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SYMMETRY ANALYSIS OF CONFORMABLE FRACTIONAL COUPLED KDV-BURGERS EQUATION

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Abstract The efficacy of the Lie group technique has been demonstrated in this study by using it to solve the fractional coupled KdV-Burgers problem with conformable fractional time derivative. An investigation has been made to find the infinitesimal generators, Lie algebra, and symmetry groups. Under the obtained group invariant transformations involving arbitrary parameters, reduced coupled ordinary differential equations have been created from the time fractional coupled KdV-Burgers equation. Particular cases corresponding to certain specific values of the coefficients involved and those infinitesimal generators for which the equation can be reduced to ODEs are presented. Moreover, some exact solutions are obtained.

1 Introduction

Though a great rejuvenation of interest in fractional derivative modelling have taken place in past few decades, yet their applications has not been fully uncovered due to the fact that the concept of fractional calculus is unusual. Some of the applications can be found in many diverse areas, for example: theory of viscoelasticity , transference theory, mathematical physics, electric communication of biological systems, production of chapped grooves on metal surfaces, diffusion processes,and damping laws [4, 25, 42, 10]. Inherent fractional order description has been reported in various physical phenomenon such as nonlinear oscillation of earthquake, electromagnetics, acoustics electrochemistry, and phenomena in material science [20, 27, 43, 46, 21, 19]. Fractional models have been used to model non-conservative forces. As a result, because the majority of the physical universe is made up of non-conservative forces, it would be more beneficial to explain them using fractional models rather than integer models. In the theory of dynamical framework control, fractional derivatives and integrals also appear, when the controlled framework as well as the controller is captured by fractional models[27].

In recent years, many researchers have used various numerical and analytical methods to achieve the solutions of fractional differential equations [5, 4, 16, 25, 13, 24, 13]. Methods for finding exact solutions for fractional order non-linear equations have recently attracted a lot of attention in the mathematical community. The Lie group analysis method is well understood as one of the most effective techniques to obtain exact solutions of ordinary and partial differential equations of fractional order [6, 26, 7]. However, a few papers (for examples [16, 3, 12, 14, 17, 18, 41, 9, 45, 40, 15, 44, 28, 8]) are based on this approach in which symmetry transformations along with similarity solutions are discussed for some fractional differential equations containing time fractional derivaive. The time fractional KdV-Burgers equation is one of such nonlinear equations. The integer order model for the equation was initially proposed for the fluid stream in the flexible pipeline and air pockets [1], which was further generalised in the study of turbulence regarded as a simplest dissipation model [2]. A great theoretical and applied significance of the coupled KdV-Burgers equation made the investigation of conformable fractional model of the equation important for solitary wave solutions along with the similarity solutions to check the exactness and dependability of of numerical algorithm. Recently, The Lie symmetry analysis of some comformable fractional partial differential equations were presented in [38, 39]. However the approach has not yet been applied on a coupled equation with conformable fractional derivative. So, our motive here is to study whether the inclusion of fractional parameters will change the properties of the conformable fractional coupled KdV-Burgers equation.

The paper consists of the following sections: After the introduction section which presents a brief sketch of the historical background of the study made here, section 2 provides some preliminary knowledge required for this study. In section 3 infinitesimals and Lie symmetry for the coupled fractional KdV-Burgers are studied. Section 4 consist of the symmetries of the fractional KdV-Burgers equation. Further an investigation for symmetry reduction along with the exact analytical solutions has been performed in section 5 followed by conclusion and references in sec 6 and 7 respectively.

2 Preliminaries

It is worth noting that there are several ways to define the fractional derivative. Various fractional derivative definitions, such as the Caputo, the Riesz, the Grunwald–Letnikov, and the Riemann–Liouville, can be seen in the writing. The Riemann–Liouville and Caputo derivatives are the most well-known. Each fractional derivative has its own set of advantages and disadvantages. [47, 27, 43]. Recently, Khalil et. al. [29] proposed a simple alternative definition of fracional derivative by extending the familiar limit definition. The work carried out in this paper considers conformable type fractional derivatives [29].

