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ON A CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY GEGENBAUER POLYNOMIALS

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Abstract The aim of this article is to introduce and investigate a new subclass of analytic functions involving Gegenbauer polynomials. We obtain for the introduced class various geometric properties giving the coefficient inequalities, distortion theorem, radius of close-to-convexity, starlikeness, convex linear combination, partial sums and convolution properties. Further, we obtain a neighborhood result for the class defined in the present paper.

1 Introduction

Let A specify the category of analytical functions f represent on the unit disc $U = \{z : |z| < 1\}$ with normalization $f(0) = 0$ and $f'(0) = 1$, such a function has the extension of the Taylor series on the origin in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Indicated by S, the subclass of A be composed of functions that are univalent in U.

Then a $f(z)$ function of A is known as starlike and convex of order ϑ if it delights the pursing

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \vartheta, \quad (z \in U), \tag{1.2}$$

$$\text{and } \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \vartheta, \quad (z \in U), \tag{1.3}$$

for specific $\vartheta(0 \le \vartheta < 1)$ respectively and we express by $S^*(\vartheta)$ and $K(\vartheta)$ the subclass of A be expressed by aforesaid functions respectively. Also, indicate by T the subclass of A made up of functions of this form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \ (a_n \ge 0, \ z \in U) \tag{1.4}$$

and let $T^*(\vartheta) = T \cap S^*(\vartheta), C(\vartheta) = T \cap K(\vartheta)$. There are interesting properties in the $T^*(\vartheta)$ and $C(\vartheta)$ classes and were thoroughly studied by Silverman [6] and others.

The class $\mathcal{T}(\wp), \wp \ge 0$ has been implemented and analyzed by the subclass Szynal [10] of A consisting of type functions

$$f(z) = \int_{-1}^{1} K(z,\ell)d\mu(\ell), \tag{1.5}$$

where

$$K(z,\ell) = \frac{z}{(1 - 2\ell z + z^2)^{\wp}}, \quad (z \in U, \ell \in [-1,1]) \tag{1.6}$$

and μ is a probability measure at the interval $[-1,1]$. The compilation of such $[a,b]$ calculation is denoted as $P[a,b]$.

The function expansion of the Taylor series in (1.6) gives

$$K(z,\ell) = z + c_1^{\wp}(\ell)z^2 + c_2^{\wp}(\ell)z^3 + \cdots . \tag{1.7}$$

The coefficients for (1.7) and those for (1.7) are given below:

$$\begin{aligned} &c_0^{\wp}(\ell) = 1;\ c_1^{\wp}(\ell) = 2\wp\ell; c_2^{\wp}(\ell) = 2\wp(\wp+1)\ell^2 - \wp; \\ &c_3^{\wp}(\ell) = \frac{4}{3}\wp(\wp+1)(\wp+2)\ell^3 - 2\wp(\wp+1)\ell \ \ \cdots , \end{aligned} \tag{1.8}$$

where $c_n^{\wp}(\ell)$ corresponds to the Gegenbauer degree polynomial n. Varying the $\wp$ parameter in (1.7), we get a class of usually real functions studied by (1.7) (see [1, 3, 5, 8] and [9]).

Let $\mathscr{G}_{\wp}^{\ell} : A \to A$ is defined by convolution

$$\mathscr{G}_{\wp}^{\ell} f(z) = K(z,\ell) * f(z),$$

we have

$$\mathscr{G}_{\wp}^{\ell} f(z) = z + \sum_{n=2}^{\infty} c_{n-1}^{\wp}(\ell) a_n z^n. \tag{1.9}$$

In this paper, we are consider as $|a_n| = a_n$ and $|c_{n-1}^{\wp}(\ell)| = c_{n-1}^{\wp}(\ell)$.

