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MULTIPLE SOLUTIONS FOR A BI-NONLOCAL ELLIPTIC PROBLEM INVOLVING p(x)-BIHARMONIC OPERATOR

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Abstract In this work, we study bi-nonlocal elliptic problem involving p(x)-Biharmonic operator. By applying variational method and under the adequate conditions, we prove the existence of nontrivial weak solutions of our problem.

1 Introduction and main result

In this paper, we are interested in the existence of weak solutions for the following fourth order elliptic equations of Kirchhoff type,

$$\begin{cases} M\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\right)\Delta^2_{p(x)}u = \lambda f(x,u)\left[\int_\Omega F(x,u)\right]^r & \text{in} \quad \Omega, \\ \Delta u = u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N > 1)$ is bounded smooth domain, $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ and $M : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions satisfying conditions which will be stated later. $F(x,u) = \int_0^u f(x,s)ds$, $\lambda \in \mathbb{R}$, $r > 0$ is real parameter.

The study of differential equations and variational problems involving non-local operators have received more and more interest in the last few years, which arises from optimization, finance, continuum mechanics, phase transition phenomena, population dynamics, and game theory, see [2],[3],[4],[11],[14],[15],[16],[17].

Moreover, problem (1.1) involving p(x)-Laplacian operator was initially motivated by Corréa and Augusto Cézar [2],[3]. In [2], when $q^+(r+1) < p^-$ and by Genus theory, the authors proved that the energy functional associated to problem (1.1) has infinitly many solutions. In the [3], when $f(x,u) = |u|^{q(x)-2}u$ in problem (1.1) by using variational methods, they showed the existence of positive solutions for any positive λ.

Recently, F.jaafri , A. Ayoujil and M. Berrajaa in [1], proved the existence of multiple solutions for the following fourth order elliptic equations of Kirchhoff type, with an additional nonlocal term,

$$\begin{cases} M\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\right)\Delta^2_{p(x)}u = \lambda |u|^{q(x)-2}u\left[\int_\Omega |u|^{q(x)}\right]^r & \text{in} \quad \Omega, \\ \Delta u = u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

In the sequel, this paper is a generalization of the above mentioned paper [1]. More precisely, we treat our problem (1.1) when $q^+(r+1) > \alpha^-(r+1) > p^+ > p^-$, using the mountain pass theorem, and when the nonlinear term $f(x,u)$ verifies the type of Ambrosetti-Rabinowitz condition which ensures the boundness of the Palais-Smale. In addition to other suitable conditions, we will show the existence of a weak nontrivial solution. Note that, this case is different from the one in [2].

We assume the following hypotheses M and f there are positive constants m_0, m_1, A_1, A_2 and functions $\alpha(x), q(x) \in C_+(\overline{\Omega}) = \{h : h \in C(\overline{\Omega}); h(x) > 1, \forall x \in \overline{\Omega}\}$, such that

$(M1)$ $\quad m_0 \leq M(t) \leq m_1.$

(f_1) $\quad A_1 t^{\alpha(x)-1} \leq f(x,t) \leq A_2 t^{q(x)-1}, \qquad \forall x \in \overline{\Omega}, \quad \alpha(x) \leq q(x).$

(f_2) $\quad$ there exists $\theta > \dfrac{m_1}{m_0}$ such that $0 < \theta F(x,s) < (r+1) f(x,s)s$, for all $s > 0, x \in \Omega$.

(f_3) $\quad f(x,t) = -f(x,-t), \quad \forall (x,t) \in (\Omega, \mathbb{R}).$

Hereafter, let

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-2p(x)} & \text{if } p(x) < \frac{N}{2} \\ +\infty & \text{if } p(x) \geqslant \frac{N}{2}. \end{cases}$$

Now we can present our main results.

Theorem 1.1. *Suppose $p(x) < q(x) < p^*(x)$ for all $x \in \overline{\Omega}$ and $\alpha^-(r+1) > p^+$.*
Then for any $\lambda > 0$, with $(M1), (f1)$ and $(f2)$ satisfied, problem (1.1) has nontrivial solution.

