1,P_2)+2\mu\,\eta(P_1)\,\eta(P_2)\\&=2(\beta+a_1-\lambda)\,g(P_1,P_2)+2(a_2+\mu-\beta)\,\eta(P_1)\,\eta(P_2).\end{aligned} \tag{4.4}$$

From equation (4.4) it is obvious that $\mathbb{K}^3$ admits an almost η-RS with $V=\xi$, if

$$\beta+a_1-\lambda=0 \text{ and } \mu+a_2-\beta=0, \tag{4.5}$$

which implies

$$\lambda=\beta+a_1 \text{ and } \mu=\beta-a_2. \tag{4.6}$$

This completes the proof. □

Proposition 4.4. *An almost η-RS on a paraKenmotsu 3-manifold $\mathbb{K}^3$ is steady if $\mu=2$, shrinking if $\mu>2$ and expanding if $\mu<2$.*

Proof. From equations (4.3), (4.7) and taking $P_1=P_2=\xi$, we get

$$a_1+a_2=-2\beta^2.$$

Also from equation (4.6), we get

$$\lambda=\mu-2\beta^2.$$

Now for paraKenmotsu 3-manifold, we get

$$\lambda=\mu-2,$$

which gives the required result. □

Proposition 4.5. *Let us consider a β-paraKenmotsu 3-manifold $\mathbb{K}^3$ admitting a steady almost η-RS, then its scalar curvature is constant.*

Proof. By using equation (3.2) we get the required result. □

Proposition 4.6. *If a paraKenmotsu 3-manifold $\mathbb{K}^3$ admitting a steady almost η-RS then $\mathbb{K}^3$ has no metric with positive scalar curvature.*

Proof. By using equation (3.2) we get the required result. □

Case 2: When potential vector field $V\perp\xi$.

Proof of Theorem 1.2

Consider a β-paraKenmotsu 3-manifold $\mathbb{K}^3$ admitting an almost η-RS. Then equation (3.1) can be rewritten as:

$$\mathcal{S}(P_1,P_2)=-\frac{1}{2}\left\{g(\nabla_{P_1}V,P_2)+g(P_1,\nabla_{P_2}V)\right\}+\lambda\, g(P_1,P_2)-\mu\,\eta(P_1)\,\eta(P_2). \tag{4.7}$$

Put $P_1=P_2=\xi$ in the above equation, we get

$$\mathcal{S}(\xi,\xi)=\lambda-\mu.$$

Then Proposition 4.1 gives $\lambda = \mu - 2\beta^2$.
Put $P_1 = P_2 = V$ in equation (4.7), we get

$$\mathcal{S}\left(V, V\right) = \lambda = \mu - 2\beta^2. \tag{4.8}$$

Put $P_1 = P_2 = \phi V$ in equation (4.7), we get

$$\mathcal{S}\left(\phi V, \phi V\right) + g\left(\nabla_{\phi V} V, \phi V\right) = -\lambda = -\mu + 2\beta^2. \tag{4.9}$$

As we know in a 3-dimensional pseudo-Riemannian manifold, Weyl curvature tensor vanishes then

$$\begin{aligned}\mathcal{R}\left(P_1, P_2\right) P_3 =& \mathcal{S}\left(P_2, P_3\right) P_1 - \mathcal{S}\left(P_1, P_3\right) P_2 + g\left(P_2, P_3\right) \mathcal{S} X_1 \\ & - g\left(P_1, P_3\right) \mathcal{S} X_2 - \frac{\tau}{2}\left\{g\left(P_2, P_3\right) P_1 - g\left(P_1, P_3\right) P_2\right\},\end{aligned}$$

for any vector fields $P_1, P_2, P_3 \in \Gamma\left(\mathcal{T}\mathbb{M}^3\right)$. Which implies

$$\mathcal{R}\left(P_1, \xi\right)\xi = \mathcal{Q}P_1 - \left(2\beta^2 + \frac{\tau}{2}\right) P_1 + \left(4\beta^2 + \frac{\tau}{2}\right)\eta\left(P_1\right)\xi, \tag{4.10}$$

for any vector fields $P_1, P_2, P_3 \in \Gamma\left(\mathcal{T}\mathbb{K}^3\right)$. Using previous relation and Proposition 4.1, we have