Now, we propose a new subclass $\phi_{\wp}^{\ell}(\hbar,\vartheta)$ of A concerning polynomial of Geganbaur as below:

Definition 1.1. For $0 \le \hbar < 1, 0 \le \vartheta < 1, \wp > 0, \ell > 0$, we say $f(z) \in A$ is in $\phi_{\wp}^{\ell}(\hbar,\vartheta)$ if it fulfils the requirement

$$\Re\left(\frac{z\left(\mathscr{G}_{\wp}^{\ell} f(z)\right)' + \hbar z^2 \left(\mathscr{G}_{\wp}^{\ell} f(z)\right)''}{\mathscr{G}_{\wp}^{\ell} f(z)}\right) > \vartheta,\ (z \in U). \tag{1.10}$$

Also we indicate by $T\phi_{\wp}^{\ell}(\hbar,\vartheta) = \phi_{\wp}^{\ell}(\hbar,\vartheta) \cap T$.

2 Coefficient Inequalities

This section gives us an adequate requirement for a function f given by (1.1) to be in $\phi_{\wp}^{\ell}(\hbar,\vartheta)$.

Theorem 2.1. *A function $f \in A$ is assigned to the class $\phi_{\wp}^{\ell}(\hbar,\vartheta)$ if*

$$\sum_{n=2}^{\infty} [n + \hbar n(n-1) - \vartheta] c_{n-1}^{\wp}(\ell) a_n \le 1 - \vartheta. \tag{2.1}$$

Proof. Since $0 \le \vartheta < 1$ and $\hbar \ge 0$, now if we put

$$\varrho(z) = \frac{z\left(\mathscr{G}_{\wp}^{\ell} f(z)\right)' + \hbar z^2 \left(\mathscr{G}_{\wp}^{\ell} f(z)\right)''}{\mathscr{G}_{\wp}^{\ell} f(z)},\ \ (z \in U).$$

Then it's just a matter of proving it $|\varrho(z) - 1| < 1 - \vartheta, (z \in U)$.
Indeed if $f(z) = z,\ (z \in U)$, then we have $\varrho(z) = z,\ (z \in U)$.
Implies (2.1) holds.
If $f(z) \ne z,\ (|z| = r < 1)$, then there exist a coefficient $\Omega_n(\wp,\ell)a_n \ne 0$ for some $n \ge 2$. The consequence is that $\sum\limits_{n=2}^{\infty} c_{n-1}^{\wp}(\ell) a_n > 0$. Now

$$\sum_{n=2}^{\infty} [n + \hbar n(n-1) - \vartheta] c_{n-1}^{\wp}(\ell) a_n > (1-\vartheta) \sum_{n=2}^{\infty} c_{n-1}^{\wp}(\ell) a_n$$

$$\Rightarrow \sum_{n=2}^{\infty} c_{n-1}^{\wp}(\ell)a_n < 1.$$

By (2.1), we obtain

$$\begin{aligned}
|\varrho(z)-1| &= \left| \frac{\sum\limits_{n=2}^{\infty}[n+\hbar n(n-1)-1]c_{n-1}^{\wp}(\ell)a_n z^{n-1}}{1+\sum\limits_{n=2}^{\infty} c_{n-1}^{\wp}(\ell)a_n z^{n-1}} \right| \\
&< \frac{\sum\limits_{n=2}^{\infty}[n+\hbar n(n-1)-1]c_{n-1}^{\wp}(\ell)a_n}{1+\sum\limits_{n=2}^{\infty} c_{n-1}^{\wp}(\ell)a_n} \\
&\le \frac{\sum\limits_{n=2}^{\infty}[n+\hbar n(n-1)-\vartheta]c_{n-1}^{\wp}(\ell)a_n + (1-\vartheta)c_{n-1}^{\wp}(\ell)a_n}{1+\sum\limits_{n=2}^{\infty} c_{n-1}^{\wp}(\ell)a_n} \\
&\le \frac{(1-\vartheta)+(1-\vartheta)\sum\limits_{n=2}^{\infty} c_{n-1}^{\wp}(\ell)a_n}{1+\sum\limits_{n=2}^{\infty} c_{n-1}^{\wp}(\ell)a_n} \\
&= 1-\vartheta, \quad (z\in U).
\end{aligned}$$

Hence we obtain

$$\Re\left(\frac{z\left(\mathscr{G}_{\wp}^{\ell}f(z)\right)'+\hbar z^2\left(\mathscr{G}_{\wp}^{\ell}f(z)\right)''}{\mathscr{G}_{\wp}^{\ell}f(z)}\right) = \Re(\varrho(z)) > 1-(1-\vartheta)=\vartheta.$$

