Perfect Italian Domination Number of Graphs

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Abstract: In this paper, an upper bound for the perfect Italian domination number of the cartesian product of any two graphs is obtained and the exact value of this parameter for cartesian product of some special graphs are obtained. We have also proved that for any two positive integers a, b there exists a graph G and an induced subgraph H of G such that $\gamma_I^p(G) = a$ and $\gamma_I^p(H) = b$. Relationship of the perfect Italian domination number with the Roman domination number and the perfect domination number of a graph G are obtained and the corresponding realization problems are also solved. We have also obtained the perfect Italian domination number of the Mycielskian of a graph in terms of the perfect domination number of the graph. Some open problems related to this parameters are also included.

1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. If there is no ambiguity in the choice of G, then we write $V(G)$ and $E(G)$ as V and E respectively. A subset $S \subseteq V(G)$ of vertices is called a dominating set if every $v \in V(G)$ is either an element of S or is adjacent to an element of S [12]. The domination number, $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set S is a perfect dominating set if $|N(v) \bigcap S| = 1$ for each $v \in V - S$, where $N(v)$ is the collection of all vertices that are adjacent to the vertex v. The perfect domination number, $\gamma_p(G)$ is the minimum cardinality of a perfect dominating set of G [12].

The weight of a function f defined on the vertex set V of a graph G, $f(V)$ is $\sum_{u \in V} f(u)$. A map $f : V(G) \to \{0, 1, 2\}$ is a Roman dominating function for a graph G if for every vertex v with $f(v) = 0$, there exists at least one vertex $u \in N(v)$ such that $f(u) = 2$. The minimum weight of a Roman dominating function on G is called the Roman domination number of G, $\gamma_R(G)$ [4].

An Italian dominating function, of a graph G is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that for every $v \in V(G)$ with $f(v) = 0, \sum_{u \in N(v)} f(u) \geq 2$, i.e., either v is adjacent to a vertex u with $f(u) = 2$ or to at least two vertices x and y with $f(x) = f(y) = 1$. The Italian domination number of G, $\gamma_I(G)$ is the minimum weight of an Italian dominating function on G [10].

A function $f : V(G) \to \{0, 1, 2\}$ is a perfect Italian dominating function (abbreviated as PID-function) on G if for every vertex $v \in V(G)$ with $f(v) = 0, \sum_{u \in N(v)} f(u) = 2$. The perfect Italian domination number of G, $\gamma_I^p(G)$, is the minimum weight of a PID-function of G. A PID-function of G with weight $\gamma_I^p(G)$ is called a $\gamma_I^p(G)$-function of G [11].

We also denote a function $f : V(G) \to \{0, 1, 2\}$ as $f = (V_0^f, V_1^f, V_2^f)$ or simply (V_0, V_1, V_2), where V_i is the set of all vertices which are assigned the value i for $i = 0, 1, 2$. For any subgraph H of G, the sum of the weights of the vertices of H is denoted by $f(H)$. i.e., $f(H) = \sum_{u \in V(H)} f(u)$. In [9] the authors characterize the graphs G with $\gamma_I^p(G)$ equal to 2

and 3 and determined the exact value of the parameter for several simple structured graphs. It is also proved that it is NP-complete to decide whether a given bipartite graph admits a perfect Italian dominating function of weight k. The perfect Italian domination number of Sieriński graphs and generalized Sierpiński graphs are studied in [8] and [7] respectively.

For disjoint graphs G and H, the join $G + H$ has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv : u \in V(G) \; and \; v \in V(H)\}$ [12]. The Cartesian product of two graphs G and H, $G \square H$ has vertex $V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $v_1 = v_2$ and $u_1 u_2 \in E(G)$ [5]. It is a simple observation that $G \square H$ can be partitioned as $|V(H)|$ copies of G and $|V(G)|$ copies of H.

The Mycielskian of a graph G, $M(G)$ is the graph with vertex set $V(G) \cup V'(G) \cup \{w\}$ where $V'(G) = \{u_i : v_i \in V(G)\}$ and edge set $E(G) \cup \{v_i u_j : v_i v_j \in E(G)\} \cup \{wu_i : u_i \in V'(G)\}$. The double Roman domination number and the Italian domination number of the Mycielskian of a graph have been studied in [2] and [6].

The following observations are simple.

Observation 1. For a graph with no edge and n vertices, $\gamma_I^p(G) = n$.

Observation 2. For any complete bipartite graph $K_{p,q}$,

$$\gamma_I^p(K_{p,q}) = \begin{cases} 4, \; p, q \geq 3, \\ 2, \; otherwise. \end{cases}$$

Observation 3. For complete graph K_m, $\gamma_I^p(K_m) = 2$.