$$\mathcal{Q}P_1 = \left(\beta^2 + \frac{\tau}{2}\right) P_1 + \left(3\beta^2 + \frac{\tau}{2}\right)\eta\left(P_1\right)\xi \tag{4.11}$$

for any vector fields $P_1, P_2, P_3 \in \Gamma\left(\mathcal{T}\mathbb{K}^3\right)$. This gives the relation $\mathcal{Q} \circ \phi = \phi \circ \mathcal{Q}$ and then from equation (4.8) and (4.9), we obtain $g\left(\nabla_{\phi V} V, \phi V\right) = 0$.
From equation(4.7), we get

$$div(V) + \tau = 3\lambda - \mu.$$

By using calculated value of λ, we get the scalar curvature as:

$$\tau = 2\mu - 6\beta^2.$$

Which is the complete proof of Theorem 1.2. □

Corollary 4.7. *Let a β-paraKenmotsu 3-manifold $\mathbb{K}^3$ admits an almost η-RS having potential vector field orthogonal to ξ and $\mu = \beta^2$, then the scalar curvature of $\mathbb{K}^3$ is -3.*

Corollary 4.8. *Let a paraKenmotsu 3-manifold $\mathbb{K}^3$ admits an almost η-RS having potential vector field orthogonal to ξ, then the scalar curvature of $\mathbb{K}^3$ is $\tau = 2\mu - 6$.*

Corollary 4.9. *Let a paraKenmotsu 3-manifold $\mathbb{K}^3$ admits an almost Ricci soliton having potential vector field orthogonal to ξ, then $\tau = -6$, is the scalar curvature of $\mathbb{K}^3$.*

5 Gradient almost η-RS

In this section, firstly we define gradient and Hessian operators on $\mathbb{M}^{2m+1}$. So, consider a manifold $\mathbb{M}^{2m+1}$ with pseudo-Riemannian metric g and $f : \mathbb{M}^{2m+1} \longrightarrow \mathbb{R}$ is a smooth function over $\mathbb{M}^{2m+1}$. Then, the gradient (first order differential operator) $\nabla : C^1\left(\mathbb{M}^{2m+1}\right) \longrightarrow \Gamma\left(\mathcal{T}\mathbb{M}^{2m+1}\right)$ of a function f is given as:

$$g\left(\nabla f\left(x\right), P_1\right) = P_1 f\left(x\right),$$

for any vector field $P_1 \in \mathcal{T}_x\mathbb{M}^{2m+1}$ and Hessian (covariant derivative of the gradient operator) of a function f is given as:

$$\nabla^2 f\left(P_1, P_2\right) = P_1 X_2 f - \left(\nabla_{P_1} P_2\right) f,$$

for any vector fields $P_1, P_2 \in \Gamma\left(\mathcal{T}\mathbb{M}^{2m+1}\right)$.

Definition 5.1. An almost η-RS is a *gradient almost η-RS* if V of equation (3.1) is of gradient type, *i.e.*. $V = \text{grad}\,(\text{f})$ and satisfies:

$$\left(Hess\,(f) + \mathcal{S} - \lambda\, g + \mu\, \eta \otimes \eta\right)(P_1, P_2) = 0, \tag{5.1}$$

where $P_1, P_2, P_3 \in \Gamma\left(\mathcal{T}\mathbb{M}^{2m+1}\right)$ and the Hessian of f is given as: $Hess\,(f)\,(P_1, P_2) := g\,(\nabla_{P_1}\xi, P_2)$.

Theorem 5.2. *Let $\mathbb{M}^3$ be a normal a.p.c.m. 3-manifold. If a gradient almost η-RS on $\mathbb{M}^3$ is defined by equation (5.1) with $\xi := \text{grad}\,(\text{f})$ and $\eta = df$ is the g-dual of ξ, then*