Then $f\in\phi_{\wp}^{\ell}(\hbar,\vartheta)$. . □

Theorem 2.2. *Let f be given by (1.4). Then the function $f\in T\phi_{\wp}^{\ell}(\hbar,\vartheta)$ if and only if*

$$\sum_{n=2}^{\infty}[n+\hbar n(n-1)-\vartheta]c_{n-1}^{\wp}(\ell)a_n \le 1-\vartheta. \tag{2.2}$$

Proof. In view of Theorem 2.1, to examine it $f\in T\phi_{\wp}^{\ell}(\hbar,\vartheta)$ fulfils the coefficient inequality (2.1). If $f\in T\phi_{\wp}^{\ell}(\hbar,\vartheta)$ then the function

$$\varrho(z)=\frac{z\left(\mathscr{G}_{\wp}^{\ell}f(z)\right)'+\hbar z^2\left(\mathscr{G}_{\wp}^{\ell}f(z)\right)''}{\mathscr{G}_{\wp}^{\ell}f(z)}, \quad (z\in U)$$

satisfies $\Re(\varrho(z))>\vartheta$. This implies that

$$\mathscr{G}_{\wp}^{\ell}f(z)=z-\sum_{n=2}^{\infty}c_{n-1}^{\wp}(\ell)a_n z^n \neq 0, (z\in U\setminus\{0\}).$$

Noting that $\frac{\mathscr{G}_{\wp}^{\ell}f(r)}{r}$ in the open interval $(0,1)$, this is the real continuous function with $\eta(0)=1$, we have

$$\frac{\mathscr{G}_{\wp}^{\ell}f(r)}{r}=1-\sum_{n=2}^{\infty}c_{n-1}^{\wp}(\ell)a_n r^{n-1}>0, \quad (0<r<1). \tag{2.3}$$

Now $\vartheta<\varrho(r)=\frac{1-\sum\limits_{n=2}^{\infty}[n+\hbar n(n-1)]c_{n-1}^{\wp}(\ell)a_n r^{n-1}}{1-\sum\limits_{n=2}^{\infty}c_{n-1}^{\wp}(\ell)a_n r^{n-1}}$ and consequently by (2.3),

we get $\sum\limits_{n=2}^{\infty}[n+\hbar n(n-1)-\vartheta]c_{n-1}^{\wp}(\ell)a_n r^{n-1}\le 1-\vartheta$.

Letting $r \to 1$, we get $\sum\limits_{n=2}^{\infty}[n+\hbar n(n-1)-\vartheta]c_{n-1}^{\wp}(\ell)a_n \leq 1-\vartheta$.
This proves the converse part. □

Remark 2.3. If a function f of the form (1.4) belongs to the class $T\phi_{\wp}^{\ell}(\hbar,\vartheta)$ then

$$a_n \leq \frac{1-\vartheta}{[n+\hbar n(n-1)-\vartheta]c_{n-1}^{\wp}(\ell)}, \quad (n \geq 2).$$

3 Distortion Theorem

In the section, the distortion limits of the functions owned by the class $T\phi_{\wp}^{\ell}(\hbar,\vartheta)$.

Theorem 3.1. *Let $\eta \in T\phi_{\wp}^{\ell}(\hbar,\vartheta)$ and $|z|=r<1$. Then*

$$r-\frac{1-\vartheta}{[2\hbar-\vartheta+2]c_{n-1}^{\wp}(\ell)}r^2 \leq |f(z)| \leq r+\frac{1-\vartheta}{[2\hbar-\vartheta+2]c_{n-1}^{\wp}(\ell)}r^2 \tag{3.1}$$

and

$$1-\frac{2(1-\vartheta)}{[2\hbar-\vartheta+2]c_{n-1}^{\wp}(\ell)}r \leq |f'(z)| \leq 1+\frac{2(1-\vartheta)}{[2\hbar-\vartheta+2]c_{n-1}^{\wp}(\ell)}r. \tag{3.2}$$

