Palestine Journal of Mathematics

Vol. 12(1)(2023) , 222–230

Note on Extended Generalized Bessel Function

Farhatbanu H. Patel, R. K. Jana, A. K. Shukla*

Communicated by Lopez-Bonilla

MSC 2010 Classifications: 33C10, 26A33, 44A99.

Keywords and phrases: Generalized Bessel function, Generalized Wright Hypergeometric function, Generalized fractional integral operator, Modified Bessel function, P- transform or Pathway transform.

Abstract An attempt is made to obtain some properties of Extended Generalized Bessel function. Extended Modified Bessel Function has also been discussed.

1 Introduction

In 1824, F. W. Bessel gave the systematic study of Bessel function which is also known as special kinds of cylinder functions. In the current era, Bessel functions play a vital character for resolving the problems of Mathematical Physics, Atomic Physics, Nuclear Physics, Engineering Sciences such as a Flux Distribution in a Nuclear Reactor, Fluid Mechanics, Heat Transfer, Vibrations, Hydrodynamics, Stress Analysis etc.

The Bessel function of first kind $J_\nu(z)$ [4, P. 109] is represented as,

$$J_\nu(z)=\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(1+\nu+k)k!}\left(\frac{z}{2}\right)^{2k+\nu}, \tag{1.1}$$

where $|z|<\infty$, $|\arg z|<\pi$.

In 1935, Wright [6] introduced generalized Bessel function in following form

$$J_\nu^h(z)=\sum_{k=0}^{\infty}\frac{(-1)^k z^k}{\Gamma(1+\nu+hk)k!}, \tag{1.2}$$

where $h>0$, $|z|<\infty$, $|\arg z|<\pi$.

Galue [9, P. 395] generalized Bessel function as,

$$_hJ_\nu(z)=\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(1+\nu+hk)k!}\left(\frac{z}{2}\right)^{2k+\nu}, \tag{1.3}$$

where $h>0$, $|z|<\infty$, $|\arg z|<\pi$.

In 2013, Salehbhai et al. [8, P. 2] introduced Extended Generalized Bessel Function in following manner,

$$_h^mJ_v\left(z\right)=\sum_{k=0}^{\infty}\frac{\left(-1\right)^k}{\Gamma\left(1+m\nu+hk\right)\ k!}\left(\frac{z}{2}\right)^{2k+v}, \tag{1.4}$$

where, $h\in N$, $|z|<\infty$, $|\arg z|<\pi$, $m\ and\, v\in C$.

and also discussed Extended Generalized Bessel Function which is defined [8, P. 2] as,

$$_hJ_v^m\left(z\right)=\sum_{k=0}^{\infty}\frac{\left(-1\right)^k}{\Gamma\left(1+m\nu+hk\right)\ k!}\left(z\right)^k, \tag{1.5}$$

where, $h \in N,\ |z| < \infty,\ |\arg z| < \pi,\ m\ and\, v \in C$.

Some following facts are needed for our further study.

The well known generalized hypergeometric function [4, P. 73] is defined by

$$ {}_pF_q\left[\begin{array}{c|}\alpha_1, \alpha_2, ..., \alpha_p\\ \beta_1, \beta_2, ..., \beta_q\end{array}\, z\right] = 1 + \sum_{k=1}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k...(\alpha_p)_k}{(\beta_1)_k(\beta_2)_k...(\beta_q)_k}\frac{z^k}{k!},\quad |z| < 1, \tag{1.6}$$

where p and q are nonnegative integers and no $\beta_j\ (j = 1, 2, \ldots, q)$ is zero or a negative integer. Here, $(\alpha)_k$ is a Pochhammer symbol [4, P. 22] and defined as

$$(\alpha)_k := \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \begin{cases} 1 & (k = 0;\ \alpha \in \mathbf{C} \setminus \{0\}) \\ \alpha(\alpha+1)\ldots(\alpha+k-1) & (k \in \mathbf{N};\ \alpha \in \mathbf{C}). \end{cases} \tag{1.7}$$