Observation 4. For every graph G, $\gamma(G) \leq \gamma_I(G) \leq \gamma_I^p(G)$.

Observation 5. Let G be a graph. $\gamma_I^p(G) = 2$ if and only if $G = H_1 \vee H_2$ where $H_1 = K_1, K_2$ or $2K_1$.

Proof. If $\gamma_I^p(G) = 2$, in a PID-function of G, either a vertex v is assigned the value 2 and all the remaining vertices are adjacent to v or two vertices v and w are assigned the value 1 and all the remaining vertices are adjacent to both v and w. The adjacency between v and w is optional. Therefore, G is $K_1 \vee H_2$, $K_2 \vee H_2$ or $2K_1 \vee H_2$. The converse is a simple observation. $\qquad \square$

All notations and terminology not mentioned here are from [3].

2 Cartesian Product

In this section, we have obtained an upper bound for the Cartesian product of two graphs in terms of the original graph. Exact values for some special classes are also obtained.

Theorem 2.1. *For any graphs G and H*

$$\gamma_I^p(G \square H) \leq min\{|V(H)|\gamma_I^p(G), |V(G)|\gamma_I^p(H)\}.$$

Proof. Let g be γ_I^p-function of G. Let $f : V(G) \times V(H) \to \{0, 1, 2\}$ be γ_I^p-function of $G \square H$ defined by $f(u, v) = g(u)$, for every $u \in V(G)$ and $v \in V(H)$. Then a vertex (u, v) has weight zero, then it has neighbors with weight exactly two and all other vertices which are adjacent to (u, v) has weight zero. Therefore, f is a γ_I^p-function, and $\gamma_I^p(G \square H) \leq |V(H)|\gamma_I^p(G)$. Using the same arguments we can prove that $\gamma_I^p(G \square H) \leq |V(G)|\gamma_I^p(H)$. Therefore,

$$\gamma_I^p(G \square H) \leq min\{|V(H)|\gamma_I^p(G), |V(G)|\gamma_I^p(H)\}.$$

$\qquad \square$

There are examples of pairs of graphs for which equality and strict inequality of the above theorem are attained. For instance, let $G = P_4$ and $H = P_2$. Then $\gamma_I^p(G\Box H) = 4 < 6 = min\{|V(H)|\gamma_I^p(G), |V(G)|\gamma_I^p(H)\}$ and let $G = K_{1,3}$ and $H = P_3$. Then $\gamma_I^p(G\Box H) = 6 = min\{|V(H)|\gamma_I^p(G), |V(G)|\gamma_I^p(H)\}$.

The following theorem proved in [1] is used in the proof of Theorem 2.3

Theorem 2.2. *([1].)* $\gamma_I(P_2\Box P_n) = n$.

Theorem 2.3.

$$\gamma_I^p(P_2\Box P_n) = \begin{cases} n+1; \ if \ n = 1,3,5 \\ n; \ otherwise. \end{cases}$$

Proof. Let $f = (V_0, V_1, V_2)$ be the γ_I^p function of $P_2\Box P_n$. Let $u_1, u_2, u_3, ..., u_n$ be the vertices of the first copy of P_n and $v_1, v_2, v_3, ..., v_n$ be the vertices of the second copy of P_n. We know that $\gamma_I(P_2\Box P_n) = n$ and by observation 4, $\gamma_I(G) \leq \gamma_I^p(G)$. Therefore, $\gamma_I^p(P_2\Box P_n) \geq n$.

When $n = 2$, $P_2\Box P_n$ is C_4 and $\gamma_I^p(C_4) = 2$.

When $n = 3$, define f as follows.

$$f(u) = \begin{cases} 2; \ u = v_3, \\ 1; \ u = u_1, v_2, \\ 0; \ otherwise. \end{cases}$$

Then $\gamma_I^p(P_2\Box P_n) = 4$.

When $n = 4$, define f as follows.

$$f(u) = \begin{cases} 1; \ u = u_2, u_3, v_1, v_4, \\ 0; \ otherwise. \end{cases}$$

Then $\gamma_I^p(P_2\Box P_n) = 4$.

When $n = 5$, define f as follows.

$$f(u) = \begin{cases} 2; \ u = u_5 \\ 1; \ u = u_1, u_4, v_2, v_3, \\ 0; \ otherwise. \end{cases}$$

Then $\gamma_I^p(P_2\Box P_n) = 6$.

When $n = 6$, define f as follows.

$$f(u) = \begin{cases} 1; \ u = u_2, u_3, u_6, v_1, v_4, v_5, \\ 0; \ otherwise. \end{cases}$$

Then $\gamma_I^p(P_2\Box P_n) = 6$.