Theorem 1.2. *Suppose $p(x) < q(x) < p^*(x)$ for all $x \in \overline{\Omega}$ and $p^- > q^+(r+1)$.*
Then for any $\lambda > 0$, with $(M1), (f1)$ and $(f3)$ satisfied, problem (1.1) has infinitely many solutions.

Remark 1.3. Hypothesis $(f2)$ is type of Ambrosetti-Rabinowitz condition (see [12]). Moreover, condition $(f2)$ ensures that the Euler-Lagrange functional associated with problem (1.1) possesses the geometry of Mountain Pass theorem and it also guarantees the boundedness of the Palais-Smale sequence corresponding to the Euler-Lagrange's functional.

Problem in the form (1.1), are associated with the energy functional.

$$J_\lambda(u) = \tilde{M}\left(\int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx\right) - \frac{\lambda}{r+1}\left[\int_\Omega F(x,u)dx\right]^{r+1},$$

for all $u \in X = \left\{u \in W^{2,p(x)}(\Omega) : u = 0 \text{ and } \Delta u = 0 \text{ in } \partial\Omega\right\}$, more precise estimates concerning this space will be established in Section 2 and $\tilde{M}(t) = \displaystyle\int_0^t M(s)ds.$

The functional J_λ is differentiable and its Fréchet-derivative is given by

$$J'_\lambda(u)(v) = M\left(\int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx\right) \int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta v dx - \lambda \left[\int_\Omega F(x,u)dx\right]^r \int_\Omega f(x,u)uvdx$$

for all $u, v \in X$.
Thus, the weak solution of problem (1.1), coincides with the critical point of J_λ.

This paper is organized as follows: In section 2, we present some preliminaries on the variable exponent spaces. In section 3, we give the proof of our main results.

2 Preliminaries

We start with some preliminary basic results for the variable exponent Lebesgue-Sobolev spaces. For details, see [7],[8]. Define

$$\forall h \in C_+(\overline{\Omega}), h^- = \min_{x \in \overline{\Omega}} h(x) \leq h^+ = \max_{x \in \overline{\Omega}} h(x).$$

For $p \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \ \ mesurable; \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

with the norm

$$|u|_{p(x)} = \inf \left\{ \mu > 0 : \int_\Omega \left| \frac{u}{\mu} \right|^{p(x)} dx \leq 1 \right\},$$

and it is a separable and reflexive Banach space.

Proposition 2.1. *([7]) For $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have*

$$\left| \int_\Omega uv dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(x)} |v|_{q(x)},$$

where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$.

Proposition 2.2. *([7]) Let $\rho(u) = \displaystyle\int_\Omega |u|^{p(x)} dx$. For $u, u_n \in L^{p(x)}(\Omega)$, we have*

1. $|u|_{p(x)} < 1$ *(resp* $= 1, > 1$*)* $\Leftrightarrow \rho(u) < 1$ *(resp* $= 1, > 1$*).*
2. $min(|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}) \leqslant \rho(u) \leqslant max(|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+})$.
3. $|u_n(x)|_{p(x)} \to 0$ *(resp* $\to \infty$*)* $\Leftrightarrow \rho(u_n) \to 0$*(resp*$\to \infty$*).*

Define the variable exponent Sobolev space, for any positive integer k, set

$$W^{k,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : \ \ D^\alpha u \in L^{p(x)}(\Omega), \ \ |\alpha| \leq k \right\},$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ is a multi-index, $|\alpha| = \displaystyle\sum_{i=1}^N \alpha_i$ and $D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} ......\partial x_N^{\alpha_N}}$,

with the norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha| \leq k} | \ D^\alpha u \ |_{p(x)} \ .$$

Then, $W^{k,p(x)}(\Omega)$ also becomes a seperable, reflexive and Banach space. We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$.

