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Two reduction formulas for the Srivastava-Daoust double hypergeometric function

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Abstract We derive two double-series identities involving a bounded sequence employing the Gessel-Stanton and Andrews summation theorems for terminating ${}_3F_2$ hypergeometric series with arguments 4/3 and 3/4, respectively. Using these double-series identities, we establish two reduction formulas for the Srivastava-Daoust double hypergeometric function with arguments $z, -4z/3$ and $z, -3z/4$ expressed in terms of a single generalised hypergeometric function of argument proportional to z^3.

1 Introduction and Preliminaries

We use the following standard notation: $\mathbf{Z}_0^- := \{0, -1, -2, -3, \cdots\}$. and the symbols $\mathbf{C}$, $\mathbf{R}$ for the sets of complex and real numbers, respectively. The Pochhammer symbol (or the *shifted* factorial) is given by $(\alpha)_n = \alpha(\alpha+1)\dots(\alpha+n-1)$= $\Gamma(\alpha+n)/\Gamma(\alpha)$, it being understood conventionally that $(0)_0 = 1$. In what follows we shall adopt the usual convention of writing the sequence $(\alpha_1, \alpha_2, \dots, \alpha_p)$ simply by (α_p).

The generalized hypergeometric function ${}_pF_q(z)$ is defined by

$$ {}_pF_q\left[\begin{matrix}(\alpha_p)\\(\beta_q)\end{matrix};z\right] = {}_pF_q\left[\begin{matrix}\alpha_1,\alpha_2,\dots,\alpha_p\\ \beta_1,\beta_2,\dots,\beta_q\end{matrix};z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\dots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\dots(\beta_q)_n}\frac{z^n}{n!}, \tag{1.1}$$

where p and q are non-negative integers and the variable $z \in \mathbf{C}$. The numerator parameters $\alpha_1, \alpha_2, \dots, \alpha_p$ and the denominator parameters $\beta_1, \beta_2, \dots, \beta_q$ can, in general, take on complex values, provided that

$$\beta_j \neq 0, -1, -2, \dots, \quad (j = 1, 2, \dots, q).$$

Assuming that none of the numerator and denominator parameters is zero or a negative integer, the ${}_pF_q(z)$ function defined by equation (1.1) converges for $|z| < \infty$ $(p \le q)$, $|z| < 1$ $(p = q+1)$ and $|z| = 1$ $(p = q+1$ and $\Re(s) > 0)$, where s is the parametric excess defined by $s := \sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j$.

In [9, p.199], Srivastava and Daoust defined a generalization of the Kampé de Fériet function [2, p.150] by means of the double hypergeometric series (see also [5] and [6]):

$$F^{A:\,B;\,B'}_{C:\,D;\,D'}\left(\begin{matrix}[(\alpha_A):\vartheta,\varphi]:[(\beta_B):\psi];[(\beta'_{B'}):\psi'];\\ [(\gamma_C):\xi,\varepsilon]:[(\delta_D):\eta];[(\delta'_{D'}):\eta'];\end{matrix}x,y\right)$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{A}(\alpha_j)_{m\vartheta_j+n\varphi_j}\prod_{j=1}^{B}(\beta_j)_{m\psi_j}\prod_{j=1}^{B'}(\beta'_j)_{n\psi'_j}}{\prod_{j=1}^{C}(\gamma_j)_{m\xi_j+n\varepsilon_j}\prod_{j=1}^{D}(\delta_j)_{m\eta_j}\prod_{j=1}^{D'}(\delta'_j)_{n\eta'_j}}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{1.2}$$

where the quantities

$$\begin{cases} \vartheta_1, ..., \vartheta_A;\ \varphi_1, ..., \varphi_A;\ \psi_1, ..., \psi_B;\ \psi'_1, ..., \psi'_{B'};\ \xi_1, ..., \xi_C; \\ \quad \varepsilon_1, ..., \varepsilon_C;\ \eta_1, ..., \eta_D;\ \eta'_1, ..., \eta'_{D'} \end{cases}$$