When $n \geq 7$, and n is odd, define f as follows.

$$f(u) = \begin{cases} 2; \ u = u_j, \ j \equiv 4(mod6) \\ 1; \ u = u_j, \ j \equiv 1(mod6), \\ \quad\ u = v_j, \ j \equiv 0(mod2); \\ 0; \ otherwise. \end{cases}$$

When n is even, define f as follows

$$f(u) = \begin{cases} 1; & u = u_j, \ j \equiv 0, 1 \pmod{4}, \\ & u = v_j, \ j \equiv 2, 3 \pmod{4}; \\ 0; & otherwise. \end{cases}$$

Clearly, in each case, f is a γ_I^p-function and $f(V) = n$. Hence the theorem.

$\square$

Theorem 2.4. *If m and n are positive integers then*

$$\gamma_I^p(K_m \square K_n) = \begin{cases} n; & if \ m = n \\ min\{2m, 2n\}; & otherwise. \end{cases}$$

Proof. Let $f = (V_0, V_1, V_2)$ be the γ_I^p-function of $K_m \square K_n$. As we have already mentioned in the introduction $K_m \square K_n$ can be viewed as m rows of K_n and n columns of K_m. Let $u_{i,j}$, $i = 1, 2, ..m$ and $j = 1, 2, ...n$ be the vertices of $K_m \square K_n$.

Case 1: $m = n$.

Define f as follows.

$$f(u_{ij}) = \begin{cases} 1; & i = j, \\ 0; & otherwise. \end{cases}$$

Then $\gamma_I^p(K_n \square K_n) \leq n$.

Claim: Exactly one vertex in each copy of K_n has weight 1.

If possible assume that there exists a copy of K_n in which all vertices have weight 0. Then these vertices are dominated by vertices from corresponding columns. Then each column should have weight 2, i.e., $f(V) = 2n > n$.

If possible assume that there exist a copy of K_n which has weight at least 2. Then either there is a vertex with weight 2 or two vertices with weight 1 each in that row.

Case (a): Let u_{ij} and u_{ik} be the two vertices with weight 1.

Then in the ith row either all other vertices have weight 1 or all other vertices have weight 0. If all other vertices are assigned zero then vertices in the corresponding column is zero. In order to dominate these vertices we have to assign weight 2 in each row. Then $f(V) = 2n > n$. If all other vertices are assigned weight 1, then to dominate any vertex with weight 0 in any other row we have to assign a vertex with weight 1 in each row. Then $f(V) = 2n - 1 > n$.

Case(b): Let u_{ij} be a vertex in i^{th} row that has weight 2.

Similar to case(a), we can prove that, in this case also $f(V) = 2n > n$.
Therefore, weight of each row is one and hence, $\gamma_I^p(K_n \square K_n) = n$.

Case 2: $m \neq n$.

Without loss of generality, let $m < n$. Define f as follows.

$$f(u_{ij}) = \begin{cases} 2; & i = 1, 2, ...m \ and \ j = 1, \\ 0; & otherwise. \end{cases}$$

Then $\gamma_I^p(K_m \square K_n) \leq 2m = min\{2m, 2n\}$.

If for every γ_I^p-function f, $\sum_{j=1}^n f(u_{ij}) = 2$, for each i, then $\gamma_I^p(K_m \square K_n) = 2m$. Therefore, assume that there exists a γ_I^p-function f such that $\sum_{j=1}^n f(u_{ij}) < 2$ for some $i = k$. Therefore, $\sum_{j=1}^n f(u_{kj}) = 0$ or 1.

If $\sum_{j=1}^n f(u_{kj}) = 0$ then to dominate u_{kj} for $j = 1, 2, ...n$, $\sum_{i=1}^{\cup} f(u_{ij}) = 2$ which implies $f(V) = 2n > 2m$, which is a contradiction to the fact that f is a γ_I^p-function.

If $\sum_{j=1}^n f(u_{kj}) = 1$ then there exists l such that $f(u_{kj}) = 0$, if $j \neq l$ and $f(u_{kl}) = 1$. But then to dominate u_{kj}, $j \neq l$, $\sum_{i=1}^m f(u_{ij}) = 1$. i.e., exactly one vertex in each column has weight 1 and all other vertices have weight 0. But number of rows is less than number of columns. Therefore, there are more than one vertex with weight 1 in at least one row, say $i = k'$. But, then $\sum_{j=1}^n f(u_{k'j}) = 2$ or n. If $\sum_{j=1}^n f(u_{k'j}) = 2$ then exactly two vertices in the $(k')^{th}$ row have weight 1 and all others have weight 0. Also, the column containing this 0's must be full of 0's. But this contradicts the fact that $\sum_{i=1}^m f(u_{ij}) = 1$, for all $j \neq l$. Therefore, $\sum_{j=1}^n f(u_{k'j}) = n$. But then $f(V) = n + m - 1 \geq m + 1 + m - 1$ (since, $n \geq m + 1$) $= 2m$.