Fox-Wright function [7] defined as following form,

$$ {}_p\Psi_q(z) = {}_p\Psi_q\left[\begin{array}{c|}(\alpha_i, A_i)_{1,p}\\ (\beta_j, B_j)_{1,q}\end{array}\, z\right] = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha_1 + A_1k)\cdots\Gamma(\alpha_p + A_pk)}{\Gamma(\beta_1 + B_1k)\cdots\Gamma(\beta_q + B_qk)}\frac{z^k}{k!}, \tag{1.8}$$

where $i = 1, 2, \cdots, p;\quad j = 1, 2, \cdots, q$ and $z, \alpha_i, \beta_j \in \mathbf{C}$, and the coefficients $A_1, ..., A_p \in \mathbf{R}^+$ and $B_1, ..., B_q \in \mathbf{R}^+$ satisfying the following condition

$$\sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i > -1. \tag{1.9}$$

The generalized Jacobi polynomial is defined by [10, P. 3] as,

$$P_n^{(\alpha,\,\beta,\,c,\,d)}(x) = \frac{(1+\alpha)_n}{\Gamma(n+1)}\, {}_3F_2\left[\begin{array}{ccc|}-n, & 1+\alpha+\beta+n, & c\\ 1+\alpha, & d, & \end{array}\, \frac{1-x}{2}\right], \tag{1.10}$$

where $d \in C - z^- \cup (0)\,;\ \alpha,\ \eta \in C - z^-;\ \beta \in C,\ \Re\,(d - \beta - c) > 0$.

Saigo et al. [11, P. 869] introduced left and right-sided generalized integral transforms defined respectively as,
for $x > 0$ and $\alpha,\ \beta,\ \eta \in C,\ \Re\,(\alpha) > 0$

$$\left(I_{0,x}^{\alpha,\,\beta,\,\eta} f\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma\,(\alpha)} \int\limits_0^x (x-t)^{\alpha-1}\, {}_2F_1\left(\alpha+\beta, -\eta;\ \alpha;\ 1 - \frac{t}{x}\right) f\,(t)\,dt\,(x > 0) \tag{1.11}$$

and

$$\left(I_{x,\infty}^{\alpha,\,\beta,\,\eta} f\right)(x) = \frac{1}{\Gamma\,(\alpha)} \int\limits_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta}\, {}_2F_1\left(\alpha+\beta, -\eta;\ \alpha;\ 1 - \frac{x}{t}\right) f\,(t)\,dt\,(x > 0)\,. \tag{1.12}$$

We need following lemmas [1, P. 871-872] for our study.

Lemma 1.1. *If* $x > 0$ *and* $\alpha,\ \beta,\ \eta \in C,\ \Re\,(\alpha) > 0$ *respectively, then*

$$\left(I_{0,x}^{\alpha,\,\beta,\,\eta} f\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma\,(\alpha)} \int\limits_0^x (x-t)^{\alpha-1}\, {}_2F_1\left(\alpha+\beta,\ -\eta;\ \alpha;\ 1 - \frac{t}{x}\right) f\,(t)\,dt, (x > 0) \tag{1.13}$$

Using (1.13), for $\alpha,\ \beta,\ \eta \in C$ *be such that* $\Re\,(\alpha) > 0,\ \Re\,(\sigma) > \max\,[0, \Re\,(\beta - \eta)]$*, we get*

$$\left(I_{0,x}^{\alpha,\,\beta,\,\eta} t^{\sigma-1}\right)(x) = \frac{\Gamma\,(\sigma)\,\Gamma\,(\sigma+\eta-\beta)}{\Gamma\,(\sigma-\beta)\,\Gamma\,(\sigma+\alpha+\eta)} x^{\sigma-\beta-1}. \tag{1.14}$$

In particular,

$$\left(\mathcal{R}_{0,x}^{\alpha}\, t^{\sigma-1}\right)(x) = \frac{\Gamma(\sigma)}{\Gamma(\sigma+\alpha)}\, x^{\sigma+\alpha-1},\ \Re(\alpha)>0,\ \Re(\sigma)>0, \tag{1.15}$$

$$\left(\mathcal{K}_{0,x}^{\mu,\alpha}\, t^{\sigma-1}\right)(x) = \frac{\Gamma(\sigma+\mu)}{\Gamma(\sigma+\alpha+\mu)}\, x^{\sigma-1},\ \Re(\alpha)>0,\ \Re(\sigma)>-\Re(\mu). \tag{1.16}$$