Define $\|u\|_X = |u|_{p(x)} + |\nabla u|_{p(x)} + \sum_{|\alpha|=2} | \ D^\alpha u \ |_{p(x)} \ \forall u \in X$, the norm associad with the space X, which is equivalent to the norm $|\Delta u(x)|_{p(x)}$ (see [18]).

Let us choose on X the norm defined by

$$\|u\| = |\Delta u(x)|_{p(x)}.$$

Note that $(X, \|.\|)$ is also a separable and reflexive Banach space. Similar to Proposition 2.1, we have the following proposition:

Proposition 2.3. *[13] Let $I(u) = \displaystyle\int_\Omega |\Delta u|^{p(x)} dx$. For $u, u_n \in L^{p(x)}(\Omega)$, we have*

1. $\|u\| < 1$ *(resp* $= 1, > 1$*)* $\Leftrightarrow I(u) < 1$ *(resp* $= 1, > 1$*).*
2. $min(\|u\|^{p^-}, \|u\|^{p^+}) \leqslant I(u) \leqslant max(\|u\|^{p^-}, \|u\|^{p^+})$.
3. $\|u_n - u\| \to 0 \Leftrightarrow I(u_n - u) \to 0$.

Remark 2.4. Let $h \in C_+(\overline{\Omega})$ and $h(x) < p^*(x)$ for any $x \in \overline{\Omega}$. Then,
by ([13], Theorem 3.2), we deduce that X, is continuously and compactly embedded in $L^{h(x)}(\Omega)$.

3 Proofs

3.1 Proof of Theorem 1.1

We apply the Mountain Pass Theorem,

$$\begin{aligned} J_\lambda(u) &= \tilde{M}\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\right) - \frac{\lambda}{r+1}\left[\int_\Omega F(x,u)dx\right]^{r+1} \\ &\geq \frac{m_0}{p^+}\int_\Omega |\Delta u|^{p(x)}dx - \frac{\lambda}{r+1}\left(\frac{A_2}{q^-}\right)^{r+1}\left[\int_\Omega |u|^{q(x)}dx\right]^{r+1}, \end{aligned}$$

$\|u\|$ is small enough, such that $\|u\| = \rho \in (0,1)$,

$$\begin{aligned} J_\lambda(u) &\geq \frac{m_0}{p^+}\rho^{p^+} - \frac{\lambda}{r+1}\left(\frac{A_2}{q^-}\right)^{r+1} C\rho^{q^-(r+1)} \\ &= \rho^{p^+}\left[\frac{m_0}{p^+} - \frac{\lambda}{r+1}\left(\frac{A_2}{q^-}\right)^{r+1} C\rho^{q^-(r+1)-p^+}\right]. \end{aligned}$$

Since $q^-(r+1) > \alpha^-(r+1) > p^+$, we find positive a, ρ such that

$$J_\lambda(u) \geq a > 0$$

for any $u \in X$ with $\|u\| = \rho$.

Now we choose $\phi \in X$, $\phi > 0$. For $t > 1$, we have

$$J_\lambda(t\phi) \leq \frac{m_1}{p^-}t^{p^+}\int_\Omega |\Delta\phi|^{p(x)}dx - \frac{\lambda}{r+1}\left(\frac{A_1}{\alpha^+}\right)^{r+1} t^{\alpha^-(r+1)}\left[\int_\Omega |\phi|^{\alpha(x)}dx\right]^{r+1}.$$

Using the fact that $\alpha^-(r+1) > p^+$, we obtain $J_\lambda(t\phi) \to -\infty$ as $t \to +\infty$. It follows J_λ satisfies the geometry of The Mountain Pass Theorem.

