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ON THE ZEROS OF THE RIEMANN ZETA FUNCTION

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Abstract We express the Riemann zeta function $\zeta(z)$ in power series form, which we then use to calculate an approximate Riemann zeta function $\zeta_a(z)$. We then calculate the zeros of $\zeta_a(z)$ and obtain some findings about these zeros.

1 Introduction

In the eighteenth century Leonhard Euler studied and introduced the following series,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1.1}$$

for $s \geq 1$. He used only real numbers because complex analysis was not yet available at the time. In 1737 Euler proposed the Euler product formula [1],

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - p^{-s}\right)^{-1}, \tag{1.2}$$

which was rigorously proved by Leopold Kronecker in 1876. The Euler product formula forms a connection between prime numbers and the Euler zeta function (1.1) and therefore links number theory and analysis. In 1849 Carl Gauss revealed that he had been working on a related problem on the density of prime numbers. His observation is known as the prime number theorem and states that [2],

$$\pi(x) \approx \frac{x}{\log x}, \tag{1.3}$$

as $x \to \infty$ and where $\pi(x)$ is the number of primes at most x. Where 1 is not considered to be a prime and $\log x \equiv \ln x$. Gauss later refined the prime number theorem to,

$$\pi(x) \approx \mathrm{Li(x)} = \int_2^{\mathrm{x}} \frac{\mathrm{dt}}{\log \mathrm{t}}, \tag{1.4}$$

as $x \to \infty$. In 1850 Pafnuty Chebyshev showed that [2],

$$0.89\ \mathrm{Li(x)} < ß(\mathrm{x}) < 1.11\ \mathrm{Li(x)}, \tag{1.5}$$

for all sufficiently large x. He then extended the definition of the Euler zeta function (1.1) to complex numbers $z \in \mathbb{C}$ where $\Re(z) > 1$ and showed that [3],

$$\zeta(z) = \frac{z}{z-1} + \frac{1}{\Gamma(z)} \int_0^{\infty} \left(\frac{1}{e^x - 1} - \frac{1}{x} \right) e^{-x} x^{z-1} \mathrm{dx}. \tag{1.6}$$

In 1859 Bernhard Riemann used the extension of Euler's product formula to the complex plane and Chebyshev's result (1.6) to calculate the analytic continuation of $\zeta(z)$ to the entire complex plane given by [1],

$$\zeta(z) = 2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z) \zeta(1-z). \tag{1.7}$$

Riemann used his result (1.7) to study the zeros of $\zeta(z)$ and proposed that all non-trivial zeros of $\zeta(z)$ lie on the line $z = \frac{1}{2} + iy$, where $y \in \mathbb{R}$. Sometime after this, $\zeta(z)$ became known as the Riemann zeta function.

The analytic continuation of $\zeta(z)$ to $\Re(z) > 0$ is calculated as follows,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^z} + \sum_{n=1}^{\infty} \frac{1}{n^z} &= 2 \sum_{n=2,4,\ldots}^{\infty} \frac{1}{n^z} \\ &= 2 \sum_{k=1}^{\infty} \frac{1}{(2k)^z} \\ &= 2^{1-z} \sum_{k=1}^{\infty} \frac{1}{k^z},\end{aligned}$$

then rewriting this in terms of $\zeta(z)$ gives,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^z} + \zeta(z) = 2^{1-z}\zeta(z), \tag{1.8}$$

which ultimately gives [4],

$$\zeta(z) = \frac{1}{1-2^{1-z}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}. \tag{1.9}$$

In this work we use (1.9) as a starting point and we derive a power series representation of $\zeta(z)$ in section 2. This allows us to use the tools of power series in order to study the zeros of the Riemann zeta function. In section 3 we calculate a few coefficients of this power series and use the results to approximate a general formula for the coefficients. In section 4 we use the results of section 3 to calculate an approximation of the Riemann zeta function that we denote with $\zeta_a(z)$ and we compare some values of $\zeta_a(z)$ and $\zeta(z)$. In section 5 we calculate the zeros of $\zeta_a(z)$ and discuss some of their properties. Then we compare some of these zeros to some of the non-trivial zeros of $\zeta(z)$. In section 6 we do a calculation in order to answer a question we have about the zeros of $\zeta_a(z)$.