Proof. Since $f \in T\phi_{\wp}^{\ell}(\hbar,\vartheta)$, we apply Theorem 2.2 to attain

$$\begin{aligned}[2\hbar-\vartheta+2]c_{n-1}^{\wp}(\ell)\sum_{n=2}^{\infty}a_n &\leq \sum_{n=2}^{\infty}[n+\hbar n(n-1)-\vartheta]c_{n-1}^{\wp}(\ell)a_n \\ &\leq 1-\vartheta.\end{aligned}$$

$$\text{Thus } |f(z)| \leq |z|+|z|^2\sum_{n=2}^{\infty}a_n \leq r+\frac{1-\vartheta}{[2\hbar-\vartheta+2]c_{n-1}^{\wp}(\ell)}\,r^2.$$

$$\text{Also we have, } |f(z)| \leq |z|-|z|^2\sum_{n=2}^{\infty}a_n \leq r-\frac{1-\vartheta}{[2\hbar-\vartheta+2]c_{n-1}^{\wp}(\ell)}\,r^2,$$

and (3.1) follows. In similar way for f', the inequalities

$$|f'(z)| \leq 1+\sum_{n=2}^{\infty}na_n|z|^{n-1} \leq 1+|z|\sum_{n=2}^{\infty}na_n$$

and

$$\sum_{n=2}^{\infty}na_n \leq \frac{2(1-\vartheta)}{[2\hbar-\vartheta+2]c_{n-1}^{\wp}(\ell)}$$

are satisfied, which leads to (3.2). □

4 Radii of close-to-convexity and starlikeness

A close-to-convex and star-like radius of this class $T\phi_{\wp}^{\ell}(\hbar,\vartheta)$ is obtained in this section.

Theorem 4.1. *Let f be specified by (1.4) is in $T\phi_{\wp}^{\ell}(\hbar,\vartheta)$. Then f is a close-to-convex of order ℓ, $(0 \leq \ell < 1)$ in the disc $|z|<t_1$, where*

$$t_1 = \inf_{n\geq 2}\left[\frac{(1-\ell)[n+n\hbar(n-1)-\vartheta]\Omega_n(\wp,\ell)}{n(1-\vartheta)}\right]^{\frac{1}{n-1}}. \tag{4.1}$$

Proof. If $f \in T$ and f is a close-to-convex of order ℓ then we get

$$|f'(z) - 1| \leq 1 - \ell. \tag{4.2}$$

For the left hand side of (4.2), we obtain

$$|f'(z) - 1| \leq \sum_{n=2}^{\infty} n a_n |z|^{n-1} < 1 - \ell$$

$$\Rightarrow \quad \sum_{n=2}^{\infty} \frac{n}{1-\ell} a_n |z|^{n-1} \leq 1.$$

We know that $f(z) \in T\phi_{\wp}^{\ell}(\hbar, \vartheta)$ if and only if

$$\sum_{n=2}^{\infty} \frac{[n + n\hbar(n-1) - \vartheta]\Omega_n(\wp, \ell)}{(1-\vartheta)} a_n \leq 1.$$

Thus (4.2) holds true if

$$\frac{n}{1-\ell}|z|^{n-1} \leq \frac{[n + n\hbar(n-1) - \vartheta]\Omega_n(\wp, \ell)}{(1-\vartheta)}$$

$$\text{then} \quad |z| \leq \left[\frac{(1-\ell)[n + n\hbar(n-1) - \vartheta]\Omega_n(\wp, \ell)}{n(1-\vartheta)}\right]^{\frac{1}{n-1}}$$

hence the proof. □

Theorem 4.2. *Let $f \in T\phi_{\wp}^{\ell}(\hbar, \vartheta)$. Then f is a starlike of order ℓ, $(0 \leq \ell < 1)$ in the disc $|z| < t_2$, where*

$$t_2 = \inf_{n \geq 2} \left[\frac{(1-\ell)[n + n\hbar(n-1) - \vartheta]\Omega_n(\wp, \ell)}{(n-\ell)(1-\vartheta)}\right]^{\frac{1}{n-1}}. \tag{4.3}$$

Proof. We have $f \in T$ and f is a starlike of order ℓ, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \ell. \tag{4.4}$$

For the left hand side of (4.4), we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum\limits_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1 - \sum\limits_{n=2}^{\infty} a_n |z|^{n-1}}$$