are real and positive. The double power series in (1.2) converges for all complex values of x and y when $\Delta_1 > 0$, $\Delta_2 > 0$; for suitably constrained values of $|x|$ and $|y|$ when $\Delta_1 = \Delta_2 = 0$; and diverges (except in the trivial case $x = y = 0$) when $\Delta_1 < 0$, $\Delta_2 < 0$, where

$$\Delta_1 = 1 + \sum_{j=1}^{C} \xi_j + \sum_{j=1}^{D} \eta_j - \sum_{j=1}^{A} \vartheta_j - \sum_{j=1}^{B} \psi_j, \quad \Delta_2 = 1 + \sum_{j=1}^{C} \varepsilon_j + \sum_{j=1}^{D'} \eta'_j - \sum_{j=1}^{A} \varphi_j - \sum_{j=1}^{B'} \psi'_j.$$

Motivated by the work of Srivastava *et al.* [7, 8, 10], we derive two double series identities involving a bounded sequence of arbitrary complex numbers in Section 2 by making use of two known summation theorems for the terminating ${}_3F_2$ series of arguments $\frac{4}{3}$ and $\frac{3}{4}$. For non-negative integer n, these are given by the Gessel-Stanton summation theorem [3, Eq. (5.21)]

$${}_3F_2\left[\begin{matrix} -n, 3a+\frac{1}{2}, 3a+1 \\ 6a+1, 2a+1-\frac{1}{3}n \end{matrix}; \frac{4}{3}\right] = \begin{cases} 0 & n \neq 3m \\ \dfrac{(\frac{1}{3})_m(\frac{2}{3})_m}{(2a+1)_m(-2a)_m} & n = 3m \end{cases} \tag{1.3}$$

and the Andrews theorem [1, Eq. (1.12)] (see also [3, Eq. (1.1)])

$${}_3F_2\left[\begin{matrix} -n, a, 3a+n \\ \frac{3}{2}a, \frac{3}{2}a+\frac{1}{2} \end{matrix}; \frac{3}{4}\right] = \begin{cases} 0 & n \neq 3m \\ \dfrac{(3m)!(a+1)_m}{m!\,(3a+1)_{3m}} & n = 3m, \end{cases} \tag{1.4}$$

where $m = 0, 1, 2, \ldots$.

We now present some preliminary results necessary for our investigation. First, we state Cauchy's double series identity [4, p. 56]

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Theta(m,n) = \sum_{m=0}^{\infty}\sum_{n=0}^{m} \Theta(m-n,n)\ , \tag{1.5}$$

provided that the associated double series are absolutely convergent. We also have the following identities involving the Pochhammer symbol:

$$(-n)_r = \frac{(-1)^r n!}{(n-r)!} \qquad (0 \leq r \leq n), \tag{1.6}$$

and

$$(a)_{3n} = 3^{3n}\left(\frac{a}{3}\right)_n \left(\frac{a+1}{3}\right)_n \left(\frac{a+2}{3}\right)_n . \tag{1.7}$$

It should be observed that throughout we tacitly exclude any values of the parameters and arguments in Sections 2 and 3 leading to results that do not make sense.

2 Two double-series identities

In this section, we derive two double-series identities involving a bounded sequence The first identity takes the following form:

Theorem 2.1. *Let* $\{\Psi(\mu)\}_{\mu=1}^{\infty}$ *be a bounded sequence of complex (or real) numbers such that* $\Psi(0) \neq 0$. *Then, the following double-series identity holds true:*

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \Psi(n+r)\, \frac{(2a+1)_{-\frac{n}{3}-\frac{r}{3}}(3a+\frac{1}{2})_r(3a+1)_r}{(2a+1)_{\frac{2r}{3}-\frac{n}{3}}(6a+1)_r}\, \frac{(-4/3)^r z^{n+r}}{n!r!} = \sum_{n=0}^{\infty} \frac{\Psi(3n)(z/3)^{3n}}{(-2a)_n(2a+1)_n n!} \tag{2.1}$$

provided $-2a,\ 2a+1,\ 6a+1 \in \mathbf{C}\backslash\mathbf{Z}_0^-$ *and the infinite series on both sides of (2.1) are absolutely convergent.*