Lemma 1.2. *If $x>0$ and $\alpha,\ \beta,\ \eta\in C$, $\Re(\alpha)>0$, then*

$$\left(I_{x,\infty}^{\alpha,\,\beta,\,\eta} f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(t-x)^{\alpha-1}t^{-\alpha-\beta}{}_2F_1\left(\alpha+\beta,\ -\eta;\ \alpha;\ 1-\frac{x}{t}\right) f(t)\, dt\, (x>0). \tag{1.17}$$

Using (1.17), for $\alpha,\ \beta,\ \eta\in C$ be such that $\Re(\sigma)<1+\min\left[\Re(\beta),\ \Re(\eta)\right]$ we get,

$$\left(I_{x,\infty}^{\alpha,\,\beta,\,\eta}\, t^{\sigma-1}\right)(x) = \frac{\Gamma(\beta-\sigma+1)\,\Gamma(\eta-\sigma+1)}{\Gamma(1-\sigma)\,\Gamma(\alpha+\beta+\eta-\sigma+1)}\, x^{\sigma-\beta-1}. \tag{1.18}$$

In particular,

$$\left(\mathcal{S}_{x,\infty}^{\alpha}\, t^{\sigma-1}\right)(x) = \frac{\Gamma(1-\alpha-\sigma)}{\Gamma(1-\sigma)}\, x^{\sigma+\alpha-1},\ 0<\Re(\alpha)<1-\Re(\sigma), \tag{1.19}$$

$$\left(\mathcal{W}_{x,\infty}^{\mu,\,\alpha}\, t^{\sigma-1}\right)(x) = \frac{\Gamma(\mu-\sigma+1)}{\Gamma(\alpha+\mu-\sigma+1)}\, x^{\sigma-1},\ \Re(\alpha)<1+\Re(\mu). \tag{1.20}$$

P_{α}-transform:

The P_{α}-transform is a binomial type transform containing many class of transforms including the well known Laplace transform.

In 2011, Kumar [3, P. 3] introduced a fractional type integral transform called P-transform or pathway transform.

The P_{α}-transform of a function $f(t)$ of a real variable t denoted by $P_{\alpha}\left[f(t);s\right]$ is a function $F(s)$ of a complex variable s valid under certain conditions on $f(t)$ along with the condition $\alpha>1$, and is defined as,

$$P_{\alpha}\left[f(t);s\right] = F(s) = \int_{0}^{\infty}\left[1+(\alpha-1)s\right]^{\frac{-t}{\alpha-1}} f(t)\, dt. \tag{1.21}$$

Theorem 1.3. *If $\rho\in C$, $\Re(\rho)>0$ and $\alpha>1$, the $P_{\alpha}-$transform of the power function is as follows [3, P. 8 Result(1)],*

$$P_{\alpha}\left[t^{\rho-1};s\right] = \left\{\frac{\alpha-1}{\ln\left[1+(\alpha-1)s\right]}\right\}^{\rho}\Gamma(\rho). \tag{1.22}$$

2 Integral involving Extended Generalized Bessel functions

In this section, we derive two integrals involving the function ${}_{h}^{m}J_{v}(z)$ [8].

Theorem 2.1. *If ${}_{h}^{m}J_{v}(z)$ is defined as (1.4), for $a>0$, $b>0$, $\Re(v)>-1$, $h>0$, $m\in N\cup\{0\}$; $v\in C$, then*

$$\int_{0}^{\infty} e^{-ax}\, {}_{h}^{m}J_{v}(bx)\, dx = \frac{b^{v}}{\sqrt{\pi}a^{v+1}}\ {}_2\psi_1\left[\begin{array}{c}\left(\frac{v}{2}+\frac{1}{2},1\right),\left(\frac{v}{2}+1,1\right)\\ (1+mv,h)\end{array}\middle|\left(\frac{-b^{2}}{a^{2}}\right)\right] \tag{2.1}$$

Proof. Let us denote the L.H.S. of (2.1) by I and using the definition (1.4),

$$I = \int_0^\infty e^{-ax} \sum_{k=0}^\infty \frac{(-1)^k}{\Gamma(1+m\nu+hk)\ k!} \left(\frac{bx}{2}\right)^{2k+v} dx$$

$$I = \left(\frac{b}{2}\right)^\nu \sum_{k=0}^\infty \frac{\left(\frac{-b^2}{4}\right)^k}{\Gamma(1+m\nu+hk)\ k!} \int_0^\infty e^{-ax} x^{2k+v} dx.$$