Here we present some definitions and basic properties of conformable fractional calculus.

2.1 Conformable Fractional Derivative

For a continuous function $u(t) : (0, \infty) \to \mathbb{R}$, $\alpha \in (0, 1]$ the conformable fractional derivative at a point $t > 0$, is stated as [29]

$$ {}_aD_t^{\alpha}\ u(t) = \lim_{\epsilon \to 0} \frac{u(t + \epsilon(t-a)^{1-\alpha}) - u(t)}{\epsilon}. \tag{2.1}$$

If ${}_aD_t^{\alpha}\ u(t)$ exists, then u is called conformable α-differentiable at $t = a$. Also if $a = 0$ and u is α-differentiable in some $(0, x)$, and $\lim_{t\to 0}{}_0D_t^{\alpha}u(t)$ exists, then ${}_0D_t^{\alpha}u(0) = \lim_{t\to 0}{}_0D_t^{\alpha}u(t)$. We would be using $u_t^{(\alpha)}$ to denote the conformable fractinal derivative ${}_0D_t^{\alpha}u(t)$ throughout this work.

2.2 Conformable Integral of Fractional Order

Consider a continuous real valued funcion $u(t)$, then conformable fractional integral of $u(t)$ of order α is given by [29].

$$ I_t^{\alpha}\ (u)(t) = I\ (t^{\alpha-1}u)(t) = \int_0^t \frac{u(x)}{x^{1-\alpha}}\, dx,\ \ 0 < \alpha \le 1, \tag{2.2}$$

with I as the usual Riemann integral.

2.3 Useful Properties of Conformable Fractional Derivative

The product rule, quotient rule and some other properties for conformable fractional derivative are given below:

(i) $D_t^{\alpha}\,(au(t) + bv(t)) = a(D_t^{\alpha}u)(t) + b(D_t^{\alpha}v)(t),\quad \forall a, b \in \mathbb{R}$, provided $u(t)$ & $v(t)$ are α-differentiable.

(ii) $D_t^{\alpha}(uv)(t) = u(t)D_t^{\alpha}v(t) + D_t^{\alpha}u(t)\,v(t)$.

(iii) $D_t^{\alpha}t^{\beta} = \beta t^{\beta-\alpha},\quad \forall\beta \in \mathbb{R}$.

(iv) $D_t^{\alpha}(C) = 0$, where C is a constant.

(v) $D_t^\alpha\left(\frac{u}{v}\right)(t) = \frac{u(t)D_t^\alpha v(t) - D_t^\alpha u(t)\,v(t)}{v^2(t)}$.

(vi) $D_t^\alpha u(t) = t^{1-\alpha}u'(t)$, given $u(t)$ is differentiable.

For complete description of these properties with their limitations and scope of applications, the reader may refer to [29, 30, 31, 32, 33, 34, 35, 36, 37].

2.4 Invariance Condition for a System of PDEs

A system of PDEs

$$F^\mu(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \partial^2\mathbf{u}, ..., \partial^k\mathbf{u}) = 0, \tag{2.3}$$

with $\mu \leq N$ a positive integer, $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{u} = (u^1, u^2, ..., u^m)$, admits a Lie group of one parameter point transformations if the point transformations leave the system invariant. Let the kth-extended transformation be

$$\tilde{x_i} = x_i + \xi_i(\mathbf{x}, \mathbf{u})\epsilon + o(\epsilon^2) \tag{2.4}$$

$$\tilde{u^\mu} = u^\mu + \eta^\mu(\mathbf{x}, \mathbf{u})\epsilon + o(\epsilon^2), \tag{2.5}$$

$$\tilde{u_i^\mu} = u_i^\mu + \eta_i^{(1)\mu}(\mathbf{x}, \mathbf{u}, \partial\mathbf{u})\epsilon + o(\epsilon^2), \tag{2.6}$$