Proof. Let

$$G(z) := \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(n+r)\,\frac{(2a+1)_{-\frac{n}{3}-\frac{r}{3}}(3a+\frac{1}{2})_r(3a+1)_r}{(2a+1)_{\frac{2r}{3}-\frac{n}{3}}(6a+1)_r}\,\frac{(-4/3)^r z^{n+r}}{n!r!}$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(n+r)\,\frac{(3a+\frac{1}{2})_r(3a+1)_r}{(2a+1-\frac{1}{3}(n+r))_r(6a+1)_r}\,\frac{(-4/3)^r z^{n+r}}{n!r!}. \tag{2.2}$$

Replacing n by $n-r$ in (2.2) and using Cauchy's double series identity (1.5), we have

$$G(z) = \sum_{n=0}^{\infty}\sum_{r=0}^{n}\Psi(n)\,\frac{(3a+\frac{1}{2})_r(3a+1)_r}{(2a+1-\frac{1}{3}n)_r(6a+1)_r}\,\frac{(-4/3)^r z^{n}}{(n-r)!r!}$$

$$= \sum_{n=0}^{\infty}\frac{\Psi(n)z^n}{n!}\sum_{r=0}^{n}\frac{(-n)_r(3a+\frac{1}{2})_r(3a+1)_r(4/3)^r}{(2a+1-\frac{1}{3}n)_r(6a+1)_r r!}$$

by (1.6). Identification of the inner sum as a ${}_3F_2(\frac{4}{3})$ hypergeometric series then leads to

$$G(z) = \sum_{n=0}^{\infty}\frac{\Psi(n)z^n}{n!}\,{}_3F_2\left[\begin{matrix}-n, 3a+\frac{1}{2}, 3a+1\\ 2a+1-\frac{1}{3}n, 6a+1\end{matrix};\frac{4}{3}\right]. \tag{2.3}$$

We now apply the decomposition identity

$$\sum_{n=0}^{\infty}\Phi(n) = \sum_{n=0}^{\infty}\Phi(3n) + \sum_{n=0}^{\infty}\Phi(3n+1) + \sum_{n=0}^{\infty}\Phi(3n+2), \tag{2.4}$$

provided that each of the sums is absolutely convergent, to the right-hand side of (2.3). This produces

$$G(z) = \sum_{n=0}^{\infty}\frac{\Psi(3n)z^{3n}}{(3n)!}\,{}_3F_2\left[\begin{matrix}-3n, 3a+\frac{1}{2}, 3a+1\\ 2a+1-n, 6a+1\end{matrix};\frac{4}{3}\right] + \sum_{n=0}^{\infty}\frac{\Psi(3n+1)z^{3n+1}}{(3n+1)!}\times$$

$$\times {}_3F_2\left[\begin{matrix}-3n-1, 3a+\frac{1}{2}, 3a+1\\ 2a+\frac{2}{3}-n, 6a+1\end{matrix};\frac{4}{3}\right] + \sum_{n=0}^{\infty}\frac{\Psi(3n+2)z^{3n+2}}{(3n+2)!}\,{}_3F_2\left[\begin{matrix}-3n-2, 3a+\frac{1}{2}, 3a+1\\ 2a+\frac{1}{3}-n, 6a+1\end{matrix};\frac{4}{3}\right].$$

Finally, use of the Gessel-Stanton summation theorem (1.3) leads to

$$G(z) = \sum_{n=0}^{\infty}\frac{\Psi(3n)z^{3n}}{(3n)!}\,\frac{(\frac{1}{3})_n(\frac{2}{3})_n}{(-2a)_n(2a+1)_n},$$

which, after suitable simplification, yields the required result (2.1). □

The second identity is given by the following theorem:

Theorem 2.2. *Let $\{\Psi(\mu)\}_{\mu=1}^{\infty}$ be a bounded sequence of complex (or real) numbers such that $\Psi(0) \neq 0$. Then, the following double-series identity holds true:*

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(n+r)\,\frac{(3b)_{n+2r}(b)_r}{(3b)_{n+r}(3b)_{2r}}\,\frac{(-3)^r z^{n+r}}{n!r!} = \sum_{n=0}^{\infty}\frac{\Psi(3n)(z/3)^{3n}}{(b+\frac{1}{3})_n(b+\frac{2}{3})_n n!} \tag{2.5}$$

provided $3b$, $3b+1 \in \mathbf{C}\backslash\mathbf{Z}_0^-$ and the infinite series on both sides of (2.5) are absolutely convergent.