$(1-\ell)$ is greater than the right hand side of the left relation if

$$\sum_{n=2}^{\infty} \frac{n-\ell}{1-\ell} a_n |z|^{n-1} < 1.$$

We know that $f \in T\phi_{\wp}^{\ell}(\hbar, \vartheta)$ if and only if

$$\sum_{n=2}^{\infty} \frac{[n + n\hbar(n-1) - \vartheta]\Omega_n(\wp, \ell)}{(1-\vartheta)} a_n \leq 1.$$

Thus (4.4) is true if

$$\frac{n-\ell}{1-\ell}|z|^{n-1} \leq \frac{[n + n\hbar(n-1) - \vartheta]\Omega_n(\wp, \ell)}{(1-\vartheta)}$$

$$\text{then} \quad |z| \leq \left[\frac{(1-\ell)[n + n\hbar(n-1) - \vartheta]\Omega_n(\wp, \ell)}{(n-\ell)(1-\vartheta)}\right]^{\frac{1}{n-1}}$$

hence the proof. □

5 Convex Linear combinations

Theorem 5.1. *Let* $f_1(z) = z$ *and*

$$f_n(z) = z - \frac{1-\vartheta}{[n + \hbar n(n-1) - \vartheta]c_{n-1}^{\wp}(\ell)} z^n, \quad (z \in U, n \geq 2). \tag{5.1}$$

Then $f \in T\phi_{\wp}^{\ell}(\hbar, \vartheta)$ *if and only if* f *is the form of*

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z), \quad (\mu_n \geq 0) \tag{5.2}$$

and $\sum\limits_{n=1}^{\infty} \mu_n = 1$.

Proof. If a function f is of the form $f(z) = \sum\limits_{n=1}^{\infty} \mu_n f_n(z), \mu_n \geq 0$ and $\sum\limits_{n=1}^{\infty} \mu_n = 1$ then

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n + \hbar n(n-1) - \vartheta]c_{n-1}^{\wp}(\ell)a_n \\
\leq &\sum_{n=2}^{\infty}[n + \hbar n(n-1) - \vartheta]c_{n-1}^{\wp}(\ell)\frac{(1-\vartheta)\mu_n}{[n + \hbar n(n-1) - \vartheta]c_{n-1}^{\wp}(\ell)} \\
= &\sum_{n=2}^{\infty}(1-\vartheta)\mu_n = (1-\mu_1)(1-\vartheta) \\
= &1-\vartheta
\end{aligned}$$

which provides (2.2), hence $f \in T\phi_{\wp}^{\ell}(\hbar, \vartheta)$, by Theorem 2.2.

On the other hand, if f is in the class $f \in T\phi_{\wp}^{\ell}(\hbar, \vartheta)$, then we may set

$$\mu_n = \frac{[n + \hbar n(n-1) - \vartheta]c_{n-1}^{\wp}(\ell)}{1-\vartheta} a_n, \quad (n \geq 2),$$

and $\mu_1 = 1 - \sum\limits_{n=2}^{\infty} \mu_n$.

Then the function f is of the form (5.2) . □

6 Partial Sums

Silverman [7] examined partial sums f for the function $f \in A$ given by (1.1) established through

$$f_1(z) = z \text{ and } f_m(z) = z + \sum_{n=2}^{m} a_n z^n, m = 2, 3, 4, \cdots . \tag{6.1}$$

In this paragraph, in the class $\phi_{\wp}^{\ell}(\hbar, \vartheta)$, partial function sums can be considered and sharp lower limits can be reached for the function. True component ratios of f to f_m and f' to f'_m.

Theorem 6.1. *Let* $f \in \phi_{\wp}^{\ell}(\hbar, \vartheta)$ *and fulfils (2.1). Then*

$$\Re\left(\frac{f(z)}{f_m(z)}\right) \geq 1 - \frac{1}{d_{m+1}}, (z \in U, m \in N), \tag{6.2}$$

where

$$d_n = \frac{[n + \hbar n(n-1) - \vartheta]}{1-\vartheta}. \tag{6.3}$$

Proof. Clearly, $d_{n+1} > d_n > 1, n = 2, 3, 4, \cdots$.