$\vdots$

$$\tilde{u}^\mu_{i_1 i_2 \ldots i_k} = u^\mu_{i_1 i_2 \ldots i_k} + \eta^{(k)\mu}_{i_1 i_2 \ldots i_k}(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \partial^2\mathbf{u}, ..., \partial^k\mathbf{u})\epsilon + o(\epsilon^2). \tag{2.7}$$

Let

$$V = \xi_i(\mathbf{x}, \mathbf{u})\frac{\partial}{\partial x_i} + \eta^\mu(\mathbf{x}, \mathbf{u})\frac{\partial}{\partial u^\mu}, \tag{2.8}$$

and

$$\begin{aligned} V^{(k)} = \xi_i(\mathbf{x}, \mathbf{u})\frac{\partial}{\partial x_i} + \eta^\mu(\mathbf{x}, \mathbf{u})\frac{\partial}{\partial u^\mu} + \eta_i^{(1)\mu}(\mathbf{x}, \mathbf{u}, \partial\mathbf{u})\frac{\partial}{\partial u_i^\mu} + \\ \eta^{(2)\mu}_{i_1 i_2}(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \partial^2\mathbf{u})\frac{\partial}{\partial u^\mu_{i_1 i_2}} + ... + \eta^{(k)\mu}_{i_1 i_2 \ldots i_k}(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \partial^2\mathbf{u}, ..., \partial^k\mathbf{u})\frac{\partial}{\partial u^\mu_{i_1 i_2 \ldots i_k}}, \end{aligned} \tag{2.9}$$

be the infinitesimal generator and the kth-extended infinitesimal generator of point transformations (2.4)–(2.5), with the extended infinitesimals as

$$\eta_i^{(1)\mu} = D_i\eta^\mu - (D_i\xi_j)u_j^\mu$$

$$\eta^{(k)\mu}_{i_1 i_2 \ldots i_k} = D_{i_k}\eta^{(k-1)\mu}_{i_1 i_2 \ldots i_{k-1}} - (D_{i_k}\xi_j)u^\mu_{i_1, i_2, \ldots, i_{k-1}, i_k},$$

where $i_j \leq n$, $j \leq k$ are positive integers and $k \geq 2$ with D_i as total derivative operator

$$D_i = \frac{\partial}{\partial x_i} + u_i^\mu\frac{\partial}{\partial u^\mu} + u_{ij}^\mu\frac{\partial}{\partial u_j^\mu} + u^\mu_{i i_1 i_2}\frac{\partial}{\partial u^\mu_{i_1 i_2}} + ... + u^\mu_{i i_1 i_2 \ldots i_n}\frac{\partial}{\partial u^\mu_{i_1 i_2 \ldots i_n}} + ...,$$

also the infinitesimals corresponding to the conformable fractional derivative is defined as [39]

$$\eta_\alpha^{1,t} = D_t^\alpha\eta^1 - (D_t^\alpha\xi)u_x - (D_t^\alpha\tau)u_t + (1-\alpha)\tau t^{-\alpha}u_t,$$

and

$$\eta_\alpha^{2,t} = D_t^\alpha\eta^2 - (D_t^\alpha\xi)v_x - (D_t^\alpha\tau)v_t + (1-\alpha)\tau t^{-\alpha}v_t,$$

with D_t^α as the total fractional derivative operator such that $D_t^\alpha = t^{1-\alpha}D_t$ whenever the function is differentiable. Then the point transformations (2.4)–(2.5) are the admitted symmetries for the system (2.3) iff

$$V^{(k)}F^\mu(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \partial^2\mathbf{u}, ..., \partial^k\mathbf{u}) = 0, \tag{2.10}$$

provided $F^\mu(\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \partial^2\mathbf{u}, ..., \partial^k\mathbf{u}) = 0$.