On using property of gamma function, we get

$$I = \frac{b^v}{\sqrt{\pi} a^{v+1}} \sum_{k=0}^\infty \frac{\Gamma\left(k+\frac{v}{2}+\frac{1}{2}\right)\Gamma\left(k+\frac{v}{2}+1\right)}{\Gamma(1+m\nu+hk)\, k!} \left(\frac{-b^2}{a^2}\right)^k,$$

which can be represented in terms of generalized Wright hypergeometric function (1.8), leads to the right side of (2.1). □

Corollary 2.2. *On putting $v=0$ in (2.1), this reduces to*

$$\int_0^\infty e^{-ax}\, {}_h^m J_0(bx)\, dx = \frac{1}{\sqrt{\pi} a}\, {}_2\psi_1 \left[\begin{array}{c} \left(\frac{1}{2}, 1\right), (1,1) \\ (1,h) \end{array} \middle| \left(\frac{-b^2}{a^2}\right) \right]. \tag{2.2}$$

Corollary 2.3. *On setting $h=1$ in (2.1), this yields*

$$\int_0^\infty e^{-ax}\, {}_1^m J_v(bx)\, dx = \frac{b^v}{\sqrt{\pi} a^{v+1}} \frac{\Gamma\left(\frac{v}{2}+\frac{1}{2}\right)\Gamma\left(\frac{v}{2}+1\right)}{\Gamma(1+m\nu)}\, {}_2F_1\left(\frac{v}{2}+\frac{1}{2}, \frac{v}{2}+1;\ 1+v;\ \frac{-b^2}{a^2}\right). \tag{2.3}$$

Corollary 2.4. *On taking $h=1$ in equation (2.2), we get*

$$\int_0^\infty e^{-ax}\, {}_1^m J_0(bx)\, dx = \frac{1}{a}\, {}_1F_0\left(\frac{1}{2};\ -;\ \frac{-b^2}{a^2}\right)$$

$$\int_0^\infty e^{-ax}\, {}_1^m J_0(bx)\, dx = \left(a^2+b^2\right)^{\frac{-1}{2}};\ a>0,\ b>0. \tag{2.4}$$

Theorem 2.5. *If ${}_h^m J_v(z)$ is defined as (1.4), for $a>0$, $b>0$, $\Re(v)>-1$, $h>0$, $m \in N \cup \{0\}$; $v \in C$, then*

$$\int_0^\infty e^{-a^2x^2} x^{v+1}\, {}_h^m J_v(bx)\, dx = \frac{b^v}{2^{v+1} a^{2v+2}}\, {}_1\psi_1 \left[\begin{array}{c} (k+v,1) \\ (1+mv,h) \end{array} \middle| \left(\frac{-b^2}{4a^2}\right) \right] \tag{2.5}$$

Proof. Let us denote the L.H.S. of (2.5) by I and using the definition (1.4),

$$I = \sum_{k=0}^\infty \frac{(-1)^k}{\Gamma(1+m\nu+hk)\ k!} \left(\frac{b}{2}\right)^{2k+v} \int_0^\infty e^{-a^2x^2} x^{2k+2v+1} dx,$$

after simplification, we obtain

$$I = \frac{b^v}{2^{v+1} a^{2v+2}} \sum_{k=0}^\infty \frac{\Gamma(k+v+1)}{\Gamma(1+m\nu+hk)\ k!} \left(\frac{-b^2}{4a^2}\right)^k,$$

which can be represented in terms of generalized Wright hypergeometric function (1.8), leads to the right side of (2.5). □

Corollary 2.6. *On setting $h = 1$ and $m = 1$ in (2.5), this becomes,*

$$I = \int_0^\infty e^{-a^2x^2} x^{v+1} J_v(bx)\, dx,$$

one can easily obtain by simplification,

$$I = \frac{b^v}{2^{v+1} a^{2v+2}} e^{\frac{-b^2}{4a^2}}. \tag{2.6}$$

3 Generalized fractional integration of a Extended Generalized Bessel functions

In this section, we give some results using fractional integration of Extended Generalized Bessel Function [8] in terms of Generalized Wright hypergeometric function [7].