Therefore, if $m \neq n$ then $\gamma_I^p(K_m \square K_n) = 2m$, where $m < n$.

$\square$

3   Realization problems

Theorem 3.1. *Given any two positive integers a, $b \geq 3$, there exist a graph G and induced subgraph H of G such that $\gamma_I^p(G) = a$ and $\gamma_I^p(H) = b$.*

Proof. We consider the following three cases.
Case 1: $b \leq a$.

Let $G = P_{2a-1}$ and $H = P_{2b-1}$. Then $\gamma_I^p(G) = \lceil \frac{2(a-1)+1}{2} \rceil = a$, and $\gamma_I^p(H) = \lceil \frac{2(b-1)+1}{2} \rceil = b$.

Case 2: $b > a$.

Let $v_1, v_2, ...v_{2b-1}$ be a path on $2b - 1$ vertices. Construct G as follows. Let u and v be two vertices adjacent to $v_{2a-3}, v_{2a-2}, ...v_{2b-1}$ and let v_{2a-4} be adjacent to v alone. Clearly, $\gamma_I^p(G) = \lceil \frac{2a-5+1}{2} \rceil + 2 = a$. Also $H = P_{2b-1}$ is an induced subgraph and $\gamma_I^p(H) = b$. $\square$

Lemma 3.2. *For any graph G, $\gamma_R(G) \leq 2\gamma_I^p(G) - 1$.*

Proof. Let $f = (V_0^f, V_1^f, V_2^f)$ be a γ_I^p-function of G. Let $u \in V_1^f$. Define $g = (V_0^f, u, V_1^f \cup V_2^f - u)$. Since every vertex in V_0^f is adjacent to exactly one vertex in V_2^f or two vertices in V_1^f, in g every vertex in V_0^g will have at least one neighbor with weight 2. Therefore, this assignment gives a Roman dominating function. Now $g(V) = 2(|V_1^f \cup V_2^f|) - 1 \leq 2\gamma_I^p(G) - 1$. Therefore, $\gamma_R(G) \leq 2\gamma_I^p(G) - 1$. $\square$

If $\gamma_R(G) = 1$, G is K_1, and $\gamma_I^p(G) = 1$ and vice versa. Similarly, if $\gamma_R(G) = 2$ then G has a universal vertex and $\gamma_I^p(G) = 2$, but the converse is not true. If $\gamma_R(G) = 3$ and $\gamma_I^p(G) = 2$, then let $G = K_{2,n}$.

Theorem 3.3. *Given $a, b \geq 3$ such that $a \leq 2b - 1$, then there exists a graph G such that $\gamma_R(G) = a$ and $\gamma_I^p(G) = b$.*

Proof. We consider the following cases.

Case a: $a \geq 3$, $b \geq a + 1$ and a is odd.

Consider $K_p^c \vee P_{b-3}$, where p is arbitrarily large. Attach a vertex u to every vertex of $K_p^c \vee P_{b-3}$ and a pendent vertex v to an end vertex of P_{b-3}. Then $\gamma_R(G) = 3$ where γ_R-function f can be

defined as $f(u) = 2$, $f(v) = 1$ and $f(v_i) = 0$ for all other vertices. Also, $\gamma_I^p(G) = b$ where γ_I^p-function g can be defined as $g(u) = 2$, $g(v) = 1$ and all the vertices of P_{b-3} has weight 1. If we attach a P_3 to the vertex u by an edge then $\gamma_R(G) = 3 + 2 = 5$ and $\gamma_I^p(G) = b + 2$. Similarly, by attaching P_{3k} to the already attached P_3, we can get $\gamma_R(G) = 3 + 2k$ and $\gamma_I^p(G) = b + 2k, b > a$.

Case b: $a \geq 4$, $b \geq a + 1$ and a is even.

Consider $K_p^c \vee P_{b-4}$, where p is arbitrarily large. Attach a vertex u to every vertex of $K_p^c \vee P_{b-4}$ and attach both the vertices p and q of K_2 to an end vertex of P_{b-4}. Then $\gamma_R(G) = 4$, where γ_R-function f can be defined as $f(u) = 2$ and $f(p) = f(q) = 1$. Also, $\gamma_I^p(G) = b$, where γ_I^p-function g can be defined as $g(u) = 2$, $g(p) = g(q) = 1$ and all the vertices of P_{b-4} has weight 1. Attach a P_3 with u by an edge then $\gamma_R(G) = 4 + 2$ and $\gamma_I^p(G) = b + 2$. Similarly, by attaching P_{3k} as in the previous case we can get $\gamma_R(G) = 4 + 2k$ and $\gamma_I^p(G) = b + 2k, b > a$.