2.5 Method of Invariant Forms

Here we begin by explicitly solving the associated characteristics equations for u, which are given for $\mathbf{u} = \theta(\mathbf{x})$ as

$$\frac{dx_1}{\xi_1(\mathbf{x},\mathbf{u})} = \frac{dx_2}{\xi_2(\mathbf{x},\mathbf{u})} = ... = \frac{dx_n}{\xi_n(\mathbf{x},\mathbf{u})} = \frac{du^1}{\eta^1(\mathbf{x},\mathbf{u})} = \frac{du^2}{\eta^2(\mathbf{x},\mathbf{u})} = ... = \frac{du^m}{\eta^m(\mathbf{x},\mathbf{u})}. \quad (2.11)$$

On solving the characteristic equations (2.11), similarity transformations provide the invariant form as

$$u^\nu(\mathbf{x},\mathbf{u}) = \Phi^\nu(y_1, y_2, ..., y_{n-1}), \quad (2.12)$$

with Φ^ν a differentiable function of its arguments that can take any value, for $\nu = 1, 2, ..., m$. Specifically, these invariant arrangements are found by settling a diminished arrangement of DEs. The independent variables $y_1, y_2, ..., y_{n-1}$ are commonly called similarity variables which can be further used . Then one can find the reduced system of DEs using the obtained invariant forms of the PDE system (2.3).

3 Application of Lie Symmetry Analysis on Conformable Fractional Coupled KdV-Burgers Equation

Here, we made an investigation to find the Lie symmetries of a conformable fractional KdV-Burgers coupled system:

$$u_t^{(\alpha)} + uu_x + av_{xx} + bu_{xxx} = 0,$$

$$v_t^{(\alpha)} + vv_x + cv_{xx} + du_{xxx} = 0, \quad (3.1)$$

where $t > 0,\ \ x \in (0, \infty),\ \ 0 < \alpha < 1$, with a, b, c and d, as arbitrary constants, is examined for explicit exact solutions of various type. Here, $u_t^{(\alpha)}$ and $v_t^{(\alpha)}$ are the time fractional conformable derivatives.

3.1 Evaluation of Symmetries of Fractional Coupled KdV-Burgers Equation

Herein, we present an investigation of the Lie symmetries of fractional coupled KdV-Burgers equation (3.1)

We suppose the admitted symmetries of equation (3.1) in the form

$$\tilde{t} = t + \epsilon\tau + o(\epsilon^2) \quad (3.2)$$

$$\tilde{x} = x + \epsilon\xi + o(\epsilon^2) \quad (3.3)$$

$$\tilde{u} = u + \epsilon\eta^1 + o(\epsilon^2) \quad (3.4)$$

$$\tilde{v} = v + \epsilon\eta^2 + o(\epsilon^2) \quad (3.5)$$

where ξ, τ are the infinitesimals of the transformations for the independent variables and η^1, η^2 are the infinitesimals for the dependent ones, and ϵ the parameter of group.

The associated infinitesimals then provide the vector field

$$V = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta^2\frac{\partial}{\partial v} + \eta^1\frac{\partial}{\partial u}, \quad (3.6)$$

with third order prolongation as

$$pr^{(3)}V = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta^1\frac{\partial}{\partial u} + \eta_\alpha^{1(t)}\frac{\partial}{\partial u_t^\alpha} + \eta^{1(x)}\frac{\partial}{\partial u_x} + \eta^{1(xx)}\frac{\partial}{\partial u_{xx}} + \eta^{1(xxt)}\frac{\partial}{\partial u_{xxt}} + \eta^{1(xxx)}\frac{\partial}{\partial u_{xxx}} + \eta^2\frac{\partial}{\partial v} + \eta_\alpha^{2(t)}\frac{\partial}{\partial v_t^\alpha} + \eta^{2(x)}\frac{\partial}{\partial v_x} + \eta^{2(xx)}\frac{\partial}{\partial v_{xx}} + \eta^{2(xxt)}\frac{\partial}{\partial (v_{xxt})} + \eta^{2(xxx)}\frac{\partial}{\partial v_{xxx}}. \tag{3.7}$$