2 Expressing $\zeta(z)$ in power series form

Let $z \in \mathbb{C}$, then the Riemann zeta function is given by,

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \tag{2.1}$$

and is defined for $\Re(z) > 1$. However as we stated in the previous section an analytic continuation of this function exists and is given by,

$$\zeta(z) = \frac{1}{1-2^{1-z}} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^z}, \tag{2.2}$$

which is defined for $\Re(z) > 0$ and diverges at $z = 1$. The Mclaurin series for e^z is given by,

$$e^z = \sum_{m=0}^{\infty} \frac{z^m}{m!}. \tag{2.3}$$

We can also write $n^z = e^{\log n^z} = e^{z \log n}$. This then implies that $n^{-z} = e^{-z \log n}$ and if we substitute this into (2.3) we get the following,

$$n^{-z} = \sum_{m=0}^{\infty} \frac{(-1)^m \, z^m \, (\log n)^m}{m!}. \tag{2.4}$$

This then implies that,

$$\sum_{n=1}^{\infty}(-1)^{n-1}\, n^{-z} = \sum_{n=1}^{\infty}(-1)^{n-1} \sum_{m=0}^{\infty} \frac{(-1)^m\, z^m\, (\log n)^m}{m!}. \tag{2.5}$$

If we expand the right-hand side of (2.5) we get the following,

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n-1}\, n^{-z} &= \frac{1}{0!}\left[\sum_{m=1}^{\infty}(-1)^{m-1}\right] z^0 - \frac{1}{1!}\left[\sum_{m=1}^{\infty}(-1)^{m-1}\, \log m\right] z \\ &+ \frac{1}{2!}\left[\sum_{m=1}^{\infty}(-1)^{m-1}\, (\log m)^2\right] z^2 - ...,\end{aligned} \tag{2.6}$$

and we can then express (2.6) in the following simplified form,

$$\sum_{n=1}^{\infty}(-1)^{n-1}\, n^{-z} = \sum_{n=0}^{\infty}(-1)^n\, k_n\, \frac{z^n}{n!}, \tag{2.7}$$

where,

$$k_n = \sum_{m=1}^{\infty}(-1)^{m-1}\, (\log m)^n, \tag{2.8}$$

for all $n = 0, 1, 2, ...$ So we can now write the Riemann zeta function (2.2) as follows,

$$\zeta(z) = \frac{1}{1-2^{1-z}} \sum_{n=0}^{\infty}(-1)^n\, k_n\, \frac{z^n}{n!}. \tag{2.9}$$

3 Calculating the k_n's

In the previous section we expressed the Riemann zeta function in power series form, which will allow us to use the techniques of power series to study the Riemann zeta function. Our task now is to calculate the k_n's. We will calculate 30 of them and study the pattern they follow in order to write a general formula for them. The k_n's are given by (2.8), so in order to calculate them we have to calculate the sum $\sum_{m=1}^{\infty}(-1)^{m-1}\, (\log m)^n$ for $n = 0, 1, 2, ...$. In order to calculate these sums we use Cesàro summation defined below.

Definition 3.1. A sequence (a_m) is called Cesàro summable with Cesàro sum $A \in \mathbb{R}$ if as m tends to infinity, the arithmetic mean of its first m partial sums $s_1, s_2, ..., s_m$ tends to A. That is [5],

$$\lim_{m\to\infty} \frac{s_1 + s_2 + ... + s_m}{m} = A.$$

Two of the properties of Cesàro summation are:

(i) If a series converges then it's also Cesàro summable and its sum is equal to its Cesàro sum.

(ii) If a series diverges but is Cesàro summable then its Cesàro sum is its sum.

We first calculate $\sum_{m=1}^{\infty}(-1)^{m-1}\, (\log m)^n$ for $n = 0$, which when expanded gives,

$$\sum_{m=1}^{\infty}(-1)^{m-1} = 1 - 1 + 1 - 1 + ..., \tag{3.1}$$

which is known as Grandi's series. We let $a_m = (-1)^{m-1}$, then the m-th partial sum of this series is given by $s_m = a_1 + a_2 + ... + a_m$. The arithmetic mean of its first m partial sums is given by,

$$c_m = \frac{s_1 + s_2 + ... + s_m}{m}. \tag{3.2}$$

Using formula (3.2) we calculate a few c_m's for Grandi's series analytically and find that they quickly converge to $\frac{1}{2}$. Therefore $k_0 = \frac{1}{2}$. We notice that the bigger n gets the slower c_m converges for $\sum_{m=1}^{\infty}(-1)^{m-1}\,(\log m)^n$. We also find that for $n = 1, 2, 3, ...$, c_m can be expressed as follows,

$$c_m = \frac{\sum_{i=2}^{m}(-1)^{i+1}\,(m-(i-1))\,(\log i)^n}{m}. \tag{3.3}$$

Table 1.

n	1	2	3	4	5	6	7	8
k_n	-0.2250	-0.0600	0.0200	0.0500	0.0374	0.0034	-0.0300	-0.0583

Table 2.

n	9	10	11	12	13	14	15	16
k_n	-0.0500	-0.0090	0.0694	0.1586	0.2082	0.1438	-0.1142	-0.6011

Table 3.

n	17	18	19	20	21	22	23	24
k_n	-1.2207	-1.6221	-1.0871	1.4560	7.0560	15.6420	23.8103	21.4361

We use formula (3.3) to write computer code to calculate k_n for $n = 1, 2, ..., 30$. The results of this are shown in the tables 1 to 4. We then plot the k_n's in order to study their behaviour, see figure 1. We find that the k_n's generally have a behaviour that is oscillatory with increasing amplitude. We then fit a curve to this k_n data, see figure 2.

