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Generalized Notion of Conjugacy in Semigroups

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Abstract In this paper, we study the new notion $\sim_r$ notion of conjugacy in subsemigroups of partial transformation semigroup through restricted homomorphism of the digraphs.

1 introduction

If G is a group and $a, b \in G$ then a is said to be conjugate to b if there exists $g \in G$ such that $a = gbg^{-1}$ which is equivalent to $ag = gb$. Due to this fact $\sim_l$ notion was introduced in a semigroup S as

$$x \sim_l y \Leftrightarrow \exists\, p \in S^1 \text{ such that } xp = py$$

where S^1 is S with an identity adjoined. If $x \sim_l y$, we say x is left conjugate to y (see [4], [13] and [14]). The relation $\sim_l$ is always reflexive and transitive in any semigroup but not symmetric in general. Lallement in [7] has defined the conjugate elements of a free semigroup S as those related by $\sim_l$ and showed that $\sim_l$ is equal to the following equivalence on the free semigroup S:

$$x \sim_p y \Leftrightarrow \exists\, u, v \in S^1 \text{ such that } x = uv \text{ and } y = vu$$

The relation $\sim_p$ is always reflexive and symmetric but not transitive in general.

The relation $\sim_l$ has been restricted to $\sim_o$ in [4], and $\sim_p$ has been extended to $\sim_p^*$ in [5] and in [6], in such a way that the modified relations are equivalences on an arbitrary semigroup S:

$$x \sim_o y \Leftrightarrow \exists\ \ p, q \in S^1 \text{ such that } xp = py \text{ and } yq = qx.$$

$$\sim_p^* = \text{ the transitive closure of } \sim_p \text{ (i.e., the smallest transitive relation on } S \text{ containing } \sim_p).$$

The relation $\sim_o$ is not useful for semigroups S with zero since for every such S, we have $\sim_o = S \times S$. This deficiency has been remedied in [8] by Araujo et al., where the following relation has been defined on an arbitrary semigroup S,

$$x \sim_c y \Leftrightarrow \exists p \in \mathbb{P}^1(x), q \in \mathbb{P}^1(y) \text{ such that } xp = py \text{ and } yq = qx,$$

where for $x \neq 0$, $\mathbb{P}(x) = \{p \in S : (mx)p \neq 0 \text{ for all } mx \in S^1x \setminus \{0\}\}$ denotes the left principal ideal generated by x and $\mathbb{P}(0) = \{0\}$. The relation $\sim_c$ is an equivalence relation in any semigroup and does not reduce to $S \times S$ if S has a zero, and it is equal to $\sim_o$ if S does not have a zero.

Furthermore, J. Konieczny in [10] introduced the $\sim_n$ notion of conjugacy in semigroup S as

$$x \sim_n y \Leftrightarrow \exists\, p, q \in S^1 \text{ such that } xp = py, yq = qx, x = pyq \text{ and } y = qxp.$$

This relation is an equivalence relation in any semigroup and does not reduce to universal relation in a semigroup S with zero.

For a non-empty set X, $\mathcal{P}(X)$ denotes the set of all *partial transformations* on X and it forms a semigroup under operation as composition of maps and is known as partial transformation semigroup. For each $\rho \in \mathcal{P}(X)$, the *domain* of ρ is denoted by

$$\text{dom}(\rho) = \{x \in X : \text{ there exists } y \in X \text{ with } (x, y) \in \rho\},$$

the *image* of ρ is denoted by

$$\text{im}(\rho) = \{y \in X : \text{ there exists } x \in X \text{ with } (x, y) \in \rho\},$$

and the *span* of ρ is denoted by

$$\text{span}(\rho) = \text{dom}(\rho) \cup \text{im}(\rho).$$

By $\sigma \neq 0$ we mean ρ such that dom(ρ) $\neq \emptyset$.

A semigroup S is called an *inverse semigroup* if for every $a \in S$, there is a unique $a^{-1} \in S$ (called the inverse of a) such that

$$aa^{-1}a = a \text{ and } a^{-1}aa^{-1} = a^{-1}.$$

For a non-empty set X, denote by $\mathcal{I}(X)$ the symmetric inverse semigroup on X, which is the subsemigroup of $\mathcal{P}(X)$ consisting of all partial injective transformations on X. Both $\mathcal{P}(X)$ and $\mathcal{I}(X)$ have the symmetric group Sym(X) of permutations on X as their group of units, and the zero in P(X) is an element of $\mathcal{I}(X)$. The semigroup $\mathcal{I}(X)$ is universal for the class of inverse semigroups because of the Vagner-Preston theorem that states that every inverse semigroup can be embedded in some $\mathcal{I}(X)$ [12, Theorem 5.1.7]. This is analogous to the Cayley theorem for groups that states that every group can be embedded in some symmetric group Sym(X).

Next we discuss $\sim_i$ notion of conjugacy in inverse semigroups. This was introduced by Araujo et al. in [11].