Invoking the invariance of (3.1) under the prolongation vector field (3.7), the coefficients of the first order of ϵ provides the following relations:

$$\eta_\alpha^{1(t)} + u_x\eta^1 + u\eta^{1(x)} + a\eta^{2(xx)} + b\eta^{1(xxx)} = 0,$$

$$\eta_\alpha^{2(t)} + v_x\eta^2 + v\eta^{2(x)} + c\eta^{2(xx)} + d\eta^{1(xxx)} = 0, \tag{3.8}$$

where $\eta_\alpha^{1(t)}, \eta_\alpha^{2(t)}, \eta^{1(x)}, \eta^{2(x)}, \eta^{2(xx)}$ and $\eta^{1(xxx)}$ are prolonged infinitesimals functioning in the realm of derivatives of various kinds for dependent variables u and v in relation to the independent ones x and t. The next step requires finding the infinitesimals from the invariance conditions, on vanishing the coefficients of different differentials. It leads to several PDEs in τ, ξ, η^1 and η^2. The set of determining equations for the group infinitesimals for the infinitesimals ξ, τ, η^1 and η^2 is derived after some algebraic computations, which after solving provides the following infinitesimals $\xi, \tau, \eta^1, \eta^2$; along with the permissible values of the coefficients in the system (3.1):

$$\xi = k_3\frac{t^\alpha}{\alpha} + k_2, \tag{3.9}$$

$$\tau = k_1 t^{1-\alpha} \tag{3.10}$$

$$\eta^1 = k_3, \tag{3.11}$$

$$\eta^2 = k_3, \tag{3.12}$$

where k_1, k_2, k_3 are arbitrary constants. The fractional coupled KdV-Burgers equation thus attains the below mentioned infinitesimal generators

$$V_1 = t^{1-\alpha}\frac{\partial}{\partial t}$$

$$V_2 = \frac{\partial}{\partial x}$$

$$V_3 = \frac{t^\alpha}{\alpha}\frac{\partial}{\partial x} + \frac{\partial}{\partial u} + \frac{\partial}{\partial v}. \tag{3.13}$$

Subsequently, the Lie algebra of associated infinitesimal symmetries of the fractional coupled KdV-Burgers equation is spanned by the set $\{V_1, V_2, V_3,\}$ of vector fields along with the commutator table as follows:

Table 1. Commutator Table

	V_1	V_2	V_3
V_1	0	0	V_2
V_2	0	0	0
V_3	$-V_2$	0	0

The commutator table of Lie algebra determined by $V_1; V_2; V_3$, shows that $V_1; V_2; V_3$, is closed under the Lie bracket.

4 Symmetry Groups Admitted by Conformable Fractional Coupled KdV-Burgers Equation

The one-parameter symmetry groups are discussed in this section corresponding to various infinitesimal generators. Some exact solutions from a known solution of coupled KdV-Burgers equation could be obtained using these symmetries. The initial value problems:

$$\frac{d\tilde{t}}{d\epsilon} = \tau \tag{4.1}$$

$$\frac{d\tilde{x}}{d\epsilon} = \xi \tag{4.2}$$

$$\frac{d\tilde{u}}{d\epsilon} = \eta^1 \tag{4.3}$$

$$\frac{d\tilde{v}}{d\epsilon} = \eta^2 \tag{4.4}$$

subject to the conditions that at $\epsilon = 0$

$$\tilde{x} = x \tag{4.5}$$

$$\tilde{t} = t \tag{4.6}$$

$$\tilde{u} = u \tag{4.7}$$

$$\tilde{v} = v \tag{4.8}$$

provides the admitted Lie symmetry groups. We examine the reduced systems of ODEs for the following vector fields:
(i) $V_1 = t^{1-\alpha}\frac{\partial}{\partial t}$,
(ii) $V_2 = \frac{\partial}{\partial x}$,
(iii) $V_3 = \frac{t^\alpha}{\alpha}\frac{\partial}{\partial x} + \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$,
(iv) $V_4 = V_2 + V_3 = (\frac{t^\alpha}{\alpha} + 1)\frac{\partial}{\partial x} + \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$,
(v) $V_5 = V_1 + V_3 = t^{1-\alpha}\frac{\partial}{\partial t} + \frac{t^\alpha}{\alpha}\frac{\partial}{\partial x} + \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$,
(vi) $V_6 = cV_2 + kV_1 = c\frac{\partial}{\partial x} + kt^{1-\alpha}\frac{\partial}{\partial t}$
(vii) $V_7 = V_1 + V_2 + V_3 = t^{1-\alpha}\frac{\partial}{\partial t} + (\frac{t^\alpha}{\alpha} + 1)\frac{\partial}{\partial x} + \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$,
with r and s as arbitrary constant parameters.

We acquire the following one parameter groups $g_i(\epsilon)$ created by $V_i, 1 \leq i \leq 7$ on exponentiating the obtained infinitesimal symmetries of equation (3.1) mapping the vector (x,t,u,v) to the vectors as below:

$$g_1 : \left(x, t + \epsilon t^{1-\alpha}, u, v\right)$$
$$g_2 : (x + \epsilon, t, u, v)$$
$$g_3 : \left(x + \epsilon\frac{t^\alpha}{\alpha}, t, u + \epsilon, v + \epsilon\right)$$
$$g_4 : \left(x + \epsilon\left(\frac{t^\alpha}{\alpha} + 1\right), t, u + \epsilon, v + \epsilon\right)$$
$$g_5 : \left(x + \epsilon\frac{t^\alpha}{\alpha}, t + \epsilon t^{1-\alpha}, u + \epsilon, v + \epsilon\right)$$
$$g_6 : \left(x + c\epsilon, t + k\epsilon t^{1-\alpha}, u, v\right)$$
$$g_7 : \left(x + \epsilon\left(\frac{t^\alpha}{\alpha} + 1\right), t + \epsilon t^{1-\alpha}, u + \epsilon, v + \epsilon\right),$$

where g_3; g_4; g_5; g_7 representing Galilean transformations, g_1 is a time translation, g_2 as a translation of space variable, and with ϵ as an arbitrary constant, g_6 is a space-time translation.

5 Symmetry Reduction of the Fractional Coupled KdV-Burgers System With Some Exact Solutions

Here utilising the invariant form method as given in section 2.5, we perform the similarity reduction on the coupled fractional KdV-Burgers equation corresponding to some of the infinitesimal generators. It is worth mentioning here that an attempt was made to derive a further system of ODEs of lower order through Lie group method, however, in almost all the cases, the symmetries obtained turned out to be the trivial ones. Therefore, we focus on attempting some specific sorts of exact explicit solutions for the reduced system of ODEs.

Vector field (i) For the infinitesimal generator

$$V_1 = t^{1-\alpha}\frac{\partial}{\partial t},$$

the similarity transformations, as well as the form of similarity solutions, are obtained as $X(x,t) = x, \quad u = \phi(X)$ and $\quad v = \psi(X)$, This reduces the fractional coupled KdV-Burgers system to the coupled system

$$b\phi'''(X) + a\psi''(X) + \phi'(X)\phi(X) = 0 \tag{5.1}$$

$$d\phi'''(X) + \psi'(X)\psi(X) + c\psi''(X) = 0. \tag{5.2}$$

To solve the reduced system, we look for a special form of solution as

$$\phi(X) = A_0 + A_1X + A_2X^2 + A_3\frac{1}{X} + A_4\frac{1}{X^2},$$

$$\psi(X) = B_0 + B_1X + B_2X^2 + B_3\frac{1}{X} + B_4\frac{1}{X^2},$$

with A_i, B_i, $i = 0, 1, ..., 4$ as arbitrary constants.