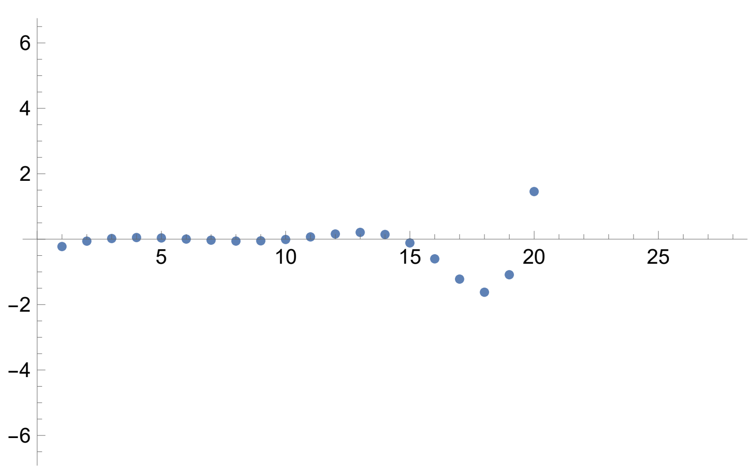

Figure 1. The horizontal axis displays the values of n and the vertical axis displays the values of k_n.

Table 4.

n	25	26	27	28	29	30
k_n	-11.7646	-105.7842	-286.9028	-535.0485	-690.4405	-297.3133

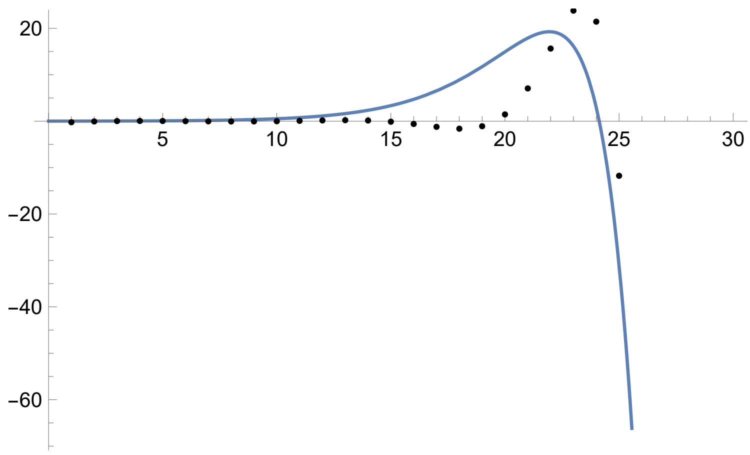

Figure 2. We fit a curve to the k_n values plotted in figure 1.

As we can see in figure 2, the curve doesn't fit the data perfectly but it approximates it and models its general behavior. Therefore we can use this curve to predict future values of k_n. Therefore we have that,

$$k_n \approx e^{0.458187n} \sin(28.2652 + 0.000378965n), \tag{3.4}$$

which is the function that generates the curve in figure 2.

4 Approximating the Riemann Zeta Function

If we let $a = 0.458187$, $b = 28.2652$ and $c = 0.000378965$ and then substitute (3.4) into (2.9) we get,

$$\zeta(z) \approx \frac{1}{1-2^{1-z}}\left[\frac{1}{2} + \sum_{n=1}^{\infty}(-1)^n\, e^{an}\, \sin(b+cn)\, \frac{z^n}{n!}\right]. \tag{4.1}$$

Using the ratio test we find that (4.1) converges and has radius of convergence $R = \infty$. Therefore we write computer code to calculate (4.1) and we get,

$$\zeta(z) \approx \frac{1}{1-2^{1-z}}\left[\frac{1}{2} + Ae^{\alpha_1 z}\left(B + Ce^{\alpha_2 z} - De^{\alpha_3 z}\right)\right], \tag{4.2}$$

where,

$$\begin{aligned}
A &= 0.00456688 - 0.49997i,\\
B &= -0.999833 + 0.0182667i,\\
C &= 1,\\
D &= 0.000166851 + 0.0182667i,\\
\alpha_1 &= -1.5812 - 0.000599221i,\\
\alpha_2 &= 0.00119844i,\\
\alpha_3 &= 1.5812 + 0.000599221i.
\end{aligned} \tag{4.3}$$