Proof. Let

$$H(z)=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(n+r)\,\frac{(3b)_{n+2r}(b)_r}{(3b)_{n+r}(3b)_{2r}}\,\frac{(-3)^r z^{n+r}}{n!r!}$$
$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(n+r)\,\frac{(3b+n+r)_r(b)_r}{(\frac{3}{2}b)_r\left(\frac{3}{2}b+\frac{1}{2}\right)_r}\,\frac{(-3/4)^r z^{n+r}}{n!r!}. \tag{2.6}$$

Replacing n by $n-r$ in (2.6) and using Cauchy's double series identity (1.5) and (1.6), we have

$$H(z)=\sum_{n=0}^{\infty}\frac{\Psi(n)z^n}{n!}\sum_{r=0}^{n}\frac{(-n)_r(b)_r(3b+n)_r(3/4)^r}{(\frac{3}{2}b)_r(\frac{3}{2}b+\frac{1}{2})_r r!}$$
$$=\sum_{n=0}^{\infty}\frac{\Psi(n)z^n}{n!}\,{}_3F_2\left[\begin{matrix}-n,b,3b+n\\ \frac{3}{2}b,\frac{3}{2}b+\frac{1}{2}\end{matrix};\frac{3}{4}\right].$$

Application of (2.4) as in Theorem 1 then produces

$$H(z)=\sum_{n=0}^{\infty}\frac{\Psi(3n)z^{3n}}{(3n)!}\,{}_3F_2\left[\begin{matrix}-3n,b,3b{+}3n\\ \frac{3}{2}b,\frac{3}{2}b+\frac{1}{2}\end{matrix};\frac{3}{4}\right]+\sum_{n=0}^{\infty}\frac{\Psi(3n{+}1)z^{3n+1}}{(3n{+}1)!}\times$$
$$\times\,{}_3F_2\left[\begin{matrix}-3n{-}1,b,3b{+}3n{+}1\\ \frac{3}{2}b,\frac{3}{2}b{+}\frac{1}{2}\end{matrix};\frac{3}{4}\right]+\sum_{n=0}^{\infty}\frac{\Psi(3n{+}2)z^{3n+2}}{(3n{+}2)!}\,{}_3F_2\left[\begin{matrix}-3n{-}2,b,3b{+}3n{+}2\\ \frac{3}{2}b,\frac{3}{2}b{+}\frac{1}{2}\end{matrix};\frac{3}{4}\right].$$

Finally, use of the Andrews summation theorem (1.4) leads to

$$H(z)=\sum_{n=0}^{\infty}\frac{\Psi(3n)z^{3n}}{(3n)!}\,\frac{(3n)!(b+1)_n}{(3b+1)_{3n}\,n!}=\sum_{n=0}^{\infty}\frac{\Psi(3n)(b+1)_n z^{3n}}{(3b+1)_{3n}\,n!},$$

which, after suitable simplification, yields the required result (2.5). □

3 Application of Theorems 1 and 2 to the Srivastava-Daoust function

In this section we establish two results concerning the reducibility of the Srivastava-Daoust double hypergeometric function defined in (1.2). We have