Definition 1.1. *let S be an inverse semigroup and $a, b \in S$. Then $a \sim_i b$ if and only if there exists $g \in S^1$ such that*

$$g^{-1}ag = b \textit{ and } gbg^{-1} = a.$$

We refer the reader to Howie [12] for any unexplained terminology in semigroups.

2 The notion $\sim_r$

In [2] and [3] we introduced $\sim_r$ notion of conjugacy in semigroups. The notion $\sim_r$ in semigroups is defined as in the following.

Definition 2.1. *Define a relation $\sim_r$ on a semigroup S by*

$$a \sim_r b \Leftrightarrow \exists\ g, h, u, v \in S^1 \textit{ such that } ag = gb, bh = ha, a = gbu \textit{ and } b = hav.$$

The r-notion of conjugacy is an equivalence relation in any semigroup and it does not reduce to a universal relation in a semigroup with zero ([2, Theorem 2.1]). In case S is a group, then $\sim_r$ reduces to the usual notion of conjugacy ([2, Theorem 2.1]). Also $\sim_n \subseteq \sim_r \subseteq \sim_c \subseteq \sim_o$ [2, Theorem 2.2].

Theorem 2.2. *[10, Theorem 2.6] Let S be an inverse semigroup and let $a, b \in S$. Then $a \sim_n b$ if and only if $a \sim_i b$.*

In the next theorem we show $\sim_r$ coincides with $\sim_n$ in an inverse semigroup.

Theorem 2.3. *Let S be an inverse semigroup and let $a, b \in S$. Then $a \sim_r b$ if and only if $a \sim_n b$.*

Proof. By definition of $\sim_n$ and $\sim_r$, we have $\sim_n \subseteq \sim_r$. So for any $a, b \in S$, $a \sim_n b$ implies $a \sim_r b$.

For the converse, we may assume by the Vagner-Preston Theorem that S is a subsemigroup of some symmetric inverse semigroup $\mathcal{I}(X)$. Let $a \sim_r b$ in S, then there exist $g, h, u, v \in S^1$ such that

$$ag = gb, bh = ha, a = gbu \text{ and } b = hav.$$

We claim $agg^{-1} = a$. Clearly dom(agg^{-1}) $\subseteq$ dom(a). Let $x \in$ dom(a) implies $xa \in$ im(a) $\subseteq$ dom(g) implies $(xa) \in$ dom(g), which implies $(xa)g \in$ dom(g^{-1}). Hence $x \in$ dom(agg^{-1}), which implies dom(a) $\subseteq$ dom(agg^{-1}). Thus dom(a) = dom(agg^{-1}). Next for every $x \in$ dom(a),

$x(agg^{-1}) = (xa)gg^{-1} = xa$. So $agg^{-1} = a$. Since $ag = gb$ implies $agg^{-1} = gbg^{-1}$ and so $a = gbg^{-1}$.

Next we claim that $g^{-1}gb = b$. For if

$$\begin{aligned} & g^{-1}gb \neq b \\ \Rightarrow\ & g^{-1}ag \neq b \\ \Rightarrow\ & g^{-1}agg^{-1} \neq bg^{-1} \\ \Rightarrow\ & g^{-1}a \neq bg^{-1} \\ \Rightarrow\ & g^{-1}gbu \neq bg^{-1} \\ \Rightarrow\ & gg^{-1}gbu \neq gbg^{-1} \\ \Rightarrow\ & gbu \neq gbg^{-1} \\ \Rightarrow\ & a \neq gbg^{-1} \end{aligned}$$

which is a contradiction. Hence $g^{-1}gb = b$. Since $ag = gb$, we have $g^{-1}ag = g^{-1}gb$ we have $g^{-1}ag = b$. Thus $a \sim_i b$ and so by Theorem 2.2, $a \sim_n b$. □

Due to Theorem 2.2 and Theorem 2.3 we have the following corollary.

Corollary 2.4. *Let S be an inverse semigroup. Then $\sim_n = \sim_r = \sim_i$ in S.*

3 $\sim_r$ in general subsemigroups of $\mathcal{P}(X)$

Definition 3.1. *Let A be any set (not necessarily finite and possibly empty) and E be a binary relation on A, then $\Gamma = (A, E)$ is called a directed graph (or a digraph). We call any $p \in A$ a vertex and any $(p, q) \in E$ an arc of Γ.*

For example, let $A = \{1, 2, 3, 4\}$ and $E = \{(1, 2), (1, 4), (2, 3), (4, 1)\}$. Then the digraph Γ is as under,

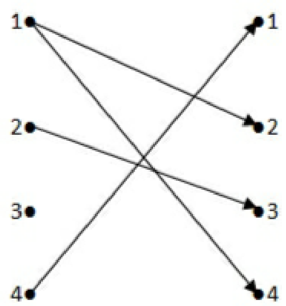

Figure 1. Digraph

Definition 3.2. *A vertex $p \in A$ is said to be an initial vertex if there is no $q \in A$ for which $(q, p) \in E$ while a vertex $p \in A$ is said to be a non-initial vertex if $(q, p) \in E$ for some $q \in A$.*