Table 5.

z	$\zeta_a(z)$	$\zeta(z)$
0	$-\frac{1}{2}$	$-\frac{1}{2}$
2	0.98261	1.64493
3	0.65462	1.20206
4	0.56101	1.08232
5	0.52360	1.03693
6	0.50670	1.01734
7	0.49866	1.00835
8	0.49473	1.00408
9	0.49279	1.00201
10	0.49183	1.00099

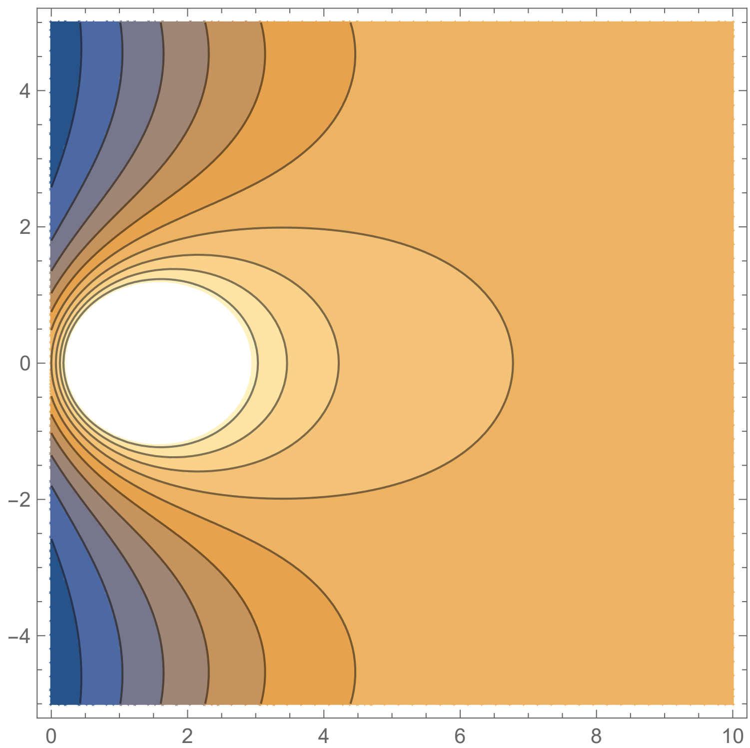

Figure 3. Contour-plot of $|\zeta_a(x+iy)|$ for $0 \leq x \leq 10$ and $-5 \leq y \leq 5$.

The function (4.2) is defined for all $z \in \mathbb{C}$ accept for at $z = 1$ and the approximation is valid for $\Re(z) \geq 0$. If we denote the approximate Riemann zeta function (4.2) with $\zeta_a(z)$ and compare it with the exact Riemann zeta function $\zeta(z)$ for
$z = 0, 2, 3, .., 10$, we get the results in table 5. Figure 3 and 5 show contour plots for $\zeta_a(z)$ and figure 4 and 6 show contour plots for $\zeta(z)$. The results in table 5 and the contour plots of $\zeta_a(z)$ and $\zeta(z)$ show that $\zeta_a(z)$ approximates $\zeta(z)$ not perfectly but nonetheless $\zeta_a(z)$ models the behaviour of $\zeta(z)$ and we can possibly improve the precision of the approximation.

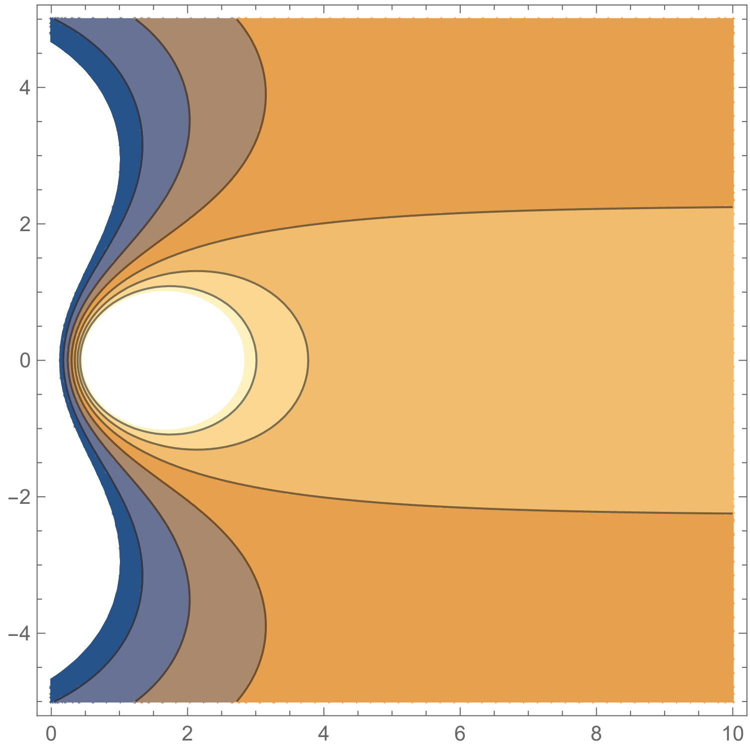

Figure 4. Contour-plot of $|\zeta(x+iy)|$ for $0 \le x \le 10$ and $-5 \le y \le 5$.