Theorem 3.1. *The following results hold true:*

$$F^{A+1:\,0;\,2}_{B+1:\,0;\,1}\left(\begin{matrix}[(d_A):1,1],[2a+1:-\frac{1}{3},-\frac{1}{3}]:-;[3a+\frac{1}{2}:1],[3a+1:1];\\ [(e_B):1,1],[2a+1:-\frac{1}{3},\frac{2}{3}]:-;[6a+1:1];\end{matrix}\;z,-\frac{4}{3}z\right)$$
$$={}_{3A}F_{3B+2}\left[\begin{matrix}\frac{d_1}{3},\frac{d_1+1}{3},\frac{d_1+2}{3},\ldots\frac{d_A}{3},\frac{d_A+1}{3},\frac{d_A+2}{3}\\ \frac{e_1}{3},\frac{e_1+1}{3},\frac{e_1+2}{3}\ldots\frac{e_B}{3},\frac{e_B+1}{3},\frac{e_B+2}{3},-2a,2a+1\end{matrix};\frac{z^3}{27^{B-A+1}}\right] \tag{3.1}$$

and

$$F^{A+1:\,0;\,1}_{B+1:\,0;\,2}\left(\begin{matrix}[(d_A):1,1],[3b:1,2]:-;[b:1];\\ [(e_B):1,1],[3b:1,1]:-;[\frac{3}{2}b:1],[\frac{3}{2}b+\frac{1}{2}:1];\end{matrix}\;z,-\frac{3}{4}z\right)$$
$$={}_{3A}F_{3B+2}\left[\begin{matrix}\frac{d_1}{3},\frac{d_1+1}{3},\frac{d_1+2}{3},\ldots\frac{d_A}{3},\frac{d_A+1}{3},\frac{d_A+2}{3}\\ \frac{e_1}{3},\frac{e_1+1}{3},\frac{e_1+2}{3}\ldots\frac{e_B}{3},\frac{e_B+1}{3},\frac{e_B+2}{3},b+\frac{1}{3},b+\frac{2}{3}\end{matrix};\frac{z^3}{27^{B-A+1}}\right], \tag{3.2}$$

where $e_1,e_2,\ldots,e_B,-2a,2a+1,b+\frac{1}{3},b+\frac{2}{3}\in\mathbf{C}\backslash\mathbf{Z}_0^-$. When $A\le B$ both sides of (3.1) and (3.2) are convergent for $|z|<\infty$, but when $A=B+1$ the above hypergeometric functions are convergent for suitably constrained values of $|z|$.

Proof. Put

$$\Psi(\mu)=\frac{(d_1)_\mu(d_2)_\mu\dots(d_A)_\mu}{(e_1)_\mu(e_2)_\mu\dots(e_B)_\mu}=\frac{\prod_{j=1}^{A}(d_j)_\mu}{\prod_{j=1}^{B}(e_j)_\mu}\qquad(\mu=0,1,2,\dots)$$

on both sides of the double-series identity (2.1) to obtain

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{\prod_{j=1}^{A}(d_j)_{n+r}}{\prod_{j=1}^{B}(e_j)_{n+r}}\frac{(2a+1)_{-\frac{n}{3}-\frac{r}{3}}(3a+\frac{1}{2})_r(3a+1)_r}{(2a+1)_{\frac{2r}{3}-\frac{n}{3}}(6a+1)_r}\frac{(-4/3)^r z^{n+r}}{n!r!}$$

$$=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{A}(d_j)_{3n}}{\prod_{j=1}^{B}(e_j)_{3n}}\frac{(z/3)^{3n}}{(-2a)_n(2a+1)_n n!}. \tag{3.3}$$

Applying the definition of the Srivastava-Daoust function in (1.2) to the left-hand side of (3.3) and the definition of the generalised hypergeometric function in (1.1), together with the Pochhammer identity (1.7), to the right-hand side of (3.3), we obtain the desired result (3.1).

The proof of (3.2) follows exactly the same procedure and will be omitted. This completes the proof of Theorem 3. □

4 Concluding remarks

We have obtained two general double-series identities by employing the summation theorems of Gessel-Stanton and Andrews for the terminating hypergeometric series ${}_3F_2$ with arguments $4/3$ and $3/4$, respectively. These results have been used to derive two reduction formulas for the Srivastava-Daoust double hypergeometric function with arguments $(z,-4z/3)$ and $(z,-3z/4)$ in terms of a single generalized hypergeometric function ${}_{3A}F_{3B+2}$ with argument $z^3/27^{B-A+1}$. It is hoped that the results derived in this paper will find useful application.

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