Theorem 3.1. *Let $\alpha,\ \beta,\ \mu,\ \sigma,\ m,\ \nu \in C,\ h > 0$, such that*

$$\Re(\nu) > -1,\ \Re(\alpha) > 0,\ \Re(\sigma + v) > \max[0, \Re(\beta - \eta)]. \tag{3.1}$$

Then,

$$\left(I_{0,x}^{\alpha,\beta,\mu}\, t^{\sigma-1}\, {}_{h}^{m}J_v(t)\right)(x) = \frac{x^{\sigma+v-\beta-1}}{2^v}\, {}_2\psi_3\left[\begin{array}{c}(\sigma+v,2),\ (\sigma+\mu+v-\beta,2)\\ (\sigma+v-\beta,2),\ (\sigma+v+\alpha+\mu,2),\ (1+mv,h)\end{array}\middle|\ \frac{-x^2}{4}\right] \tag{3.2}$$

Proof. From (1.4), we have,

$$\left(I_{0,x}^{\alpha,\beta,\mu}\, t^{\sigma-1}\, {}_{h}^{m}J_v(t)\right)(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{2}\right)^{v+2k}}{\Gamma(1+m\nu+hk)\, k!} \left(I_{0,x}^{\alpha,\beta,\mu} t^{v+\sigma+2k-1-\beta}\right)(x).$$

On applying Lemma 1.1 and σ replaced by $\sigma + v + 2k$, this yields,

$$\left(I_{0,x}^{\alpha,\beta,\mu}\, t^{\sigma-1}\, {}_{h}^{m}J_v(t)\right)(x)$$

$$= \frac{x^{\sigma+v-\beta-1}}{2^v} \sum_{k=0}^{\infty} \frac{\Gamma(v+\sigma+2k)\Gamma(v+\sigma+\mu-\beta+2k)}{\Gamma(\sigma+v-\beta+2k)\Gamma(\sigma+v+\alpha+\mu+2k)\Gamma(1+m\nu+hk)} \frac{\left(-x^2\right)^k}{4^k k!},$$

this can be represented in terms of generalized wright hypergeometric function (1.8), leads to the right side of (3.2). □

Corollary 3.2. *Let $\alpha,\ \mu,\ \sigma,\ \nu \in C\ \ h > 0$ such that $\Re(\nu) > -1$, $\Re(\alpha) > 0$ and $\Re(\sigma + \nu) > 0$. Then*

$$\left(\mathcal{R}_{0,x}^{\alpha}\, t^{\sigma-1}\, {}_{h}^{m}J_v(t)\right)(x) = \frac{x^{\sigma+v+\alpha-1}}{2^v}\, {}_1\psi_2\left[\begin{array}{c}(\sigma+v,2)\\ (\sigma+v+\alpha,2),\ (1+mv,h)\end{array}\middle|\ \frac{-x^2}{4}\right]. \tag{3.3}$$

Corollary 3.3. *Let $\alpha,\ \beta,\ \mu,\ \sigma,\ m,\ \nu \in C,\ h > 0$ such that $\Re(\nu) > -1$, $\Re(\alpha) > 0$ and $\Re(\sigma + \nu) > -\Re(\mu)$. Then*

$$\left(\mathcal{K}_{0,x}^{\mu,\alpha}\, t^{\sigma-1}\, {}_{h}^{m}J_v(t)\right)(x) = \frac{x^{\sigma+v-1}}{2^v}\, {}_1\psi_2\left[\begin{array}{c}(\sigma+v+\mu,2)\\ (\sigma+v+\alpha+\mu,2),\ (1+mv,h)\end{array}\middle|\ \frac{-x^2}{4}\right]. \tag{3.4}$$

Theorem 3.4. *Let $\alpha,\ \beta,\ \eta,\ \sigma,\ m,\ \nu \in C,\ h > 0$, such that,*

$$\Re(\nu) > -1, \Re(\alpha) > 0,\ \Re(\sigma - v) < 1 + \min[\Re(\beta), \Re(\eta)]. \tag{3.5}$$

Then,

$$\left(I_{x,\infty}^{\alpha,\beta,\eta}\, t^{\sigma-1}\, {}_{h}^{m}J_v\left(\tfrac{1}{t}\right)\right)(x) = \frac{x^{\sigma-v-\beta-1}}{2^v}\, {}_2\psi_3\left[\begin{array}{c}(1+\beta-\sigma+v,2),(1+\eta-\sigma+v,2)\\(1-\sigma+v,2),(1+\beta+\alpha+\eta-\sigma+v,2),(1+mv,h)\end{array}\middle|\frac{-1}{4x^2}\right] \tag{3.6}$$