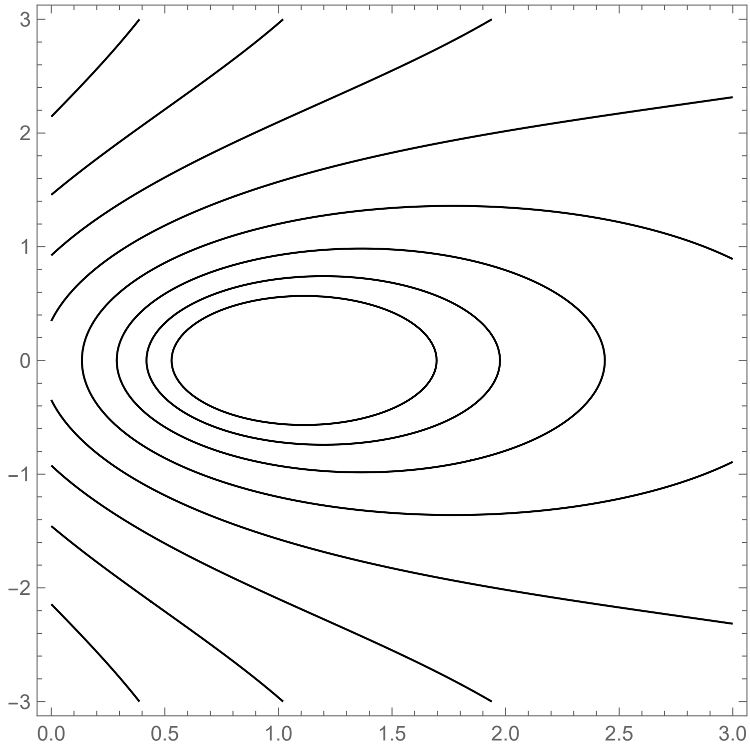

Figure 5. Unshaded contour-plot of $|\zeta_a(x+iy)|$ for $0 \le x \le 3$ and $-3 \le y \le 3$.

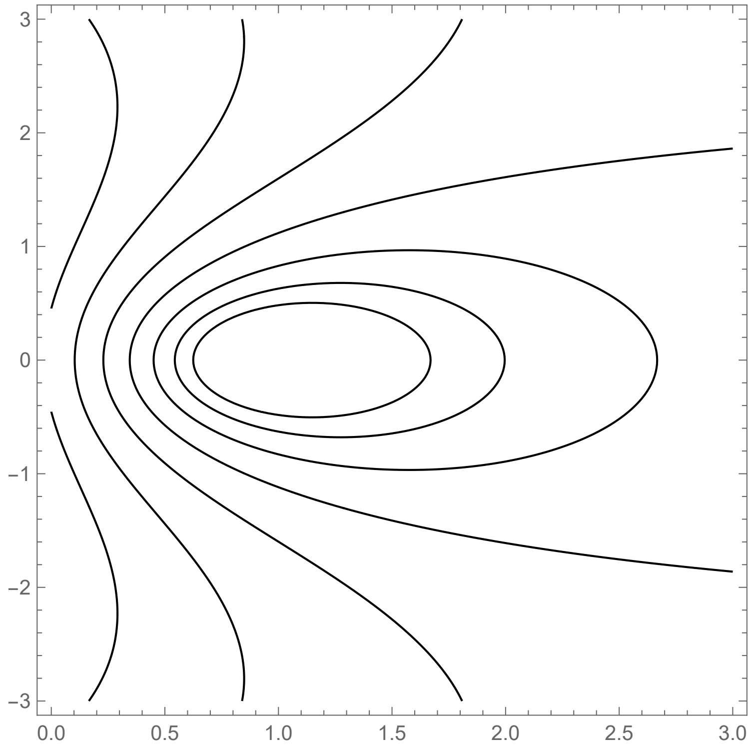

Figure 6. Unshaded contour-plot of $|\zeta(x+iy)|$ for $0 \leq x \leq 3$ and $-3 \leq y \leq 3$.