Definition 3.3. *A vertex $p \in A$ for which there exists no q in A such that $(p, q) \in E$ is called a terminal vertex of Γ.*

Remark 3.4. *Let $\rho \in P(X)$. Then ρ can be represented by the digraph $\Gamma(\rho) = (A, E)$, where $A = span(\rho)$ and for all $x, y \in A$, $(x, y) \in E$ if and only if $x \in \mathrm{dom}(\rho)$ and $x\rho = y$. For example,* *the partial transformation*

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 7 & 8 & \dots \\ 2 & 3 & 1 & 8 & 9 & \dots \end{pmatrix} \in \mathcal{P}(X),$$

where $X = \{1, 2, 3, \dots\}$ *is represented by the digraph as in figure* 2

Figure 2. The Digraph of a Transformation.

Remark 3.5. *For a non empty* X*, we fix an element* $\diamond \notin X$*. For* $\alpha \in \mathcal{P}(X)$ *and* $x \in X$*, we will write* $x\alpha = \diamond$ *if and only if* $x \notin$ *dom*(α)*. We also assume that* $\diamond\alpha = \diamond$*. With this notation it makes sense to write* $x\alpha = y\beta$ *or* $x\alpha \neq y\beta$ $(\alpha, \beta \in \mathcal{P}(X), x, y \in X)$ *even when* $x \notin$ *dom*(α) *or* $y \notin$ *dom*(β)*. For any* $\alpha \in \mathcal{P}(X)$*, by* $\alpha \neq 0$*, we mean dom*$(\alpha) \neq \emptyset$*. Thus* $\alpha = 0$ *if and only if dom*(α) *=* $\emptyset$*.*

Definition 3.6. *Let* $\Gamma = (A, E)$ *and* $\Lambda = (B, F)$ *be digraphs. A mapping* ρ *from* A *to* B *is called a* $homomorphism$ *from* Γ *to* Λ *if for all* $p, q \in A$*,* $(p, q) \in E$ *implies* $(p\rho, q\rho) \in F$*.*

Definition 3.7. *Let* $\Gamma = (A, E)$ *and* $\Lambda = (B, F)$ *be digraphs. A homomorphism* $\alpha : A \to B$ *is called a restricted homomorphism from* Γ *to* Λ*, if*

(i) *for every terminal vertex* x *of* Γ*,* $x\alpha$ *is a terminal vertex of* Λ;

(ii) *for every initial vertex* x *of* Γ*, either* $x\alpha$ *is an initial vertex of* Λ *or there are vertices* t, z, y *of* Γ *such that* $(x, y), (t, z), (z, y) \in E$*.*

Throughout this paper, by a *hom*, we shall mean a homomorphism, and by a *restricted hom*, we shall mean a restricted homomorphism.

Now in order to prove the main theorem of this section we need the following two lemmas.

Lemma 3.8. *Let* $\rho, \pi, \alpha, \beta, \sigma, \tau \in \mathcal{P}(X)$ *be such that* $\rho\alpha = \alpha\pi, \pi\beta = \beta\rho, \rho = \alpha\pi\sigma$*,* $\pi = \beta\rho\tau$*. Then* dom(α) = span(ρ)*,* dom(β) = span(π)*.*

Proof. Let $x \in$ span(ρ) which implies $x \in$ dom$(\rho) \cup$ im(ρ). If $x \in$ dom(ρ) then as $\rho = \alpha\pi\sigma$ which means $x \in$ dom$(\alpha)$and if $x \in$ im(ρ) then as $\rho\alpha = \alpha\pi$, $x \in$ dom(α). Thus span$(\rho) \subseteq$ dom(α). Next we have to show dom$(\alpha) \subseteq$ span(ρ). For if dom$(\alpha) \nsubseteq$ span(ρ), then there is some $x \in X$ such that $x \in$ dom(α) but $x \notin$ span(ρ), which implies there is some $z \in X$ such that $x\alpha\pi\sigma = z$ (as $\rho = \alpha\pi\sigma$) which implies $x\rho = z$, which is a contradiction as $x \notin$ span(ρ). Thus dom$(\alpha) \subseteq$ span(ρ). Hence dom(α) = span(ρ). Similarly we can prove that dom(β) = span(π). □

Lemma 3.9. *Let* $\rho, \pi, \alpha, \beta, \sigma, \tau \in \mathcal{P}(X)$ *be such that* $\rho\alpha = \alpha\pi, \pi\beta = \beta\rho, \rho = \alpha\pi\sigma, \pi = \beta\rho\tau$ *i.e,* $\rho \sim_r \tau$ *satisfying*

$$\rho\tau\sigma = \rho \text{ and } \pi\sigma\tau = \pi. \tag{1.1}$$

Then α *is a restricted hom from* $\Gamma(\rho)$ *to* $\Gamma(\pi)$ *and* β *is a restricted hom from* $\Gamma(\pi)$ *to* $\Gamma(\rho)$*.*

Proof. Let $\Gamma(\rho) = (A, E)$ and $\Gamma(\pi) = (B, F)$, where $A =$ span(ρ) and $B =$ span(π). Suppose that $(x, y) \in E$ i.e, $x\rho = y$. Then,

$$(x\alpha)\pi = x(\alpha\pi) = x(\rho\alpha) = (x\rho)\alpha = y\alpha.$$

Hence $(x\alpha, y\alpha) \in F$, which implies α is a hom from $\Gamma(\rho)$ to $\Gamma(\pi)$.