Now to complete the proof, we show that J_λ satisfies the (PS) condition. For all sequence $(u_n) \subset X$ such that

$$J_\lambda(u_n) \to c \ \ and \ \ J'_\lambda(u_n) \to 0. \tag{3.1}$$

From (3.1), $(M1)$ and (f_2), we have

$$\begin{aligned} c + 1 + \|u_n\| &\geq J_\lambda(u_n) - \frac{1}{\theta}J'_\lambda(u_n)u_n \\ &\geq \left(\frac{m_0}{p^+} - \frac{m_1}{\theta}\right)\int_\Omega |\Delta u_n|^{p(x)} dx \\ &+ \lambda\left[\int_\Omega F(x,u_n)dx\right]^r\left(\int_\Omega \frac{1}{\theta}f(x,u_n)u_n dx - \frac{1}{r+1}\int_\Omega F(x,u_n)dx\right) \\ &\geq \left(\frac{m_0}{p^+} - \frac{m_1}{\theta}\right)\|u_n\|^{p^-}, \end{aligned}$$

which is contradiction because $p^- > 1$. Hence $\{u_n\}$ is bounded in X. By the reflexity of X, for a subsequence still denoted (u_n), such that $u_n \rightharpoonup u$ in X.

From

$$J'_\lambda(u_n) \to 0,$$

we have

$$\begin{aligned} J'_\lambda(u_n)(u_n - u) &= M\left(\int_\Omega \frac{1}{p(x)}|\Delta u_n|^{p(x)}dx\right)\int_\Omega |\Delta u_n|^{p(x)-2}\Delta u_n \Delta(u_n - u)dx \\ &- \lambda\left[\int_\Omega F(x,u_n)dx\right]^r \int_\Omega f(x,u_n)(u_n - u)v dx \to 0 \end{aligned}$$

By the Hölder inequality, we obtain

$$\begin{aligned}\left|\int_\Omega f(x,u_n)(u_n-u)dx\right| &\leq A_2\int_\Omega |u_n|^{q(x)-1}|u_n-u|dx \\ &\leq C'\left||u_n|^{q(x)-1}\right|_{\frac{q(x)}{q(x)-1}} |u_n-u|_{q(x)}.\end{aligned}$$

Since $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, we deduce that X is compactly embedded in $L^{q(x)}$, hence (u_n) converges strongly to u in $L^{q(x)}$, then

$$\int_\Omega f(x,u_n)(u_n-u)dx \to 0.$$

From the definition of f and when (u_n) is bounded, there exist nonnegative constants C'' and C''' such that

$$C'' \leq \left[A_1\int_\Omega \frac{1}{\alpha(x)}|u_n|^{\alpha(x)}dx\right]^r \leq \left[\int_\Omega F(x,u_n)dx\right]^r \leq \left[A_2\int_\Omega \frac{1}{q(x)}|u_n|^{q(x)}dx\right]^r \leq C'''$$

we obtain

$$\left[\int_\Omega F(x,u_n)dx\right]^r \int_\Omega f(x,u_n)(u_n-u)dx \to 0.$$

From $M1$, we have also

$$L_{p(x)}(u_n)(u_n-u) = \int_\Omega |\Delta u_n|^{p(x)-2}\Delta u_n \Delta(u_n-u)dx \to 0.$$

By the Proposition 2.5(ii) in [5], $L_{p(x)}$ satisfies condition S_+, we have $u_n \to u$ in X. Hence J_λ satisfies the (PS) condition.

We denote that Proof of Theorem 1.2 is similar of [2].

3.2 Proof of Theorem 1.2

By (f_3) we know that J_λ is even, next we will prove the two important lemmas for our proof.