5 Finding The Zeros of $\zeta_a(z)$

We now calculate the zeros of $\zeta_a(z)$ and study their behavior. We therefore have to solve $\zeta_a(z) = 0$ which implies that,

$$\frac{1}{2} + Ae^{\alpha_1 z}\left(B + Ce^{\alpha_2 z} - De^{\alpha_3 z}\right) = 0. \tag{5.1}$$

If we let $z = x + iy$ and expand equation (5.1) using Euler's formula which is given by,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta), \tag{5.2}$$

we get the following simultaneous equations,

$$\begin{cases} (4.5668\times 10^{-3})\Big[F_1\cos(F_2x+F_3y)+G_1\cos(-F_2x+F_3y)\Big] \\ \quad +(0.499969927)\Big[-F_1\sin(F_2x+F_3y)+G_1\sin(-F_2x+F_3y)\Big] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = -0.4908672, \\ (4.5668\times 10^{-3})\Big[F_1\sin(F_2x+F_3y)+G_1\sin(-F_2x+F_3y)\Big] \\ \quad +(0.499969927)\Big[F_1\cos(F_2x+F_3y)-G_1\cos(-F_2x+F_3y)\Big] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = -8.3420\times 10^{-5}, \end{cases} \qquad \begin{matrix}(5.3)\\ \\ \\ (5.4)\end{matrix}$$

where,

$$\begin{aligned} F_1 &= e^{(-1.5812x+0.000599221y)}, \\ F_2 &= -0.000599221, \\ F_3 &= -1.5812, \\ G_1 &= e^{(-1.5812x-5.99219\times 10^{-4}y)}, \\ G_2 &= -F_2, \\ G_3 &= F_3. \end{aligned} \tag{5.5}$$

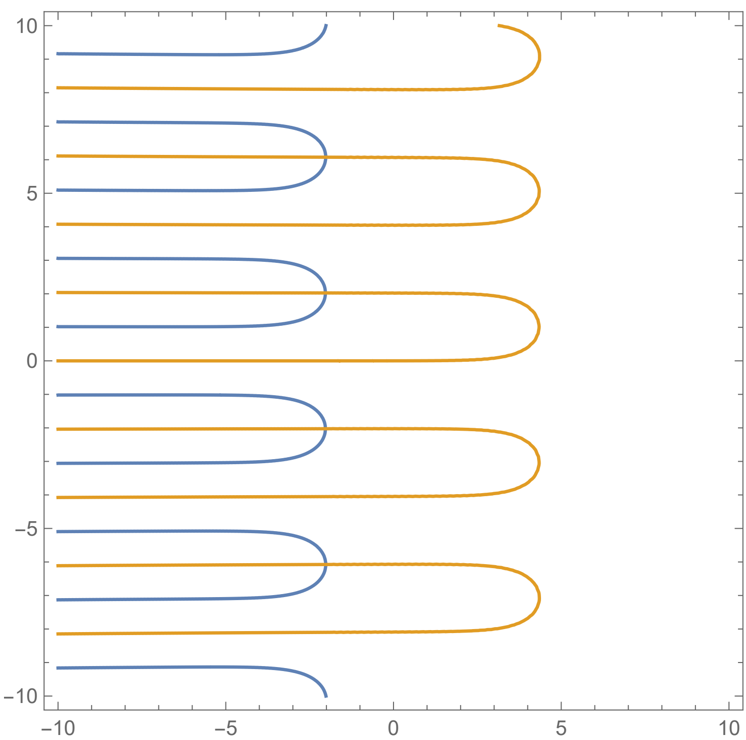

Figure 7. The solutions of the simultaneous equations (5.3) and (5.4) for -10 $\leq x \leq$ 10 and $-10 \leq y \leq 10$. Where the blue curve cuts the yellow curve is where the solutions lie.