Case c: $a = b$ and a is odd.

Consider P_{2a-1}. Let $v_1, v_2, v_3, ..., v_{2a-1}$ be the vertices of P_{2a-1}. Let u be a vertex which is attached to $v_2, v_4, v_6, ...v_{2a-2}$ and also v_1 and v_{2a-1}. Then $\gamma_R(G) = a$, where γ_R-function f can be defined as $f(u) = 2$, $f(v_3) = f(v_5) = f(v_7) = ... = f(v_{2a-3}) = 1$, $f(v) = 0$, for all other vertices and $\gamma_I^p(G) = a$, where γ_I^p-function g can be defined as $g(v_1) = g(v_3) = g(v_5) = ... = g(v_{2a-1}) = 1$, $g(v) = 0$, for all other vertices. In particular, when $a = 5$ the graph is given in Figure 1.
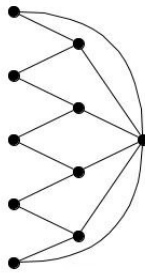

Figure 1. A graph with $a = b = 5$

Similarly, we can construct all graphs with $\gamma_R(G) = \gamma_I^p(G)$ when $\gamma_R(G)$ is odd. So we have constructed all graphs with $a \leq b$.

Case d: $a > b$.

Let G be the graph constructed as follows. Let $v_1, v_2, , ..., v_b$ be a set of independent vertices. Corresponding to every pair $(v_i, v_j), i \neq j$ let u_{ij} be a vertex adjacent to v_i and v_j alone. Then $\gamma_I^p(G) = b$, where $f(v_i) = 1$, for all $i = 1, 2, 3, ..., b$ and $f(u_{ij}) = 0$, for all $i, j \in \{1, 2, 3, ..b\}$ and $i \neq j$ is a γ_I^p-function of G. But $\gamma_R(G) = 2b - 1$, where $g(v_i) = 2$, for $i = 1, 2, 3, ..., b - 1$ and $g(v_b) = 1$ is a γ_R-function. In particular, when $a = 7$ and $b = 4$, the graph is given in Figure 2.
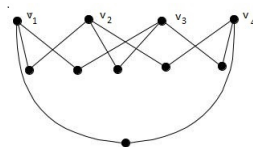

Figure 2. A graph with $a = 7$ and $b = 4$

Now, if we delete one vertex from the ${}^bC_2$ vertices, $\gamma_I^p(G)$ will not change, whereas $\gamma_R(G)$ reduces by 1. (Note that $h(v_k) = 2$ for all $k \in \{1, 2, 3, ..., b\} \setminus \{i\}$ and $h(v_i) = 1$ is a $\gamma_R(G)$-function of G). Proceeding like this we can reduce $\gamma_R(G)$ up to $\gamma_I^p + 1$. Hence the theorem.

<div align="right">□</div>

4   Relation with perfect domination number

In this section, we study the relationship between perfect Italian domination number and perfect domination number of a graph. The following are simple observations.

Observation 6: $\gamma_I^p(G) \leq 2\gamma_p(G)$.

Proof. Let P be a γ_p-set of G. Then

$$f(v) = \begin{cases} 2; & if\ v \in P, \\ 0; & otherwise. \end{cases}$$

is a PID-function. Therefore, $\gamma_I^p(G) \leq 2\gamma_p(G)$.

<div align="right">□</div>

Observation 7: If there exists a γ_I^p-function of G such that $V_1 = \phi$ then $\gamma_I^p(G) = 2\gamma_p(G)$.

Proof. If there exists a γ_I^p-function of G such that $V_1 = \phi$, then the vertices which are assigned the value 2 forms a γ_p-set. Therefore, $\gamma_p(G) \leq \frac{1}{2}\gamma_I^p(G)$. Hence, $\gamma_I^p(G) \geq 2\gamma_p(G)$ and by observation 6, $\gamma_I^p(G) = 2\gamma_p(G)$.

<div align="right">□</div>

We know that the Italian domination number of a graph G lies between $\gamma(G)$ and $2\gamma(G)$. Here we have proved that $\gamma_I^p(G) \leq 2\gamma_p(G)$. It is most expected that $\gamma_p(G)$ serves as a lower bound for $\gamma_I^p(G)$. But this is not true and $\gamma_I^p(G)$ can be arbitrarily smaller than $\gamma_p(G)$. The following theorem settles the corresponding realization problem.