$$\left.\begin{aligned}(\nabla_{P_1}\mathcal{Q})\,P_2 - (\nabla_{P_2}\mathcal{Q})\,P_1 = \Big\{&\Big(d\,(\lambda-\beta) + \Big(\beta\,\mu - \left(\alpha^2+\beta^2\right)\Big)\,df\Big) \otimes I - I \otimes (d\,(\lambda-\beta)\\ &+ \Big(\beta\,\mu - \left(\alpha^2+\beta^2\right)\Big)\,df\Big) + (\alpha\,\mu - 2\alpha\,\beta)\,(df\otimes\phi - \phi\otimes df)\\ &+ (df\otimes d\mu - d\mu\otimes df)\otimes\xi - (d\alpha\otimes\phi - \phi\otimes d\alpha)\\ &+ (d\beta\otimes df - df\otimes d\beta)\otimes\xi\Big\}\,(P_1,P_2) + 2\alpha\,\mu\, g\,(P_1, \phi P_2)\,\xi,\end{aligned}\right\} \tag{5.2}$$

for any vector fields $P_1, P_2 \in \Gamma\left(\mathcal{T}\mathbb{M}^3\right)$ and $\mathcal{Q}$ is the Ricci operator given as: $g\,(\mathcal{Q}X_1, P_2) = \mathcal{S}\,(P_1, P_2)$.

Proof. Notice that from equation (5.1), we have

$$\nabla_{P_1}\xi + \mathcal{Q}\,P_1 - \lambda\,P_1 + \mu\,df\,(P_1)\otimes\xi = 0.$$

Then

$$\begin{aligned}(\nabla_{P_1}\mathcal{Q})\,P_2 = &-\nabla_{P_1}\nabla_{P_2}\xi - \mathcal{Q}\,\nabla_{P_1}P_2 + P_1\,(\lambda)\,P_2 + \lambda\,\nabla_{P_1}P_2 - P_1\,(\mu)\,\eta\,(P_2)\,\xi - \mu\,g\,(\nabla_{P_1}P_2, \xi)\,\xi\\ &- \mu\,g\,(P_2, \nabla_{P_1}\xi)\,\xi - \mu\,\eta\,(P_2)\,\nabla_{P_1}\xi.\end{aligned}$$

By further computation and using equation (2.8) in the above relation, we get the required result. □

Proposition 5.3. *Let us consider a β-paraKenmotsu 3-manifold $\mathbb{K}^3$. If a gradient almost η-RS on $\mathbb{K}^3$ is defined by equation (5.1) with $\xi := \text{grad}\,(\text{f})$ and $\eta = df$ is the g-dual of ξ, then*

$$\begin{cases}(\nabla_{P_1}\mathcal{Q})\,P_2 - (\nabla_{P_2}\mathcal{Q})\,P_1 = & \left\{\left(d\,(\lambda) + \left(\beta\,\mu - \beta^2\right)df\right)\otimes I - I\otimes\left(d\,(\lambda) + \left(\beta\,\mu-\beta^2\right)df\right)\right.\\ & \left. + (df\otimes d\mu - d\mu\otimes df)\otimes\xi\right\}(P_1, P_2)\,,\end{cases}$$

for any vector fields $P_1, P_2 \in \Gamma\left(\mathcal{T}\mathbb{K}^3\right)$ and $\mathcal{Q}$ is the Ricci operator given as: $g\,(\mathcal{Q}X_1, P_2) = \mathcal{S}\,(P_1, P_2)$.

Proof. For a β-paraKenmotsu 3-manifold $\mathbb{K}^3$, we have $\alpha = 0$. Then by using this value of α in equation (5.2), we get the required result. □

Proposition 5.4. *For a β-paraKenmotsu 3-manifold $\mathbb{K}^3$, we have*

$$(\nabla_{P_1}\eta)\,P_2 = \beta\,\{g\,(P_1, P_2) - \eta\,(P_1)\,\eta\,(P_2)\}\,, \tag{5.3}$$

$$\mathcal{R}\,(P_1, \xi)\,P_2 = \beta^2\,\{g\,(P_1, P_2)\,\xi - \eta\,(P_2)\,P_1\}\,, \tag{5.4}$$

$$(\pounds_\xi g)\,(P_2, P_3) = 2\beta\,\{g\,(P_2, P_3) - \eta\,(P_2)\,\eta\,(P_3)\}\,, \tag{5.5}$$

for any vector fields $P_1, P_2, P_3 \in \Gamma\left(\mathcal{T}\mathbb{K}^3\right)$.