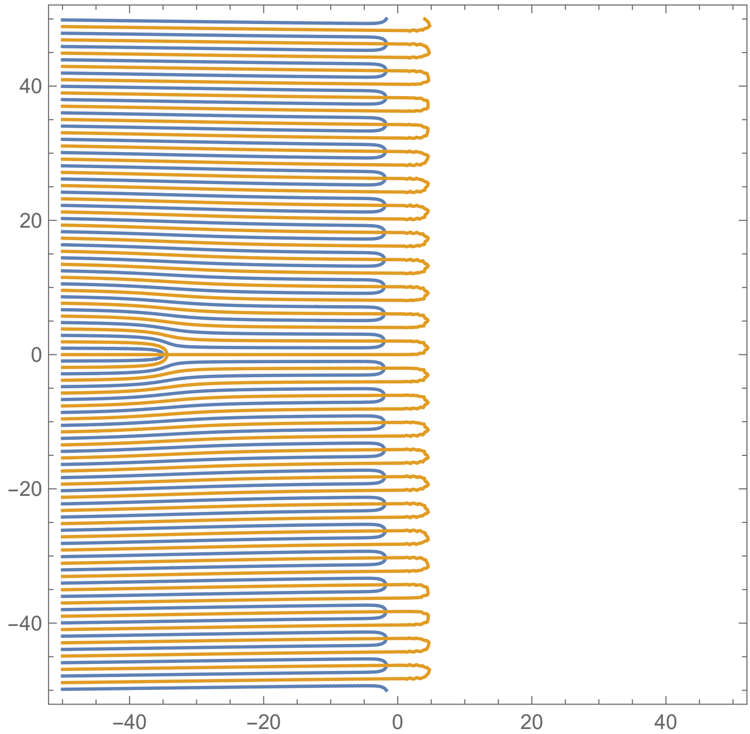

Figure 8. The solutions of the simultaneous equations (5.3) and (5.4) for -50 $\leq x \leq$ 50 and $-50 \leq y \leq 50$. Where the blue curve cuts the yellow curve is where the solutions lie.

When we plot the solutions of the simultaneous equations (5.3) and (5.4) we get the results in figure 7 and 8. In these plots, where the blue curve and the yellow curve intersect is where the zeros of $\zeta_a(z)$ lie. We can see from the plots that these two curves intersect along a vertical line. This is behavior that we expected from an approximate function of the Riemann zeta function. When we solve the simultaneous equations (5.3) and (5.4) numerically we find that the zeros of $\zeta_a(z)$ are given by,

$$z = -2 + i(4n + 10.1638), \tag{5.6}$$

for all $n \in \mathbb{Z}$. As we can see, the expression (5.6) shows that the zeros of $\zeta_a(z)$ lie on the vertical line $z = -2 + iy$, where $y \in \mathbb{R}$. As we mentioned before we expected this behavior since $\zeta_a(z)$ approximates $\zeta(z)$. We also think that if we improve the precision at which $\zeta_a(z)$ approximates $\zeta(z)$ then the vertical line where its zeros lie will coincide with $z = \frac{1}{2} + iy$, where $y \in \mathbb{R}$, which is the vertical line where the zeros of $\zeta(z)$ lie. In table 6, we compare 22 zeros of $\zeta_a(z)$ and $\zeta(z)$. The table lists the imaginary parts. Table 6 shows that even though the imaginary parts of the zeros of $\zeta_a(z)$ and $\zeta(z)$ don't exactly coincide, they're however still not far from each other.

6 More on The Zeros of $\zeta_a(z)$

Now we want to know whether $\zeta_a(z)$ has any zeros outside of (5.6). In order to find out, we first reduce equation (5.1) into a simpler approximate form. We first rearrange (5.1) and write it as follows,

$$Ce^{\alpha_2 z} - De^{\alpha_3 z} = -B - \left(\frac{1}{2}\right) A^{-1} e^{-\alpha_1 z}. \tag{6.1}$$

Table 6.

$\zeta_a(z)$	$\zeta(z)$
14.1638	14.1347
18.1995	21.0220
22.2289	25.0109
26.2523	30.4249
30.2700	32.9351
34.2826	37.5862
38.2908	40.9187
42.295	43.3271
46.2958	48.0052
50.2936	49.7738
54.289	52.9703
58.2821	56.4462
62.2733	59.3470
66.2628	60.8318
70.2509	65.1125
74.2377	67.0798
78.2234	69.5464
82.2081	72.0672
86.1919	75.7047
90.175	77.1448
94.1573	79.3374
98.139	82.9104

Now (4.3) shows that $C = 1$ and $\alpha_2 = 0.00119844i \approx 0$. This therefore implies that $Ce^{\alpha_2 z} \approx e^0 = 1$. Therefore (6.1) reduces to,

$$1 - De^{\alpha_3 z} \approx -B - \left(\frac{1}{2}\right)A^{-1}e^{-\alpha_1 z}. \tag{6.2}$$

We then note that $-\alpha_1 = \alpha_3$, which further reduces (6.2) to,

$$\left[-D + \left(\frac{1}{2}\right)A^{-1}\right]e^{\alpha_3 z} \approx -(B+1). \tag{6.3}$$

We can also see in (4.3) that $D = 0.000166851 + 0.0182667i$ which is approximately $0.0182667i$. We also have that $B = -0.999833 + 0.0182667i$ which implies that $-(B+1) \approx -0.0182667i$. When we substitute this approximate D and $-(B+1)$ into equation (6.3) we get,

$$\left[-0.0182667i + \left(\frac{1}{2}\right)A^{-1}\right]e^{\alpha_3 z} \approx -0.0182667i. \tag{6.4}$$

From (4.3) we also have that $A = 0.00456688 - 0.49997i$ which implies that $A^{-1} = 0.0182697 + 2i$. When we substitute A^{-1}into equation (6.4) we get that $e^{\alpha_3 z} \approx$