Substituting these expressions for ϕ and ψ in the reduced system, we arrive at a set of algebraic equations that, when solved, yield an exact solution to the coupled KdV-Burgers problem (3.1) in the form of

$$u(x,t) = \frac{2x\sqrt{ac} - 12b}{x^2} \tag{5.3}$$

$$v(x,t) = \frac{12\sqrt{bd} + 2cx}{x^2}. \tag{5.4}$$

Vector field (ii) Here the infinitesimal generator

$$V_2 = \frac{\partial}{\partial x},$$

comes into play and gives the following form of similarity variables along with the similarity solution:

$T(x,t) = t, \quad u(x,t) = \phi(T)$ and $\quad v(x,t) = \psi(T)$.

The reduced form of the fractional coupled KdV-Burgers equation is then

$$\phi'(T) = 0 \tag{5.5}$$

$$\psi'(T) = 0, \tag{5.6}$$

with solution as a simple one, which is stated as

$$u(x,t) = K_1, \tag{5.7}$$

$$v(x,t) = K_2, \tag{5.8}$$

where K_1 and K_2 are arbitrary constants.

Vector field (iii) The generator

$$V_3 = \frac{t^\alpha}{\alpha}\frac{\partial}{\partial x} + \frac{\partial}{\partial u} + \frac{\partial}{\partial v},$$

we get the following form of similarity variables along with the similarity solution $T(x,t) = t, \quad u(x,t) = \dfrac{(x-\zeta(T))\alpha}{T^\alpha}$ and $\quad v(x,t) = \dfrac{(x-\lambda(T))\alpha}{T^\alpha}$.
The reduced form is obtained as

$$\zeta_T^\alpha = 0 \tag{5.9}$$

$$\lambda_T^\alpha = 0, \tag{5.10}$$

which gives the solution as

$$u(x,t) = \frac{(x-K_3)\alpha}{t^\alpha}, \tag{5.11}$$

$$v(x,t) = \frac{(x-K_4)\alpha}{t^\alpha}, \tag{5.12}$$

with K_3, K_4 as arbitrary constants.

Vector field (iv) Corresponding to the generator

$$V_4 = \left(\frac{t^\alpha}{\alpha} + 1\right)\frac{\partial}{\partial x} + \frac{\partial}{\partial u} + \frac{\partial}{\partial v},$$

we get the following form of similarity variables along with the similarity solution $T(x,t) = t, \quad u = \dfrac{(x-\Phi(T))}{\dfrac{T^\alpha}{\alpha}+1}$ and $\quad v = \dfrac{(x-\Psi(T))}{\dfrac{T^\alpha}{\alpha}+1}$.
The reduced form is obtained as

$$\Phi_T^\alpha = 0 \tag{5.13}$$

$$\Psi_T^\alpha = 0, \tag{5.14}$$

which gives the solution as

$$u = \frac{(x-K_5)\alpha}{\frac{t^\alpha}{\alpha}+1}, \tag{5.15}$$

$$v = \frac{(x-K_6)\alpha}{\frac{t^\alpha}{\alpha}+1}, \tag{5.16}$$

where K_5 and K_6 are arbitrary constants.

Vector field (v) Corresponding to the generator

$$V_5 = t^{1-\alpha}\frac{\partial}{\partial t} + \frac{t^\alpha}{\alpha}\frac{\partial}{\partial x} + \frac{\partial}{\partial u} + \frac{\partial}{\partial v},$$

we get the following form of similarity variables along with the similarity solution $X(x,t) = \alpha x - \dfrac{t^{2\alpha}}{2\alpha}, \quad u(x,t) = \dfrac{t^\alpha}{\alpha} - \phi(X)$, and $\quad v(x,t) = \dfrac{t^\alpha}{\alpha} - \psi(X)$.
The reduced form is obtained as

$$1 + \alpha\phi'(X)\phi(X) - a\alpha^2\psi''(X) - b\alpha^3\phi'''(X) = 0 \tag{5.17}$$

$$1 + \alpha\psi'(X)\psi(X) - c\alpha^2\psi''(X) - d\alpha^3\phi'''(X) = 0. \tag{5.18}$$