Suppose that x is a terminal vertex of $\Gamma(\rho)$. Since $\rho\alpha = \alpha\pi$ and $x \notin$ dom(ρ), so $x\rho = \diamond$ which implies $x\rho\alpha = \diamond$ which implies $x\alpha\pi = \diamond$. Thus $x\alpha$ is a terminal vertex of $\Gamma(\pi)$.

Suppose that x is an initial vertex of $\Gamma(\rho)$ and let $u = x\alpha$ is not an initial vertex of $\Gamma(\pi)$. Then $v\pi = u$ for some $v \in$ dom(π). Let $t = v\beta$ and $z = u\beta$. Since $\pi\beta = \beta\rho$, the preceding argument for ρ and α applied to π and β shows that β is a hom from $\Gamma(\pi)$ to $\Gamma(\rho)$. Thus $(v, u) \in F$ implies that $(t, z) = (v\beta, u\beta) \in E$. Since $x \in$ span(ρ) and $x \notin$ im(ρ), we have $x \in$ dom(ρ). Setting $y = x\rho$, we have $(x, y) \in E$. Now $y = x\rho = x\alpha\pi\sigma = u\pi\sigma = u\beta\rho\tau\sigma = z\rho\tau\sigma \overset{(1.1)}{=} z\rho$ and so $(z, y) \in E$. By the similar argument, we can prove β is a restricted hom from $\Gamma(\pi)$ to $\Gamma(\rho)$. Hence proved. □

Definition 3.10. *Let S be a subsemigroup of $\mathcal{P}(X)$. We say S is closed under restriction to spans if for all $\rho, \pi \in S$ such that span(ρ)⊆ dom(π), $\pi|_{span(\rho)} \in S$.*

Note that every subsemigroup of the semigroup of $\mathcal{T}(X)$ (semigroup of full transformations on X) is closed under restrictions to spans.

Theorem 3.11. *Let S be a subsemigroup of $\mathcal{P}(X)$ such that S is closed under restrictions to spans and $\rho, \pi \in S$*

(1) *If $\rho \sim_r \pi$ satisfying (1.1), then there exist $\alpha, \beta \in S^1$ such that α is a restricted hom from $\Gamma(\rho)$ to $\Gamma(\pi)$ and β is a restricted hom from $\Gamma(\pi)$ to $\Gamma(\rho)$ with $y\alpha\sigma = y$ for all $y \in$ im(ρ) and $u\beta\tau = u$ for all $u \in$ im(π).*

(2) *Conversely, if α is a restricted hom from $\Gamma(\rho)$ to $\Gamma(\pi)$ and β is a restricted hom from $\Gamma(\pi)$ to $\Gamma(\rho)$ with $y\alpha\sigma = y$ for all $y \in$ im(ρ) and $u\beta\tau = u$ for all $u \in$ im(π). Then $\rho \sim_r \pi$.*

Proof. Let $\Gamma(\rho) = (A, E)$ and $\Gamma(\pi) = (B, F)$.

(1) Let $\rho \sim_r \pi$ then there exist $\delta, \gamma, \sigma, \tau \in S^1$ such that

$$\rho\delta = \delta\pi, \pi\gamma = \gamma\rho, \rho = \delta\pi\sigma \text{ and } \pi = \gamma\rho\tau.$$

Here span(ρ) ⊆ dom(δ) and span(π) ⊆ dom(γ). Let $\alpha = \delta|$ span(ρ), $\beta = \gamma|$ span(π). First we prove $\rho\alpha = \alpha\pi$ and $\rho = \alpha\pi\sigma$. Let $x \in X$. If $x \in$ span(ρ). Then $x\alpha\pi = x\delta\pi = x\rho\delta = x\rho\alpha$ and $x\alpha\pi\sigma = x\delta\pi\gamma = x\rho$. If $x \notin$ span(ρ), i.e., $x\rho = \diamond$, then $x\rho\alpha = \diamond$ and $x\alpha\pi = \diamond$ and also $x\alpha\pi\sigma = \diamond = x\rho$. Similarly we can prove $\pi\beta = \beta\rho$ and $\pi = \beta\rho\tau$. Therefore we have

$$\rho\alpha = \alpha\pi, \pi\beta = \beta\rho, \rho = \alpha\pi\sigma \text{ and } \pi = \beta\rho\tau.$$

Now $\rho = \rho\tau\sigma$ and $\pi = \pi\sigma\tau$. Therefore by Lemma 3.9 we have α as a restricted hom from $\Gamma(\rho)$ to $\Gamma(\pi)$ and β as a restricted hom from $\Gamma(\pi)$ to $\Gamma(\rho)$. Next for all $y \in$ im(ρ), $y\alpha\sigma = x\rho\alpha\sigma$ (for some $x \in$ dom(ρ)) = $x\alpha\pi\sigma = x\rho = y$. Similarly for all $u \in$ im(π), $u\beta\tau = u$.