Theorem 4.1. *Given any two positive integers a, and b such that $b \leq 2a$ there exists a graph G such that $\gamma_p(G) = a$ and $\gamma_I^p(G) = b$.*

Proof. Let a and b be any two positive integers such that $b \leq 2a$.

Case 1: $a \leq b \leq 2a - 1$.

Consider k copies of P_5, say $v_{i1}v_{i2}v_{i3}v_{i4}v_{i5}$ for $i = 1, 2, ...k$, where $v_{i1} = v_{j1}$, for all $i, j \in \{1, 2, ..., k\}$. Then $\gamma_I^p(G) = 2k + 1$, where γ_I^p-function f can be defined as $f(v_{i1}) = f(v_{i3}) = f(v_{i5}) = 1$, for all $i = 1, 2, ..., k$ and 0, otherwise. Also $\gamma_p(G) = k + 1$, where the perfect dominating set consist of the vertices v_{i1} and v_{i4}, $i = 1, 2, ..., k$. If we extend the path $v_{11}v_{12}v_{13}v_{14}v_{15}$ to a path of length $2l + 5$, then $\gamma_I^p(G) = 2k + 1 + l$ and $\gamma_p(G) = k + 1 + l$. Let $k = b - a$ and $l = 2a - b - 1$, so that $\gamma_p(G) = a$ and $\gamma_I^p = b$.

Case 2: $b = 2a$.

Let G be the graph P_a : $v_1v_2, ..., v_a$, with atleast two pendent vertices attached to every v_i, $i = 1, 2, ..., a$. Then $\gamma_I^p(G) = 2a$ and $\gamma_p(G) = a$.

Case 3: $a > b$.

Subcase (a): $b - a$ is even.

Let $G = K_2^c + kK_2$. Then $\gamma_I^p(G) = 2$, where vertices of K_2^c is assigned the value 1 and others 0, is the γ_I^p-function of G. But the γ_p-set contains all the vertices of the graph and hence $\gamma_p(G) = 2k + 2$. By attaching a path of length $2l$ to one of the vertices of K_2^c, as in case 1, we get $\gamma_I^p(G) = 2 + l$ and $\gamma_p(G) = 2k + 2 + l$. Let $k = \frac{a-b}{2}$ and $l = b - 2$ so that $\gamma_p(G) = a$ and

$\gamma_I^p = b$.

Subcase (b): $b - a$ is odd.

Let $G = K_2^c + (K_3 \cup kK_2)$. Then as in the previous case, $\gamma_I^p(G) = 2$ and $\gamma_p(G) = 2k + 5$. By attaching a path of length $2l$ to one of the vertices of K_2^c, we get $\gamma_I^p(G) = 2 + l$ and $\gamma_p(G) = 2k + 5 + l$. Let $k = \frac{a-b-3}{2}$ and $l = b - 2$ so that $\gamma_p(G) = a$ and $\gamma_I^p = b$.

$\square$

5 Mycielskian of a graph

In this section, we study the relationship between the perfect Italian domination number of Mycielskian of a graph and the perfect domination number of the graph.

Theorem 5.1. *For a connected graph G, $\gamma_I^p(M(G)) \leq 2\gamma_p(G) + 1$.*

Proof. Let P be a γ_p-set of G. Let $P' = \{u_i, v_i : v_i \in P\} \cup \{w\}$. Define a PID-function as follows.

$$f(v) = \begin{cases} 1; & \text{if } v \in P', \\ 0; & \text{otherwise.} \end{cases}$$

Then $f(M(G)) = 2\gamma_p(G) + 1$. Therefore, $\gamma_I^p(M(G)) \leq 2\gamma_p(G) + 1$.

$\square$

Although, many graph classes satisfy equality, it may be noted that there are infinite number of families of graph which satisfy strict inequality. If we consider the graph $G = K_2^c \bigvee kK_2$, then $\gamma_I^p(M(G)) = 6$ and $\gamma_p(G) = 2k + 2$, so that the difference can be arbitrarily large. An illustration where $k = 2$ is given in Figure 3, in which $f(v_3) = f(v_4) = f(u_1) = f(u_2) = 1$ and $f(w) = 2$ gives a γ_I^p-function of $M(G)$.
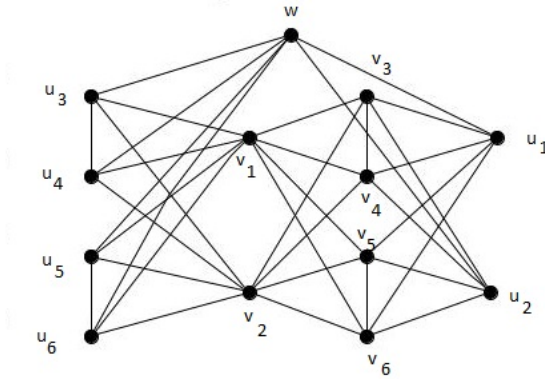

Figure 3. $M(K_2^c \vee 2K_2)$.