Proof. Take $Y = \xi$ and $Z = Y$ in equation (2.7), we get (5.3). By using (i) of Proposition 4.1, we get (5.4). Also we know that

$$(\pounds_\xi g)\,(P_2, P_3) = g\,(P_2, \nabla_{P_3}\xi) + g\,(\nabla_{P_2}\xi, P_3)\,. \tag{5.6}$$

By using (ii) of Proposition 4.1, we get (5.5). □

Proposition 5.5. *For a β-paraKenmotsu 3-manifold $\mathbb{K}^3$, we have*

$$(\pounds_\xi\mathcal{Q})\,P_2 = -2\beta\left\{\mathcal{Q}X_2 + 2\beta^2 P_2\right\} = (\nabla_\xi\mathcal{Q})\,P_2, \tag{5.7}$$

for any vector field $P_2 \in \Gamma\left(\mathcal{T}\mathbb{K}^3\right)$.

Proof. Taking covariant derivative of equation (5.5) along P_1 and using equation (5.3), we get

$$\left.\begin{aligned}(\nabla_{P_1}\pounds_\xi g)(P_2,P_3) = &-2\beta^2 g(P_1,P_2)\eta(P_3) - 2\beta^2 g(P_1,P_3)\eta(P_2)\\ &+4\beta^2\eta(P_1)\eta(P_2)\eta(P_3).\end{aligned}\right\} \tag{5.8}$$

Also from [27], we see that

$$\left(\pounds_V\nabla_{P_3}g - \nabla_{P_3}\pounds_V g - \nabla_{[V,P_3]}g\right)(P_1,P_2) = -\{g((\pounds_V\nabla)(P_3,P_1),P_2) + g((\pounds_V\nabla)(P_3,P_2),P_1)\}.$$

Now by parallelism of metric g and using equation (5.8), we get

$$\begin{aligned}g((\pounds_\xi\nabla)(P_1,P_2),P_3) + g((\pounds_\xi\nabla)(P_1,P_3),P_2) = &-2\beta^2 g(P_1,P_2)\eta(P_3) - 2\beta^2 g(P_1,P_3)\eta(P_2)\\ &+4\beta^2\eta(P_1)\eta(P_2)\eta(P_3).\end{aligned}$$

After further computation, we have

$$(\pounds_\xi\nabla)(P_2,P_3) = -2\beta^2 g(P_2,P_3)\xi + 2\beta^2\eta(P_2)\eta(P_3)\xi.$$

Taking covariant derivative of above equation and using (ii) of Proposition 4.1, we have

$$\left.\begin{aligned}(\nabla_{P_1}\pounds_\xi\nabla)(P_2,P_3) = &-2\beta^3 g(P_2,P_3)P_1 + 2\beta^3 g(P_2,P_3)\eta(P_1)\xi\\ &+2\beta^3 g(P_1,P_2)\eta(P_3)\xi + 2\beta^3 g(P_1,P_3)\eta(P_2)\xi\\ &-6\beta^3\eta(P_1)\eta(P_2)\eta(P_3)\xi + 2\beta^3\eta(P_2)\eta(P_3)P_1.\end{aligned}\right\} \tag{5.9}$$

From [27], we also have the following formula

$$(\pounds_V\mathcal{R})(P_1,P_2)P_3 = (\nabla_{P_1}\pounds_V\nabla)(P_2,P_3) - (\nabla_{P_2}\pounds_V\nabla)(P_1,P_3). \tag{5.10}$$

Using equation (5.9) in the above formula, we have

$$\begin{aligned}(\pounds_\xi\mathcal{R})(P_1,P_2)P_3 = &-2\beta^3 g(P_2,P_3)P_1 + 2\beta^3 g(P_1,P_3)P_2\\ &-2\beta^3\eta(P_1)\eta(P_3)P_2 + 2\beta^3\eta(P_2)\eta(P_3)P_1.\end{aligned}$$

Now contracting above equation over P_1

$$(\pounds_\xi\mathcal{S})(P_2,P_3) = -4\beta^3 g(P_2,P_3) + 4\beta^3\eta(P_2)\eta(P_3). \tag{5.11}$$

Now, Lie derivative of $\mathcal{S}(P_2,P_3) = g(\mathcal{Q}P_2,P_3)$ gives

$$(\pounds_\xi\mathcal{S})(P_2,P_3) = (\pounds_\xi g)(\mathcal{Q}P_2,P_3) + g((\pounds_\xi\mathcal{Q})P_2,P_3). \tag{5.12}$$