(2) Let the desired α and β exist with $y\alpha\sigma = y$ for all $y \in$ im(ρ) and $u\beta\tau = u$ for all $u \in$ im(π). We have to prove $\rho \sim_r \pi$. First we will prove that $\rho\alpha = \alpha\pi$ and $\rho = \alpha\pi\sigma$. Let $x \in X$. Two cases arise here.
Case(1): Suppose $x \notin$ dom(ρ). Then $x\rho\alpha = \diamond$.
(i) If $x \notin$ dom(α), then $x(\alpha\pi) = \diamond$ and $x\alpha\pi\sigma = \diamond$. So, $\rho\alpha = \alpha\pi$ and $\rho = \alpha\pi\sigma$ in this case.
(ii) If $x \in$ dom(α), then x is a terminal vertex of $\Gamma(\rho)$, and so $x\alpha$ is a terminal vertex in $\Gamma(\pi)$, which implies that $x(\alpha\pi) = (x\alpha)\pi = \diamond$ and $x\alpha\pi\sigma = \diamond$. Thus $\rho\alpha = \alpha\pi$ and $\rho = \alpha\pi\sigma$ in this case also.
Case(2): Suppose $x \in$ dom(ρ) and let $y = x\rho \in X$ i.e, $(x, y) \in E$ which implies $(x\alpha, y\alpha) \in F$ i.e,

$$(x\alpha)\pi = y\alpha.$$

Now, $x(\rho\alpha) = (x\rho)\alpha = y\alpha$ = $x(\alpha\pi) = (x\alpha)\pi$ and $x\alpha\pi\sigma = [(x\alpha)\pi]\sigma = y\alpha\sigma = y = x\rho$ which implies $x\alpha\pi\sigma = x\rho$. Thus in both the cases we have proved that

$$\rho\alpha = \alpha\pi, \rho = \alpha\pi\sigma \tag{1.2}$$

Similarly by using that β is restricted hom from $\Gamma(\pi)$ to $\Gamma(\rho)$ with $u\beta\tau = u$ for all $u \in \text{im}(\pi)$ we can prove that

$$\pi\beta = \beta\rho \text{ and } \pi = \beta\rho\tau. \tag{1.3}$$

Therefore by combining (1.2) and (1.3) we get $\rho \sim_r \pi$.

□

4 $\sim_r$ through the trims of digraphs

Definition 4.1. *Let $\Gamma = (A, E)$ and $\Lambda = (B, F)$ be digraphs. A mapping $\alpha : A \to B$ is called an isomorphism from Γ to Λ if α is a bijection and for all $x, y \in A, (x, y) \in E$ if and only if $(x\alpha, y\alpha) \in F$. Note that a bijection $\alpha : A \to B$ is an isomorphism if and only if both α and α^{-1} are isomorphisms. We will say that Γ and Λ are isomorphic, written $\Gamma \cong \Lambda$, if there exists an isomorphism from Γ to Λ.*

For $\rho \in \mathcal{P}(X)$ and $y \in \text{im}(\rho)$, denote by $y\rho^{-1}$ the set of all elements $x \in X$ such that $x\rho = y$. Note that $y\rho^{-1}$ is not empty (since $y \in \text{im}(\rho)$) and that it is the set of all vertices x in $\Gamma(\rho)$ such that (x, y) is an edge in $\Gamma(\rho)$.

Definition 4.2. *Let $\rho \in \mathcal{P}(X)$ and $\Gamma(\rho) = (A, E)$. Denote by A^i the set of initial vertices of $\Gamma(\rho)$. For every non-initial vertex y of $\Gamma(\rho)$ such that $y\rho^{-1} \subseteq A^i$, select $y^* \in y\rho^{-1}$. Let A_t and E_t be sets of vertices and edges respectively, defined by*

$$A_t = (A \setminus A^i) \cup \{y^* : y \text{ is a non-initial vertex of } \Gamma(\rho) \text{ and } y\rho^{-1} \subseteq A^i\},$$
$$E_t = \{(x, y) \in E : x, y \in A_t\}.$$

Then the digraph $\Gamma_t(\rho) = (A_t, E_t)$ will be called $trim$ of $\Gamma(\rho)$.