Theorem 5.2. *If G has a γ_I^p-function such that $V_1 = \phi$, then $\gamma_I^p(M(G)) = \gamma_I^p(G) + 1 = 2\gamma_p(G) + 1$.*

Proof. Assume that G has a γ_I^p-function f such that $V_1^f = \phi$. We can define a perfect Italian dominating function $g : V(M(G)) \to \{0, 1, 2\}$ as follows.

$$g(v) = \begin{cases} 1; & \text{for } v = v_i \text{ and } u_i \text{ such that } f(v_i) = 2 \text{ and } v = w, \\ 0; & \text{otherwise.} \end{cases}$$

Therefore, $\gamma_I^p(M(G)) \leq \gamma_I^p(G) + 1$.

To prove the reverse inequality, let f be a PID-function of $M(G)$.

Case 1: $\sum_{i=1}^{n} f(u_i) \neq 2$.

Define $g : V(G) \to \{0, 1, 2\}$ as follows.

$$g(v_i) = \begin{cases} 2; \; if \; f(v_i) + f(u_i) = 2, \\ 1; \; if \; f(v_i) + f(u_i) = 1, \\ 0; \; otherwise. \end{cases}$$

Note that $f(v_i) + f(u_i) \leq 2$, for every $i = 1, 2, ..., n$. Then g is a PID-function of G, since $N_G(v_i) = N_{M(G)}(u_i) \bigcap V(G)$.

Case 2: $\sum_{i=1}^{n} f(u_i) = 2$.

In this case, either there exists one vertex u_i with $f(u_i) = 2$ or there exist two vertices u_i and u_j with $f(u_i) = f(u_j) = 1$.

Subcase(a): There exists one vertex u_i with $f(u_i) = 2$.

Without loss of generality let $f(u_1) = 2$ and $f(u_i) = 0$, for all $i = 2, 3, ..., n$. If possible assume that there exists a $v_i \in N(u_1)$ such that $f(v_i) = 0$. But, we have $f(u_i) = 0$ which implies that $\sum f(N(u_i)) = 2$ and hence, $\sum f(N(v_i)) = \sum f(N(u_i)) + f(u_1) - f(w) = 2$. This implies, $f(w) = 2$. Therefore, $\sum_{x \in N(u_i) - \{w\}} f(x) = 0$, for all $i = 2, 3, ..., n$. This implies $f(x) = 0$ for $x \in N(v_i)$, for all $i = 2, 3, ..., n$. But, then $v_i's$ are perfect Italian dominated by u_i. This means v_1 is a universal vertex of G and also $f(M(G)) \geq 4$. But, $g(v_1) = g(u_1) = g(w) = 1$ is a PID-function of $M(G)$ with weight 3, which is a contradiction to the fact that f is a γ_I^p-function of $M(G)$. Therefore, none of the vertices in $V(G)$ is perfect Italian dominated by u_1. Therefore, f restricted to G is a PID-function of G and $f(G) \leq f(M(G)) - 2$. Therefore, $\gamma_I^p(G) \leq \gamma_I^p(M(G)) - 2$ so that, $\gamma_I^p(M(G)) \geq \gamma_I^p(G) + 2$, which is a contradiction to the fact that $\gamma_I^p(M(G)) \leq \gamma_I^p(G) + 1$. Therefore, such a case dose not exist.

Subcase(b): There exist two vertices u_i and u_j with $f(u_i) = f(u_j) = 1$.

Without loss of generality, let $f(u_1) = f(u_2) = 1$ and $f(u_i) = 0$, for all $i = 3, 4, ..., n$. As in the above case, there dose not exist $v_i \in N(u_1) \bigcup N(u_2), i \neq 1, 2$ such that $f(v_i) = 0$. If v_1, $v_2 \notin N(u_1) \bigcup N(u_2)$, then again f/G is a PID-function of G and hence $\gamma_I^p(G) \leq \gamma_I^p(M(G)) - 2$, which is not possible. Therefore, assume that v_1, $v_2 \in N(u_1) \bigcup N(u_2)$. i.e., $v_1 \in N(u_2)$ and $v_2 \in N(u_1)$. If $f(v_1) = f(v_2) = 0$ then define $g : V(G) \to \{0, 1, 2\}$ as follows.