Lemma 3.1. J_λ *is bounded from below.*

Proof. From $(M1)$ and (f_1), we have

$$\begin{aligned}J_\lambda(u) &= \tilde{M}\left(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\right) - \frac{\lambda}{r+1}\left[\int_\Omega F(x,u)dx\right]^{r+1} \\ &\geq \frac{m_0}{p^+}\int_\Omega |\Delta u|^{p(x)}dx - \frac{\lambda}{r+1}\left(\frac{A_2}{q^-}\right)^{r+1}\left[\int_\Omega |u|^{q(x)}dx\right]^{r+1}\end{aligned}$$

Taking $\|u\| \geq 1$, we have

$$J_\lambda(u) \geq \frac{m_0}{p^+}\|u\|^{p^-} - \frac{\lambda}{r+1}\left(\frac{A_2}{q^-}\right)^{r+1} c'\|u\|^{q^+(r+1)}$$

So J_λ is bounded from below, because $p^- > q^+(r+1)$ and the lemma is proved. □

Lemma 3.2. J_λ *satisfies the* (PS) *condition.*

Proof. Let (u_n) has a convergent subsequence in X, such that

$$J_\lambda(u_n) \to d \qquad and \qquad J'_\lambda(u_n) \to 0,$$

Then, by the ceorcivity of J_λ, the sequence (u_n) is bounded in X. By the reflexity of X, for a subsequence still denoted (u_n), such that $u_n \rightharpoonup u \ in \ X$. Similar to proof of theorem 1.1 we deduce that $u_n \to u$ in X. □

In the sequel, for each $k \in \mathbb{N}$ consider $X_k = span\{e_1, e_2, e_3, ..., e_k\}$, the subspace of X (see Theorem 4.1 in [2]). Note that $X_k \hookrightarrow L^{\alpha(x)}(\Omega)$, $1 < \alpha(x) < p^*(x)$ with continuous immersions. Thus, the norm X, $L^{\alpha(x)}(\Omega)$ are equivalent on X_k.

Note that using $(M1)$ and (f_1), we obtain

$$\begin{aligned} J_\lambda(u) &\le \frac{m_1}{p^-}\left(\int_\Omega |\Delta u|^{p(x)}\right) - \frac{\lambda}{r+1}\left(\frac{A_1}{\alpha^+}\right)^{r+1}\left(\int_\Omega |u|^{\alpha(x)}\right)^{r+1} \\ &\le \frac{m_1}{p^-}\|u\|^{p^-} - \frac{\lambda}{r+1}\left(\frac{A_1}{\alpha^+}\right)^{r+1} C(k)\|u\|^{\alpha^+(r+1)} \end{aligned}$$

where $C(k)$ is a positive constant and $\|u\|$ is small enough. Hence,

$$J_\lambda(u) \le \|u\|^{\alpha^+(r+1)}\left[\frac{m_1}{p^-}\|u\|^{p^- - \alpha^+(r+1)} - \frac{\lambda}{r+1}\left(\frac{A_1}{\alpha^+}\right)^{r+1} C(k)\right].$$

Let R be a positive constant such that

$$\frac{m_1}{p^-}R^{p^- - \alpha^+} \le \frac{\lambda}{r+1}\left(\frac{A_1}{\alpha^+}\right)^{r+1} C(k).$$

Thus, for all $0 < r_0 < R$, and considering $K = \{u \in X : ||u|| = r_0\}$, we obtain

$$\begin{aligned} J_\lambda(u) &\le r_0^{\alpha^+(r+1)}\left[\frac{m_1}{p^-}r_0^{p^- - \alpha^+(r+1)} - \frac{\lambda}{r+1}\left(\frac{A_1}{\alpha^+}\right)^{r+1} C(k)\right] \\ &< R^{\alpha^+(r+1)}\left[\frac{m_1}{p^-}R^{p^- - \alpha^+(r+1)} - \frac{\lambda}{r+1}\left(\frac{A_1}{\alpha^+}\right)^{r+1} C(k)\right] < 0 = J_\lambda(0). \end{aligned}$$

Which implies

$$\sup_K J_\lambda(u) < 0 = J_\lambda(0).$$

Because X_k and $\mathbb{R}^k$ are isomorphic and K and S^{k-1} are homeomorphic, we conclude that $\gamma(K) = k$. By the Clark theorem, J_λ has at least k different critical points. Because k is arbitrary, we obtain infinitely many critical points of J_λ.

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