In other words, a trim $\Gamma_t(\rho)$ is obtained from $\Gamma(\rho)$ by removing all initial vertices of $\Gamma(\rho)$ with the following exception: If y is a non-initial vertex of $\Gamma(\rho)$ and all vertices in $y\rho^{-1}$ are initial, then exactly one of these vertices (denoted y^* in Definition 4.2) are retained. There may be multiple trims of $\Gamma(\rho)$ since there may be multiple choices for y^*. However a trim of $\Gamma(\rho)$ is unique upto isomorphism.

For a function $f : A \to B$, the rank of f, denoted rank(f), is the cardinality of the image of f.

Lemma 4.3. *[10, Lemma 4.4] Let $\rho, \pi \in \mathcal{P}(X)$ with $\Gamma(\rho) = (A, E)$ and $\Gamma(\pi) = (B, F)$. Suppose that δ is an isomorphism from trim($\Gamma(\rho)$) $= (A_t, E_t)$ to trim($\Gamma(\pi)$) $= (B_t, F_t)$.*

(1) *Let y be a non-initial vertex of trim($\Gamma(\rho)$) and let $u = y\delta$. Then $u\pi^{-1} \neq \emptyset$. Moreover, if $y\rho^{-1} \subseteq A^i$, then $u\pi^{-1} \subseteq B^i$ and $y^*\delta = u^*$.*

(2) *there exist a restricted hom α of rank $|A_t|$ from $\Gamma(\rho)$ to $\Gamma(\pi)$ such that $x\alpha = x\delta$ for every vertex x of trim($\Gamma(\rho)$).*

Lemma 4.4. *Let $\rho, \pi, \alpha, \beta, \sigma, \tau \in \mathcal{P}(X)$ and let $\rho \sim_r \pi$ satisfying (3.1). Then $\alpha|\text{im}(\rho) = \tau$ and $\beta|\text{im}(\pi) = \sigma$.*

Proof. Let $\rho \sim_r \pi$ then there exist $\alpha, \beta, \sigma, \tau$ such that $\rho\alpha = \alpha\pi, \pi\beta = \beta\rho, \rho = \alpha\pi\sigma$ and $\pi = \beta\rho\tau$. Let $y \in \text{im}(\rho)$ then there exist $x \in X$ such that $y = x\rho$. Now

$$\begin{aligned} y\alpha &= x\rho\alpha \\ &= x\alpha\pi \\ &= x\alpha\pi\sigma\tau \\ &= x\rho\tau \\ &= y\tau. \end{aligned}$$

Similarly, $\beta|\text{im}(\pi) = \sigma$. □

Lemma 4.5. *Let $\sigma, \tau \in \mathcal{P}(X)$ with $\sigma \sim_r \tau$ satisfying (3.1). Let y be a non-initial vertex of $\Gamma(\rho)$ and $u = y\alpha \in B$. Then u is not initial in $\Gamma(\pi)$. Moreover, if $y\rho^{-1} \subseteq A^i$, then $u\pi^{-1} \subseteq B^i$ and for every $x \in y\rho^{-1}, x\alpha \in u\pi^{-1}$.*

Proof. Let $\sigma, \tau \in \mathcal{P}(X)$ with $\sigma \sim_r \tau$ satisfying (3.1), then by Theorem 3.11 there exists $\alpha, \beta \in \mathcal{P}(X)$ such that α is a restricted hom from $\Gamma(\rho)$ to $\Gamma(\pi)$ and β is a restricted hom from $\Gamma(\pi)$ to $\Gamma(\rho)$ with $y\alpha\sigma = y$ for all $y \in \text{im}(\rho)$ and $u\beta\tau = u$ for all $u \in \text{im}(\pi)$. Let $\Gamma(\rho) = (A, E)$ and $\Gamma(\pi) = (B, F)$. Since y is not initial, $(x, y) \in E$ for some $x \in A$. Then $(x\alpha, u) = (x\alpha, y\alpha) \in F$, and so u is a non-initial vertex of $\Gamma(\pi)$. Suppose to the contrary that $u\pi^{-1} \nsubseteq B^i$, that is, $(v, u) \in F$ for some non-initial $v \in B$. Then there is $t \in B$ such that $(t, v) \in F$. Since β is a hom from $\Gamma(\pi)$ to $\Gamma(\rho)$, $(t\beta, v\beta), (v\beta, u\beta) \in E$. But $u\sigma = y\alpha\sigma = y$ (since $y \in \text{im}(\rho)$). Now, by Lemma 4.4 $u\beta = y$. So $(t\beta, v\beta)$, $(v\beta, y) \in E$ which contradicts the hypothesis that $y\rho^{-1} \subseteq A^i$. Hence $u\pi^{-1} \subseteq B^i$. Finally, if $x \in y\rho^{-1}$, then $(x, y) \in E$, and so $x\alpha \in u\pi^{-1}$ since $(x\alpha, u) = (x\alpha, y\alpha) \in F$. □

Now we have the main result on r-notion of conjugacy in the semigroup $\mathcal{P}(X)$ through trims of the digraphs.