$$g(v_i) = \begin{cases} f(v_i); \; i \neq 1, 2, \\ 1; \; i = 1, \\ 0; \; i = 2. \end{cases}$$

If $f(v_1) = 0$ and $f(v_2) \neq 0$ then define $g : V(G) \to \{0, 1, 2\}$ as follows.

$$g(v_i) = \begin{cases} f(v_i); \; i \neq 2, \\ f(v_2) + 1; i = 2. \end{cases}$$

The case $f(v_1) \neq 0$ and $f(v_2) = 0$ can be delt similarly. If both $f(v_1)$ and $f(v_2)$ are non-zero then f/G is a PID-function, which again leads to a contradiction. In all the cases $\gamma_I^p(G) \leq \gamma_I^p(M(G)) - 1$. Therefore, $\gamma_I^p(M(G)) = \gamma_I^p(G) + 1$.
By Observation 7, we know that $\gamma_I^p(G) = 2\gamma_p(G)$. Therefore, $\gamma_I^p(M(G)) = 2\gamma_p(G) + 1$. □

Corollary 5.3. *For any graph G with a universal vertex, $\gamma_I^p(M(G))) = 3$.*

6 Conclusion and Open Problems

In this paper, we have already given some examples of graphs which satisfy $\gamma_I^p(G) = 2\gamma_p(G)$. Let G be a graph with n vertices and let $\Im = \{H_i : i = 1, 2, ..., n\}$ be a family of n graphs (not necessarily non-isomorphic). We define the corona of G with $\Im$, $G \odot \Im$ as the graph with vertex set $V(G) \cup V(H_i)$, $i = 1, 2, ..., n$ and edge set $E(G) \cup E(H_i) \cup \{v_i u : u \in V(H_i)$, for all $i = 1, 2, ..., n\}$. When $H_i = H$, for all i, $G \odot \Im$ reduces to the usual corona of G and H, $G \odot H$. $G \odot \Im$ satisfies, $\gamma_I^p(G \odot \Im) = 2\gamma_p(G \odot \Im)$, if $|V(H_i)| > 1$ for all $i = 1, 2, ...n$. Any supergraph of the above graph obtained by adding edges between $H_i's$, to some extend, also satisfy the above equality.
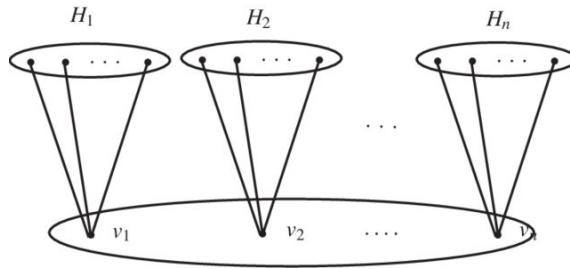

Figure 4. Structure of $G \odot \Im$.

Though we have infinitely many graphs which satisfy this equality, the charecterization problem is still open.

Problem 1: Characterize graphs for which $\gamma_I^p(G) = 2\gamma_p(G)$.

In Theorem 5.2 we have proved that, if G has a γ_I^p-function such that $V_1 = \phi$, then $\gamma_I^p(M(G)) = \gamma_I^p(G) + 1 = 2\gamma_p(G) + 1$. Let G be the graph $P_3 \odot K_1$. Here, $\gamma_p(G) = 3$, $\gamma_I^p(G) = 4$ and $\gamma_I^p(M(G)) = 7 = 2\gamma_p(G) + 1$. But there dose not exist a γ_I^p-function of G in which $V_1 = \phi$. Therefore, the converse of the theorem is not true for the equality $\gamma_I^p(M(G)) = 2\gamma_p(G) + 1$. But we strongly belive that the converse of the Theorem 5.2 is true for $\gamma_I^p(M(G)) = \gamma_I^p(G) + 1$. So we have the following open problems.

Problem 2: Characterize graphs for which $\gamma_I^p(M(G)) = 2\gamma_p(G) + 1$.

Problem 3: Prove the converse of Theorem 5.2 for the equality $\gamma_I^p(M(G)) = \gamma_I^p(G) + 1$. ie; if $\gamma_I^p(M(G)) = \gamma_I^p(G) + 1$, then there exists a γ_I^p-function of G for which $V_1 = \phi$.

We know that, if there exists a γ_I^p-function of G such that $V_1 = \phi$ then $\gamma_I^p(G) = 2\gamma_p(G)$. Therefore, if we can prove Problem 3 and if Class A and Class B are the classes of graphs which satisfies Problem 1 and 2 respectively, then the intersection of Class A and Class B is precisely the collection of graphs for which there exists a γ_I^p-function such that $V_1 = \phi$.

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