Vector field (vi) For the infinitesimal generator

$$V_6 = C\frac{\partial}{\partial x} + Kt^{1-\alpha}\frac{\partial}{\partial t},$$

the similarity transformations along with the form of similarity solutions is obtained as
Case(i) for $\alpha \neq 1$ $X = Kx - \frac{C}{\alpha}$, $\quad u = \phi(X)$ and $\quad v = \psi(X)$, This reduces the fractional coupled KdV-Burgers equation to the coupled equations

$$bK^2\phi'''(X) + \phi'(X)\phi(X) + aK\psi''(X) = 0 \tag{5.19}$$

$$dK^2\phi'''(X) + \psi'(X)\psi(X) + cK\psi''(X) = 0. \tag{5.20}$$

To solve the reduced system, we look for a special form of solution as

$$\phi(X) = A_0 + A_1X + A_2X^2 + A_3\frac{1}{X} + A_4\frac{1}{X^2},$$

$$\psi(X) = B_0 + B_1X + B_2X^2 + B_3\frac{1}{X} + B_4\frac{1}{X^2},$$

with A_i, B_i, $i = 0, 1, ..., 4$ as arbitrary constants.
Substituting these expressions for ϕ and ψ in the reduced system, we arrive at a system of algebraic equations, which on solving provide an exact solution to the coupled system (3.1) as

$$u(x,t) = \frac{2Kx\sqrt{ac} - 12bK^2}{x^2} \tag{5.21}$$

$$v(x,t) = \frac{12K^2\sqrt{b_1 d} - 2Kcx}{x^2}, \tag{5.22}$$

where $b_1 = -b$.

Case(ii) for $\alpha = 1$, $X = Kx - Ct$, $\quad u = \phi(X)$ and $\quad v = \psi(X)$, This reduces the fractional coupled KdV-Burgers equation to the coupled equations

$$-C\phi' + bK^3\phi'''(X) + K\phi'(X)\phi(X) + aK^2\psi''(X) = 0 \tag{5.23}$$

$$-C\psi' + dK^3\phi'''(X) + K\psi'(X)\psi(X) + cK^2\psi''(X) = 0. \tag{5.24}$$

This is same as obtained in [48] for integer order model.

Vector field (vii) Corresponding to the generator

$$V_7 = t^{1-\alpha}\frac{\partial}{\partial t} + \left(\frac{t^\alpha}{\alpha} + 1\right)\frac{\partial}{\partial x} + \frac{\partial}{\partial u} + \frac{\partial}{\partial v},$$

we get the following form of similarity variables along with the similarity solution
$Z(x,t) = x - \left(\frac{t^{2\alpha}}{2\alpha^2} - \frac{t^\alpha}{\alpha}\right)$, $\quad u = \Phi(Z) + \frac{t^\alpha}{\alpha}$, $\quad v = \Psi(Z) + \frac{t^\alpha}{\alpha}$,
and reduced form as

$$1 - a\alpha^2\Psi''(Z) - b\alpha^3\Phi'''(Z) + \Phi'(Z) + \alpha\Phi'(Z)\Phi(Z) = 0 \tag{5.25}$$

$$1 - c\alpha^2\Psi''(Z) - d\alpha^3\Phi'''(Z) + \Psi'(Z) + \alpha\Psi'(Z)\Psi(Z) = 0. \tag{5.26}$$

Further the one parameter point transformations corresponding to each of the infinitesimal generator can be used to generate more solutions to the fractional coupled KdV-Burgers equation.

6 Conclusion

Group invariance properties and reductions of conformable fractional coupled KdV-Burgers system is presented using the Lie group method of infinitesimal transformations. The infinitesimal generators along with commutation table of Lie algebra admitted by the fractional coupled KdV-Burgers system in comformable sense are obtained. Corresponding to various linear combinations of the infinitesimal generators, the Lie groups of transformations for the conformable fractional coupled KdV-Burgers problem are examined. Further, it is shown that the fractional coupled KdV-Burgers system reduces to coupled nonlinear ODEs. The reduced equations are further examined in order to achieve certain exact solutions.

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