Proof. On using (1.4) and interchanging the order of summation and integration, we find,

$$\left(I_{x,\infty}^{\alpha,\beta,\eta}\, t^{\sigma-1}\, {}_{h}^{m}J_v\left(\frac{1}{t}\right)\right)(x) = \sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{1}{2}\right)^{v+2k}}{\Gamma(1+m\nu+hk)\,k!}\left(I_{x,\infty}^{\alpha,\beta,\eta}t^{\sigma-v-2k-1}\right)(x).$$

On applying Lemma 1.2 and σ replaced by $\sigma-v-2k$, we have

$$\left(I_{x,\infty}^{\alpha,\beta,\mu}\, t^{\sigma-1}\, {}_{h}^{m}J_v\left(\tfrac{1}{t}\right)\right)(x) = \frac{x^{\sigma+v-1}}{2^v}$$
$$\times\sum_{k=0}^{\infty}\frac{\Gamma(\beta-\sigma+v+2k+1)\Gamma(\mu-\sigma+v+2k+1)}{\Gamma(1-\sigma+v+2k)\Gamma(\alpha+\beta+\mu-\sigma+v+2k+1)\Gamma(mv+hk+1)}\,\frac{(-1)^k}{(4x^2)^k.k!},$$

one can easily obtain the result in terms of generalized wright hypergeometric function (1.8), leads to the right side of (3.6). □

Corollary 3.5. *Let* $\alpha,\ \mu,\ \sigma,\ \nu\in C,\ h>0$ *be such that* $\Re(\nu)>-1,\ 0<\Re(\alpha)<1-\Re(\sigma-\nu)$. *Then,*

$$\left(\mathcal{S}_{x,\infty}^{\alpha}\, t^{\sigma-1}\, {}_{h}^{m}J_v\left(\frac{1}{t}\right)\right)(x) = \frac{x^{\sigma-v+\alpha-1}}{2^v}\, {}_1\psi_2\left[\begin{array}{c}(1-\alpha-\sigma+v,2)\\(1-\sigma+v,2),(1+mv,h)\end{array}\middle|\frac{-1}{4x^2}\right]. \tag{3.7}$$

Corollary 3.6. *Let* $\alpha,\ \mu,\ \sigma,\ \nu\in C,\ h>0$ *be such that* $\Re(\nu)>-1,\ \Re(\alpha)>-1,\ \Re(\sigma-\nu)<1+\min[0,\Re(\nu)]$. *Then,*

$$\left(\mathcal{W}_{x,\infty}^{\mu,\alpha}\, t^{\sigma-1}\, {}_{h}^{m}J_v\left(\frac{1}{t}\right)\right)(x) = \frac{x^{\sigma-v-1}}{2^v}\, {}_1\psi_2\left[\begin{array}{c}(\mu-\sigma+v+1,2)\\(\alpha+\mu-\sigma+v+1,2),(1+mv,h)\end{array}\middle|\frac{-1}{4x^2}\right]. \tag{3.8}$$

4 $\mathcal{P}$-transform or Pathway transform of ${}_{h}^{m}J_v(z)$

In this section, we find the Pathway transform [3] of Extended Generalized Bessel Function [8] in terms of Generalized Wright hypergeometric Function.

Theorem 4.1. *If* $\rho,\ v,\ b,\ m\in C,\ \sigma\in R^+,\ h\in N,\ \alpha>1$, *then the* $\mathcal{P}$*-transform of* ${}_{h}^{m}J_v(z)$ *is given by*

$$P_\alpha\left[t^{\rho-1}\, {}_{h}^{m}J_v(bt^\sigma);s\right] = \left\{\frac{\alpha-1}{\ln[1+(\alpha-1)s]}\right\}^{\rho+v\sigma}\left(\frac{b}{2}\right)^v \times {}_1\psi_1\left[\begin{array}{c}(\rho+v\sigma,2\sigma)\\(1+mv,h)\end{array}\middle|\frac{(\alpha-1)^{2\sigma}\left(\frac{-b^2}{4}\right)}{[\ln[1+(\alpha-1)s]]^{2\sigma}}\right] \tag{4.1}$$