Theorem 4.6. *Let $\rho, \pi \in \mathcal{P}(X)$*

(1) *Let $\rho \sim_r \pi$ satisfying (3.1). Then trim($\Gamma(\rho)$) $\cong$ trim($\Gamma(\pi)$).*

(2) *Conversely, if trim($\Gamma(\rho)$) $\cong$ trim($\Gamma(\pi)$) with $y\alpha\sigma = y$ for all $y \in$ im(ρ) and $u\beta\tau = u$ for all $u \in$ im(π),then $\rho \sim_r \pi$.*

Proof. (1) Let $\Gamma(\rho) = (A, E), \Gamma(\pi) = (B, F)$, trim($\Gamma(\rho)$) $= (A_t, E_t)$, and trim($\Gamma(\pi)$) $= (B_t, F_t)$. Suppose that $\rho \sim_r \pi$ then by Theorem 3.11, there exists $\alpha, \beta, \sigma, \tau$ such that α is a restricted hom from $\Gamma(\rho)$ to $\Gamma(\pi)$ and β is a restricted hom from $\Gamma(\pi)$ to $\Gamma(\rho)$ with $y\alpha\sigma = y$ for all $y \in \text{im}(\rho)$ and $u\beta\tau = u$ for all $u \in \text{im}(\pi)$. Suppose that x is an initial vertex of trim($\Gamma(\rho)$). Let $y = x\rho \in A$ and $u_x = y\alpha = y\tau$ (by Lemma 4.4) $\in B$. By Lemma 4.5, u_x is a non-initial vertex of trim($\Gamma(\pi)$) and $u_x\pi^{-1} \subseteq B^i$ (so $(u_x)^*$ exists). Therefore, we can define $\mu : A_t \to B_t$ by

$$x\mu = \begin{cases} x\alpha & \text{if } x \text{ is not initial in trim}(\Gamma(\rho)), \\ (u_x)^* & \text{otherwise.} \end{cases}$$

We claim that μ is an isomorphism from trim($\Gamma(\rho)$ to trim($\Gamma(\pi)$). Let $(x, y) \in E_t$. If x is not initial, then $(x\mu, y\mu) = (x\alpha, y\alpha) \in F$, and so $(x\mu, y\mu) \in F_t$ since $x\mu, y\mu \in B_t$. Suppose that x is initial and let $y = x\rho$. Then $(x\mu, y\mu) = ((u_x)^*, u_x) \in F_t$. Hence μ is a hom.

Let $x, s \in A_t$ be such that $x\mu = s\mu$. If x and s are both not initial, then $x\alpha = s\alpha$ (since $x\mu = x\alpha$ and $s\mu = s\alpha$), and so $x = x(\alpha\sigma)$(by Theorem 3.11) $= (x\alpha)\sigma = (s\alpha)\sigma = s(\alpha\sigma) = s$. Suppose that at least one of x and s, say x, is initial. Then $x\mu = (u_x)^* \in B_t$ is initial. So s must be initial since otherwise, $s\mu = s\alpha$ would not be initial (by Lemma 4.5), which would contradict $x\mu = s\mu$. Thus, $s\mu = (u_s)^*$. Let $y = x\rho$ and $z = s\rho$, so $y\alpha = u_x$ and $z\alpha = u_s$. Since y and z are not initial, we have $y\mu = y\alpha = z\alpha = z\mu$, and so $y = z$ by the preceding argument. Hence $x = y^* = z^* = s$. We have proved that μ is injective.

Let $v \in B_t$. If v is not initial, then $y = v\beta \in A$ is not initial (so $y \in A_t$), and so $y\mu = y\alpha = y\tau = v\beta\tau = v$. Suppose that v is initial and let $u = v\pi$. Then, by Lemma 4.5, $y = u\beta$ is not initial and $y\rho^{-1} \subseteq A^i$. Let $x = y^* \in A_t$, so $y = x\rho$. Then $x\mu = (u_x)^* = (y\alpha)^* = (y\tau)^*$ (by Lemma 4.4) $= ((u\beta)\tau)^* = (u\beta\tau)^* = u^* = v$, where the last equality is true since there is only one initial vertex of trim($\Gamma(\pi)$) in $u\pi^{-1}$. We have proved that μ is surjective.

Hence μ is a bijective hom from trim($\Gamma(\rho)$) to trim($\Gamma(\pi)$) such that for every non-initial $y \in A_t, y\mu = y\alpha = y\tau$ (by Lemma 4.4) and $y^*\mu = (y\alpha)^* = (y\tau)^*$ if $y\rho^{-1} \subseteq A^i$. Similarly, we can define bijective hom λ from trim($\Gamma(\pi)$) to trim($\Gamma(\rho)$) such that for every non-initial $u \in B_t, u\lambda = u\beta = u\sigma$(by Lemma 4.4) and $u^*\lambda = (u\beta)^* = (u\sigma)^*$ if $u\pi^{-1} \subseteq B^i$. Let $x \in A_t$. If x is not initial, then $x\alpha \in B_t$ is not initial and $x(\mu\lambda) = (x\mu)\lambda = (x\alpha)\lambda =$

$(x\alpha)\beta = x(\alpha\sigma) = x$. Suppose that x is initial and let $y = x\rho$. Then $x = y^*$ and $x(\mu\lambda) = (y^*\mu)\lambda = (y\alpha)^*\lambda = ((y\alpha)\beta)^* = ((y\alpha)\sigma)^*$(by Lemma 4.4) $= (y(\alpha\sigma))^* = y^* = x$. Similarly $v(\lambda\mu) = v$ for every $v \in B_t$. Hence $\lambda = \mu^{-1}$ and so μ is an isomorphism.