Replacing P_2 by $\mathcal{Q}P_2$ in equation (5.5) and using (v) of Propositon 4.1, we get

$$(\pounds_\xi g)(\mathcal{Q}P_2,P_3) = 2\beta\left\{g(\mathcal{Q}P_2,P_3) + 2\beta^2\eta(P_2)\eta(P_3)\right\}. \tag{5.13}$$

By using equation (5.12) and (5.13), equation (5.11) becomes

$$(\pounds_\xi\mathcal{Q})P_2 = -2\beta\left\{\mathcal{Q}X_2 + 2\beta^2 P_2\right\}. \tag{5.14}$$

Also, it is well known that

$$\begin{aligned}(\pounds_\xi\mathcal{Q})P_2 &= \pounds_\xi\mathcal{Q}X_2 - \mathcal{Q}(\pounds_\xi P_2)\\ &= \nabla_\xi\mathcal{Q}P_2 - \nabla_{\mathcal{Q}X_2}\xi - \mathcal{Q}(\nabla_\xi P_2) + \mathcal{Q}(\nabla_{P_2}\xi)\\ &= (\nabla_\xi\mathcal{Q})P_2 - \nabla_{\mathcal{Q}X_2}\xi + \mathcal{Q}(\nabla_{P_2}\xi).\end{aligned}$$

From (ii) and (v) of Proposition 4.1, we get

$$(\pounds_\xi\mathcal{Q})P_2 = (\nabla_\xi\mathcal{Q})P_2. \tag{5.15}$$

Thus after combining equation (5.14) and (5.15), we get the required result. □

Proposition 5.6. *Let us consider a β-paraKenmotsu 3-manifold $\mathbb{K}^3$ admitting a gradient almost η-RS, then*

$$\mathcal{R}(P_1,P_2)Df=(\nabla_{P_1}\mathcal{Q})P_2-(\nabla_{P_2}\mathcal{Q})P_1+P_2(\lambda)P_1-P_1(\lambda)P_2+P_1(\mu)\eta(P_2)\xi \\ -P_2(\mu)\eta(P_1)\xi+\beta\mu(\eta(P_2)P_1-\eta(P_1)P_2), \quad (5.16)$$

for any $P_1,P_2\in\Gamma(\mathcal{T}\mathbb{K}^3)$.

Proof. From definition 5.1, we have

$$\nabla_{P_1}Df=\mathcal{Q}X_1-\lambda P_1+\mu\eta(P_1)\xi.$$

Now using the above equation and equation (2.11) we get the required result. □

Proof of Theorem 1.3

Replacing P_2 by ξ in equation (5.16) and using Proposition 5.5, we get

$$\mathcal{R}(P_1,\xi)Df=2\beta^3P_1+\beta\mathcal{Q}P_1+\xi(\lambda)P_1-P_1(\lambda)\xi+P_1(\mu)\xi \\ -\xi(\mu)\eta(P_1)\xi+\beta\mu(P_1-\eta(P_1)\xi).$$

By using equation (5.4), the above equation becomes

$$g\left(P_1,\beta^2Df+d\lambda-D\mu+\xi(\mu)\xi\right)\xi=\beta\left(\mathcal{Q}P_1+2\beta^2P_1\right)+\left\{\beta^2\xi f+\xi(\lambda)\right\}P_1 \\ +\beta\mu(P_1-\eta(P_1)\xi). \quad (5.17)$$

Taking the inner product of above equation with ξ and using (v) of Proposition 4.1, we get

$$\beta^2Df+D\lambda-D\mu+\xi(\mu)\xi=\left\{\beta^2\xi f+\xi(\lambda)\right\}\xi.$$

Using this in equation (5.17), we get

$$\mathcal{S}(P_1,P_2)=-\left\{2\beta^2+\beta^2\xi f+\xi(\lambda)+\beta\mu\right\}g(P_1,P_2)+\left\{\beta^2\xi f+\xi(\lambda)+\beta\mu\right\}\eta(P_1)\eta(P_2). \quad (5.18)$$