$-1.7312 \times 10^{-4}i - 0.0186$ which is approximately -0.0186. Therefore through approximation equation (5.1) reduces to,

$$e^{\alpha_3 z} \approx -0.0186. \tag{6.5}$$

Let's suppose that there exists $a + bi$ where $a + bi \neq -2 + i(4n + 10.1638)$ for all $n \in \mathbb{Z}$ such that,

$$e^{\alpha_3(a+bi)} \approx -0.0186. \tag{6.6}$$

Since $a + bi \neq -2 + i(4n + 10.1638)$ for all $n \in \mathbb{Z}$, we have that $a \neq -2$ and that $b \neq 4n + 10.1638$ for all $n \in \mathbb{Z}$. Which then implies that there exists $\delta \in \mathbb{R}$ where $\delta \neq 0$ such that $a = -2 + \delta$. It also implies that for each $n \in \mathbb{Z}$ there exists $\epsilon \in \mathbb{R}$ where $\epsilon \neq 0$ such that $b = (4n + 10.1638) + \epsilon$. This then implies that,

$$e^{\alpha_3\left([-2+\delta]+i[(4n+10.1638)+\epsilon]\right)} \approx -0.0186. \tag{6.7}$$

Expanding equation (6.7) using Euler's formula gives us the following equations,

$$\begin{cases} (-2.2755)\cos(16.071 + 6.3248n) = e^{-1.5812\delta}\cos(-1.5812\epsilon), & (6.8) \\ (-2.2755)\sin(16.071 + 6.3248n) = e^{-1.5812\delta}\sin(-1.5812\epsilon). & (6.9) \end{cases}$$

Suppose that,

$$\frac{\cos(\theta_1)}{\cos(\theta_2)} = \frac{\sin(\theta_1)}{\sin(\theta_2)}, \tag{6.10}$$

which then implies,

$$\sin(\theta_1)\cos(\theta_2) - \sin(\theta_2)\cos(\theta_1) = 0. \tag{6.11}$$

Equation (6.11) then implies that,

$$\sin(\theta_1 - \theta_2) = 0. \tag{6.12}$$

If we let $\theta_1 = -1.5812\epsilon$ and $\theta_2 = 16.071 + 6.3248n$ then we get,

$$\sin\left[-1.5812\epsilon - (16.071 + 6.3248n)\right] = 0. \tag{6.13}$$

As we've stated before, for each $n \in \mathbb{Z}$ there exists $\epsilon \in \mathbb{R}$ where $\epsilon \neq 0$ such that $b = (4n + 10.1638) + \epsilon$. This implies that there exists pairs (n, ϵ_n) such that $b = (4n + 10.1638) + \epsilon_n$ for all $n \in \mathbb{Z}$. We also have that,

$$-\alpha_3 b = -1.5812\epsilon_n - 16.071 - 6.3248n, \tag{6.14}$$

for all $n \in \mathbb{Z}$ and this means that (6.13) implies the following,

$$\sin\left[-1.5812\epsilon_n - 16.071 - 6.3248n\right] = 0, \tag{6.15}$$

for all $n \in \mathbb{Z}$. This then implies that there exists $k \in \mathbb{Z}$ such that,

$$-1.5812\epsilon_n - 16.071 - 6.3248n = k\pi, \tag{6.16}$$

for all $n \in \mathbb{Z}$. Which then implies that,

$$\epsilon_n = \frac{k\pi + 16.071 + 6.3248n}{-1.5812}, \tag{6.17}$$

for all $n \in \mathbb{Z}$. So for $n = 0$ (6.17) gives,

$$\epsilon_0 = \frac{k\pi + 16.071}{-1.5812}. \tag{6.18}$$

So if we write ϵ_n in terms of ϵ_0 for all $n = 1, 2, 3, ...$ we get the following formula,

$$\epsilon_n = \epsilon_0 - 4n, \tag{6.19}$$

for all $n \in \mathbb{Z} \setminus \{0\}$. If we let $k = 1$ then (6.18) gives $\epsilon_0 \approx -12.1506$ and we use this with (6.19) to generate pairs (n, ϵ_n) for which we find that $b = -1.9868$, $\delta = -0.51998$ and $a = -2.51998$. This therefore implies that,

$$a + bi = -2.51998 - 1.9868i. \tag{6.20}$$

We find that (6.20) does indeed satisfy (6.5) and that,

$$\zeta_a(-2.51998 - 1.9868i) = -0.00638676 + 0.00651022i \approx 0. \tag{6.21}$$

So we have shown that $\zeta_a(z)$ does have at least one zero outside of (5.6) and it's given by (6.20).

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