(2) Conversely, suppose that trim($\Gamma(\rho)$) $\cong$ trim($\Gamma(\pi)$) and let $\mu : A_t \to B_t$ be an isomorphism from trim($\Gamma(\rho)$) to trim($\Gamma(\pi)$). Then $\mu^{-1} : B_t \to A_t$ is an isomorphism from trim($\Gamma(\pi)$) to trim($\Gamma(\rho)$). By Lemma 4.3, there are restricted homs α from $\Gamma(\rho)$ to $\Gamma(\pi)$ and β from $\Gamma(\pi)$ to $\Gamma(\rho)$. By given condition $y\alpha\sigma = y$ for all $y \in$ im(ρ) and $u\beta\tau = u$ for all $u \in$ im(π). Now apply Theorem 3.11, we get $\rho \sim_r \pi$.

□

5 Characterization of $\sim_r$ in the proper ideals of $\mathcal{P}(X)$

By a proper ideal of a semigroup S we mean an ideal I of S such that $I \neq S$. For a cardinal k with $0 < k \leq |X|$, denote by P_k the set of all $\rho \in \mathcal{P}(X)$ such that rank(ρ) $< k$. It is well known (see [**?**, Sec 2.2]) that the set $\{P_k : 0 < k \leq |X|\}$ is the set of proper ideals of $\mathcal{P}(X)$.

Theorem 5.1. *Let P_k be a proper ideal of $\mathcal{P}(X)$ and let $\rho, \pi \in P_k$ with $\Gamma(\rho) = (A, E), \Gamma(\pi) = (B, F)$, trim($\Gamma(\rho)$) $= (A_t, E_t)$ and trim($\Gamma(\pi)$) $= (B_t, F_t)$.*

(1) *If k is infinite, let $\rho \sim_r \pi$ satisfying 3.1. Then trim($\Gamma(\rho)$) $\cong$ trim($\Gamma(\pi)$).*

(2) *Conversely, if trim($\Gamma(\rho)$ $\cong$ trim($\Gamma(\pi)$ with $y\alpha\sigma = y$ for all $y \in$ im(ρ) and $u\beta\tau = u$ for all $u \in$ im(π), then $\rho \sim_r \pi$ in P_k.*

Proof. (1) Let $\rho \sim_r \pi$ in P_k. Then $\rho \sim_r \pi$ in $\mathcal{P}(X)$ and so by part (1) of Theorem 4.6, we have trim($\Gamma(\rho)$) $\cong$ trim($\Gamma(\pi)$).

(2) Conversely, let trim($\Gamma(\rho)$) $\cong$ trim($\Gamma(\pi)$). Suppose k is infinite. Then

$$\begin{aligned} |A_t| &= |A \setminus A^i| + |\{y^* : y \in A \setminus A^i \text{ and } y\rho^{-1} \subseteq A^i\}| \\ &= |A \setminus A^i| + |\{y : y \in A \setminus A^i \text{ and } y\rho^{-1} \subseteq A^i\}| \\ &\leq |\text{im}(\rho)| + |\text{im}(\rho)| < k + k = k. \end{aligned}$$

Thus $|A_t| < k$. let $\mu : A_t \to B_t$ be an isomorphism from trim($\Gamma(\rho)$) to trim($\Gamma(\pi)$). Then $\mu^{-1} : B_t \to A_t$ is an isomorphism from trim($\Gamma(\pi)$) to trim($\Gamma(\rho)$). By Lemma 4.3, there are r-homomorphisms $\alpha \in P_k$ from $\Gamma(\rho)$ to $\Gamma(\pi)$ and $\beta \in P_k$ from $\Gamma(\pi)$ to $\Gamma(\rho)$. By given condition $y\alpha\sigma = y$ for all $y \in$ im(ρ) and $u\beta\tau = u$ for all $u \in$ im(π). Then by Theorem 3.11, we have $\rho \sim_r \pi$ in $\mathcal{P}(X)$ i.e., $\rho\alpha = \alpha\pi, \pi\beta = \beta\rho, \rho = \alpha\pi\sigma$ and $\pi = \beta\rho\tau$. Since k is infinite, so rank(σ) $< k$ and rank(τ) $< k$. So, $\sigma, \tau \in P_k$. Thus $\rho \sim_r \pi$ in P_k.

□

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