Now, consider a local orthonormal basis $\{e_i:i=1,...,3\}$ of tangent space at each point of $\mathbb{K}^3$. Taking the inner product of equation (5.16) with P_3 and then setting $P_1=P_3=e_i$ and summing over $i:1\leq i\leq 3$, we get

$$\mathcal{S}(P_2,Df)=\sum_{i=1}^{3}\{g((\nabla_{e_i}\mathcal{Q})P_2,e_i)-g((\nabla_{P_2}\mathcal{Q})e_i,e_i)\}+2X_2\lambda+3\beta\mu\eta(P_2). \quad (5.19)$$

Also, we have from contraction of Bianchi's second identity $div\mathcal{Q}=\frac{1}{2}D\tau$ and the above equation becomes

$$\mathcal{S}(P_2,Df)=-\frac{1}{2}P_2\tau+2X_2\lambda+3\beta\mu\eta(P_2). \quad (5.20)$$

Replacing $P_2\longrightarrow\xi$, and using (v) of Proposition 4.1, we have

$$\xi\tau=4\beta^2\xi f+4\xi\lambda+6\beta\mu. \quad (5.21)$$

Again, tracing (v) of Proposition 4.1, we have

$$\xi\tau=-2\beta\left(\tau+6\beta^2\right). \quad (5.22)$$

After comparing equation (5.21) and (5.22), we get

$$\beta^2\xi f+\xi\lambda=-\frac{\beta}{2}\left(\tau+6\beta^2\right)-\frac{3}{2}\beta\mu, \quad (5.23)$$

using this in equation (5.18), we get

$$\mathcal{S}(P_1,P_2)=\left\{-2\beta^2+\frac{1}{2}\beta\tau+3\beta^3+\frac{1}{2}\beta\mu\right\}g(P_1,P_2)-\left\{\frac{1}{2}\beta\tau+3\beta^3+\frac{1}{2}\beta\mu\right\}\eta(P_1)\eta(P_2). \quad (5.24)$$

Thus, $\mathbb{K}^3$ is a η-Einstein manifold. Which is the complete proof of Theorem 1.3. □

Corollary 5.7. *Consider a β-paraKenmotsu 3-manifold $\mathbb{K}^3$ admitting a gradient almost Ricci soliton. If ξ leaves the scalar curvature invariant, then it is an Einstein manifold.*

Proof. Since $\mu = 0$, for a gradient almost Ricci soliton. If ξ leaves the scalar curvature invariant *i.e.* $\xi\tau = 0$ then $\tau = -6\beta^2$. Now, by using this value of τ in equation (5.24), we get the required result. □

6 Examples

For a β-paraKenmotsu manifold $\mathcal{K}^3$ defined in Example 2.5 we find Riemann curvature tensor field and the Ricci curvature tensor field by using equation (2.10) and (2.11) and are given as:

$$\left.\begin{aligned} \mathcal{R}(U_1,U_2)U_3 &= \mathcal{R}(U_1,U_3)U_2 = \mathcal{R}(U_2,U_3)U_1 = 0,\ \mathcal{R}(U_1,U_3)U_3 = -\frac{1}{4}U_1,\\ -\mathcal{R}(U_1,U_2)U_2 &= \mathcal{R}(U_2,U_1)U_2 = \frac{1}{4}e^zU_1,\ -\mathcal{R}(U_1,U_2)U_1 = \mathcal{R}(U_2,U_1)U_1 = \frac{1}{4}e^zU_2,\\ -\mathcal{R}(U_1,U_3)U_1 &= \mathcal{R}(U_2,U_3)U_2 = \frac{1}{4}e^zU_3,\ \mathcal{R}(U_2,U_3)U_3 = -\frac{1}{4}U_2, \end{aligned}\right\}$$

$$\mathcal{S}(U_1,U_1) = \frac{1}{2}e^z, \quad \mathcal{S}(U_2,U_2) = -\frac{1}{2}e^z, \quad \mathcal{S}(U_3,U_3) = -\frac{1}{2}.$$

Therefore,

(i) For $\lambda = \frac{1}{2}(1+e^z)$ and $\mu = \frac{1}{2}(2+e^z)$, (g,ξ,λ,μ) defines an almost η-RS on $\mathcal{K}^3$.

(ii) For $\lambda = \frac{1}{2}(1+e^z)$ and $\mu = \frac{1}{2}(2+e^z)$, $(g, \text{grad}\,(\text{f}), \lambda, \mu)$ defines a gradient almost η-RS on $\mathcal{K}^3$.

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