Proof. Let P_1 be the left-hand side of (6.3) and using the definition of (1.4), we get

$$P_1 = P_\alpha\left[t^{\rho-1}\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(1+m\nu+hk)\ k!}\left(\frac{bt^\sigma}{2}\right)^{2k+v}\right],$$

on interchanging the order of summation and integration, we have

$$P_1 = \sum_{k=0}^{\infty}\frac{(-1)^k b^{2k+v}}{2^{2k+v}\Gamma(1+m\nu+hk)\ k!}P_\alpha\left\{t^{\rho-1+2\sigma k+v\sigma}\right\},$$

on using (1.18) we get,

$$P_1 = \left\{ \frac{\alpha - 1}{\ln\left[1 + (\alpha - 1)\, s\right]} \right\}^{\rho + v\sigma} \left(\frac{b}{2} \right)^{v} \sum_{k=0}^{\infty} \frac{\Gamma\left(\rho + v\sigma + 2k\sigma\right)}{\Gamma\left(1 + m\nu + hk\right)\, k!} \left\{ \frac{(\alpha - 1)^{2\sigma} \left(\frac{-b^2}{4} \right)}{\left[\ln\left[1 + (\alpha - 1)\, s\right]\right]^{2\sigma}} \right\}^{k},$$

which can be written in terms of generalized wright hypergeometric function (1.8), leads to the right side of (6.3). □

5 Integral involving ${}^{m}_{h}J_{v}\left(z\right)$ with jacobi polynomial

In this section, we derive integral involving Extended Generalized Bessel Function [8] with Jacobi Polynomial[10].

Theorem 5.1. *Let $P_n^{(\alpha,\,\beta,\,c,\,d)}\left(x\right)$ is defined by (1.10), for $\Re\left(v\right) > -1,\ h > 0,\ m \in N \cup \{0\}\,;\, v \in C,\ \Re(\lambda) > -1,\ \alpha > -1$ and $\beta > -1$. Then*

$$\int_{-1}^{1} (1 - x)^{\lambda} (1 + x)^{\delta} P_n^{(\alpha,\beta,c,d)}\left(x\right)\, {}^{m}_{h}J_{v}\left[z\left(1 + x\right)\right] dx = \frac{2^{\lambda+\delta+1}(1+\alpha)_n \Gamma(\lambda+1)}{\Gamma(n+1)} \sum_{k=0}^{\infty} \frac{\Gamma(2k+v+\delta+1)}{\Gamma(\lambda+2+2k+v+\delta)} \times {}^{m}_{h}J_{v}\left(z\left(2\right)\right)\ {}_4F_3\left[\begin{matrix} -n, 1 + \alpha + \beta + n,\ c,\ \lambda + 1; 1 \\ \alpha + 1,\ d,\ \lambda + \delta + \nu + 2k + 2; \end{matrix} \right] \tag{5.1}$$

Proof. We denote L.H.S of (5.1) by I and using the definition of (1.4), we find that

$$I = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+v} (1+\alpha)_n (-n)_k (1 + \alpha + \beta + n)_k (c)_k}{2^{2k+v} \Gamma(1 + mv + hk) k! (1+\alpha)_k (d)_k 2^k \Gamma(n+1) k!} \int_{-1}^{1} (1 - x)^{\lambda+k} (1 + x)^{2k+v+\delta} dx. \tag{5.2}$$

The identity of Beta function [4, P. 31] is mentioned below,

$$\int_{-1}^{1} (1 + x)^{p-1} (1 - x)^{q-1} dx = 2^{p+q-1} B\left(p,\, q\right). \tag{5.3}$$

From (5.2) and (5.3), we arrive at,

$$I = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(1+mv+hk)k!} \left(\frac{z(2)}{2} \right)^{2k+v} \frac{(1+\alpha)_n}{\Gamma(n+1)} \sum_{k=0}^{\infty} \frac{(-n)_k (1+\alpha+\beta+n)_k (c)_k}{(1+\alpha)_k (d)_k k!} \times 2^{\lambda+\delta+1} \frac{(1+\lambda)_k \Gamma(\lambda+1)}{\Gamma(\lambda+\delta+1+v+k+2k)},$$

which can be representing in terms of generalized hypergeometric function (1.8), leads to the right side of (5.1). □

6 Modified Extended Generalized Bessel Functions

In this section, we discuss Modified Bessel Functions of ${}^{m}_{h}J_{v}\left(z\right)$.