Thus by Theorem 2.1 we get,

$$\sum_{n=2}^{\infty} a_n + d_{m+1} \sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} d_n a_n \leq 1. \tag{6.4}$$

$$\text{Setting } g(z) = d_{m+1} \left\{ \frac{f(z)}{f_m(z)} - \left(1 - \frac{1}{d_{m+1}}\right) \right\}$$

$$g(z) = 1 + \frac{d_{m+1} \sum\limits_{n=m+1}^{\infty} a_n z^{n-1}}{1 + \sum\limits_{n=2}^{m} a_n z^{n-1}} \tag{6.5}$$

it be good enough to show $\Re(g(z)) > 0, \ (z \in U)$. Applying (6.4) we think that

$$\left| \frac{g(z) - 1}{g(z) + 1} \right| \leq \frac{d_{m+1} \sum\limits_{n=2}^{\infty} a_n}{2 - 2\sum\limits_{n=2}^{m} a_n - d_{m+1} \sum\limits_{n=m+1}^{\infty} a_n}$$

$$\leq 1,$$

which gives,

$$\Re\left(\frac{f(z)}{f_m(z)}\right) \geq 1 - \frac{1}{d_{m+1}},$$

hence the proof. □

Theorem 6.2. *Let f in $T\phi_{\wp}^{\ell}(\hbar, \vartheta)$ and fulfils (2.1). Then*

$$\Re\left(\frac{f_m(z)}{f(z)}\right) \geq \frac{d_{m+1}}{1 + d_{m+1}}, \quad (z \in U, m \in N), \tag{6.6}$$

where

$$d_n = \frac{[n + \hbar n(n-1) - \vartheta]}{1 - \vartheta}. \tag{6.7}$$

Proof. Clearly, $d_{n+1} > d_n > 1, n = 2, 3, 4, \cdots$.

Thus by Theorem 2.1 we get,

$$\sum_{n=2}^{\infty} a_n + d_{m+1} \sum_{n=m+1}^{\infty} a_n \leq \sum_{n=2}^{\infty} d_n a_n \leq 1. \tag{6.8}$$

$$\text{Setting } h(z) = (1 + d_{m+1}) \left\{ \frac{f_m(z)}{f(z)} - \left(\frac{d_{m+1}}{1 + d_{m+1}}\right) \right\}$$

$$h(z) = 1 - \frac{(1 + d_{m+1}) \sum\limits_{n=m+1}^{\infty} a_n z^{n-1}}{1 + \sum\limits_{n=2}^{m} a_n z^{n-1}} \tag{6.9}$$

to show $\Re(h(z)) > 0, \ \ (z \in U)$. Implementing (6.8), we attain

$$\left| \frac{h(z) - 1}{h(z) + 1} \right| \leq \frac{(1 + d_{m+1}) \sum\limits_{n=2}^{\infty} a_n}{2 - 2\sum\limits_{n=2}^{m} a_n - (1 + d_{m+1}) \sum\limits_{n=m+1}^{\infty} a_n}$$

$$\leq 1,$$

which gives,

$$\Re\left(\frac{f_m(z)}{f(z)}\right) \geq \frac{d_{m+1}}{1 + d_{m+1}},$$

and hence the proof. □

Theorem 6.3. *Let f in $T\phi_{\wp}^{\ell}(\hbar,\vartheta)$ and fulfils (2.1). Then*

$$\Re\left(\frac{f'(z)}{f'_m(z)}\right) \geq 1-\frac{m+1}{d_{m+1}}, \quad (z\in U, m\in N), \tag{6.10}$$

and

$$\Re\left(\frac{f'_m(z)}{f'(z)}\right) \geq \frac{d_{m+1}}{m+1+d_{m+1}}, \quad (z\in U, m\in N) \tag{6.11}$$

where

$$d_n=\frac{[n+\hbar n(n-1)-\vartheta]}{1-\vartheta}. \tag{6.12}$$

Proof. By Setting

$$g(z)=d_{m+1}\left\{\frac{f('z)}{f'_m(z)}-\left(1-\frac{m+1}{d_{m+1}}\right)\right\}, \quad (z\in U)$$

$$\text{and } \; h(z)=(m+1+d_{m+1})\left\{\frac{f'_m(z)}{f('z)}-\left(\frac{d_{m+1}}{m+1+d_{m+1}}\right)\right\}, \quad (z\in U).$$

The evidence is close to that of the 6.1 and 6.2 theorems, so the specifics are omitted. □

7 Convolution properties

We will prove in this section that the $T\phi_{\wp}^{\ell}(\hbar,\vartheta)$ class is closed by convolution.