Definition 6.1. We define Modified Extended Generalized Bessel Function as

$${}^{m}_{h}I_{v}\left(z\right) = i^{-v}\, {}^{m}_{h}J_{v}\left(iz\right). \tag{6.1}$$

On using definition (1.4) in the Right Hand Side of (6.1), we find that

$${}^{m}_{h}I_{v}\left(z\right) = \left(\frac{z}{2} \right)^{2k+v} W\left(\frac{z^2}{4}, h, mv + 1 \right), \tag{6.2}$$

this can be written as,

$$ {}_{h}^{m}I_{v}(z)=\left(\frac{z}{2}\right)^{v}W\left(\frac{z^{2}}{4},h,mv+1\right), $$

where the Wright Function [6] is defined as,

$$ W(z,\rho,\beta)=\sum_{r=0}^{\infty}\frac{z^{r}}{r!\Gamma(\rho r+\beta)}, $$

where, $\Re(v)>-1,\ h>0,\ m\in N\cup\{0\};\ v\in C.$

Theorem 6.2. *If ${}_{h}^{m}I_{v}(z)$ is defined as (1.7) and for $\Re(v)>-1,\ h>0,\ m\in N\cup\{0\};\ v\in C$, then*

$$ D^{n}\left[z^{-v}\,{}_{h}^{m}I_{v}(z)\right]=\frac{z^{-n}}{2^{v}\sqrt{\pi}}\,{}_{1}\psi_{2}\left[\begin{array}{c}(k,1)\\(mv+1,h),\ (1-n,2)\end{array}\middle|z^{2}\right];where\sum_{j=1}^{3}\beta_{j}-\sum_{i=1}^{3}\alpha_{i}>-1. \tag{6.3} $$

Proof. On taking Modified Extended Bessel Function (6.2),

$$ D^{n}\left[z^{-v}\,{}_{h}^{m}I_{v}(z)\right]=D^{n}\left[\sum_{k=0}^{\infty}\frac{z^{2k+v-v}}{2^{2k+v}\Gamma(1+mv+hk)\ k!}\right]. $$

On differentiating term by term, we get

$$ D^{n}\left[z^{-v}\,{}_{h}^{m}I_{v}(z)\right]=\sum_{k=0}^{\infty}\frac{z^{2k-n}(2k)!}{\Gamma(1+mv+hk)\ 2^{2k+v}k!\,(2k-n)!}. $$

Or

$$ D^{n}\left[z^{-v}\,{}_{h}^{m}I_{v}(z)\right]=\sum_{k=0}^{\infty}\frac{2^{2k}\left(\frac{1}{2}\right)_{k}z^{2k-n}}{2^{2k+v}\Gamma(1+mv+hk)(2k-n)!}. $$

On using Legendre duplication formula [4, P. 23,section 19], we get

$$ D^{n}\left[z^{-v}\,{}_{h}^{m}I_{v}(z)\right]=\frac{z^{-n}}{2^{v-1}\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{\Gamma(2k)\Gamma\left(\frac{1}{2}+k\right)k!z^{2k}}{\Gamma(1+mv+hk)\Gamma k\Gamma(2k-n+1)(2k)!}. $$

Afterwards simplification, this yields,

$$ D^{n}\left[z^{-v}\,{}_{h}^{m}I_{v}(z)\right]=\frac{z^{-n}}{2^{v}\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{\Gamma(k+1)\,z^{2k}}{\Gamma(1+mv+hk)\Gamma(2k-n+1)\,k!}. $$

This can be written as,

$$ D^{n}\left[z^{-v}\,{}_{h}^{m}I_{v}(z)\right]=\frac{z^{-n}}{2^{v}\sqrt{\pi}}\,{}_{1}\psi_{2}\left[\begin{array}{c}(k,1)\\(mv+1,h),\ (1-n,2)\end{array}\middle|z^{2}\right]. $$

This completes the proof. □

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Author information

Farhatbanu H. Patel, R. K. Jana, A. K. Shukla*, Department of Applied Mathematics and Humanities, Sardar Vallabhbhai National Institute of Technology, Surat-395 007, Gujarat,, India.
E-mail: `ajayshukla2@rediffmail.com`*

Received:February 27, 2021.
Accepted: March 29, 2021.