Theorem 7.1. *Let $g(z)$ of the form*

$$g(z)=z-\sum_{n=2}^{\infty}b_nz^n$$

be regular in U. If $f\in T\phi_{\wp}^{\ell}(\hbar,\vartheta)$ then the function $f*g$ is in the class $T\phi_{\wp}^{\ell}(\hbar,\vartheta)$. Here the symbol $*$ denoted to the Hadmard product .

Proof. Since $f\in T\phi_{\wp}^{\ell}(\hbar,\vartheta)$, we have

$$\sum_{n=2}^{\infty}[n+\hbar n(n-1)-\vartheta]c_{n-1}^{\wp}(\ell)a_n \leq 1-\vartheta.$$

Employing the last inequality and the fact that

$$f(z)*g(z)=z-\sum_{n=2}^{\infty}a_nb_nz^n.$$

We obtain

$$\begin{aligned}&\sum_{n=2}^{\infty}[n+\hbar n(n-1)-\vartheta]c_{n-1}^{\wp}(\ell)a_n|b_n|\\ \leq&\sum_{n=2}^{\infty}[n+\hbar n(n-1)-\vartheta]c_{n-1}^{\wp}(\ell)a_n\\ =&1-\vartheta\end{aligned}$$

and hence, in view of Theorem 2.1, the result follows. □

8 Neighbourhood results

Following [2, 4], we defined the $\alpha-$neighbourhood of the function $f(z) \in T$ by

$$N_\alpha(f) = \left\{ g \in T : g(z) = z - \sum_{n=2}^{\infty} b_n z^n \text{ and } \sum_{n=2}^{\infty} n|a_n - b_n| \leq \alpha \right\}, \text{ where } \alpha \geq 0. \quad (8.1)$$

Definition 8.1. A function $f \in T$ is said to be in the class $T\phi_{\wp}^{\ell,\gamma}(\hbar, \vartheta)$ if there exists a function $h \in T\phi_{\wp}^{\ell}(\hbar, \vartheta)$ such that

$$\left| \frac{f(z)}{h(z)} - 1 \right| < 1 - \gamma, \quad (z \in U,\ 0 \leq \gamma < 1). \quad (8.2)$$

Theorem 8.2. *If* $h \in T\phi_{\wp}^{\ell}(\hbar, \vartheta)$ *and*

$$\gamma = 1 - \frac{\alpha(2\hbar - \vartheta + 2)\Omega_2(\wp, \hbar)}{2(2\hbar - \vartheta + 2)\Omega_2(\wp, \hbar) - (1 + \vartheta)}$$

then $N_\alpha(h) \subseteq T\phi_{\wp}^{\ell,\gamma}(\hbar, \vartheta)$.

Proof. Let $f \in N_\alpha(h)$. We then find from that

$$\sum_{n=2}^{\infty} n|a_n - b_n| \leq \alpha,$$

which is easily implies the coefficient inequality

$$\sum_{n=2}^{\infty} |a_n - b_n| \leq \frac{\alpha}{n}.$$

Since $h \in T\phi_{\wp}^{\ell}(\hbar, \vartheta)$, we have from equation (2.1) that

$$\sum_{n=2}^{\infty} a_n \leq \frac{1 - \vartheta}{(2\hbar - \vartheta + 2)\Omega_2(\wp, \hbar)}$$

and

$$\begin{aligned} \left| \frac{f(z)}{h(z)} - 1 \right| &< \frac{\sum\limits_{n=2}^{\infty} n|a_n - b_n|}{1 - \sum\limits_{n=2}^{\infty} b_n} \\ &\leq \frac{\alpha}{2} \ \frac{(2\hbar - \vartheta + 2)\Omega_2(\wp, \hbar)}{(2\hbar - \vartheta + 2)\Omega_2(\wp, \hbar) - (1 + \vartheta)} \\ &= 1 - \gamma, \end{aligned}$$

hence the proof. □

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