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A NEW SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY LAMBDA OPERATOR

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Abstract In this work, we introduce and investigate a new class $k-\widetilde{U}ST_s(\mu,\varrho,\gamma,t)$ of analytic functions in the open unit disc U with negative coefficients. The object of the present paper is to determine coefficient estimates, neighborhoods and partial sums for functions u belonging to this class.

1 Introduction

Let A denote the class of analytic functions u defined on the unit disk $U=\{z:|z|<1\}$ with normalization $u(0)=0$ and $u'(0)=1$. Such a function has the Taylor series expansion about the origin in the form

$$u(z)=z+\sum_{\eta=2}^{\infty}a_\eta z^\eta, \tag{1.1}$$

denoted by S, the subclass of A consisting of functions that are univalent in U.

For $u\in A$ given by (1.1) and $g(z)$ given by

$$g(z)=z+\sum_{\eta=2}^{\infty}b_\eta z^\eta \tag{1.2}$$

their convolution (or Hadamard product), denoted by $(u*g)$, is defined as

$$(u*g)(z)=z+\sum_{\eta=2}^{\infty}a_\eta b_\eta z^\eta=(g*u)(z),\ \ (z\in U). \tag{1.3}$$

Note that $u*g\in A$.

A function $u\in A$ is said to be in $k-UST(\gamma)$, the class of $k-$uniformly starlike functions of order $\gamma, 0\le\gamma<1$, if satisfies the condition

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\}>k\left|\frac{zu'(z)}{u(z)}-1\right|+\gamma,\ \ (k\ge 0), \tag{1.4}$$

and a function $u\in A$ is said to be in $k-UCV(\gamma)$, the class of $k-$uniformly convex functions of order $\gamma,\ 0\le\gamma<1$, if satisfies the condition

$$\Re\left\{1+\frac{zu''(z)}{u'(z)}\right\}>k\left|\frac{zu''(z)}{u'(z)}\right|+\gamma,\ \ (k\ge 0). \tag{1.5}$$

Uniformly starlike and uniformly convex functions were first introduced by Goodman [4] and studied by Ronning [6] and also see [12].

In [8], Sakaguchi defined the class ST_s of starlike functions with respect to symmetric points as follows:

Let $u\in A$. Then u is said to be starlike with respect to symmetric points in U if and only if

$$\Re\left\{\frac{2zu'(z)}{u(z)-u(-z)}\right\}>0,\ (z\in U).$$

Recently, Owa et al. [5] defined the class $ST_s(\alpha,t)$ as follows:

$$\Re\left\{\frac{(1-t)zu'(z)}{u(z)-u(tz)}\right\}>\alpha,\ (z\in U),$$

where $0\leq\alpha<1, |t|\leq 1, t\neq 1$. Note that $ST_s(0,-1)=ST_s$ and $ST_s(\alpha,-1)=ST_s(\alpha)$ is called Sakaguchi function of order α.

Let us recall lambda function [11] defined by

$$\lambda(z,\varrho)=\sum_{\eta=0}^{\infty}\frac{z^\eta}{(2\eta+1)^\varrho}$$

where $z\in U, \varrho\in\mathbb{C}$, when $|z|<1, \Re(\varrho)>1$, when $|z|=1$ and let

$$\lambda^{(-1)}(z,\varrho)=1+\sum_{\eta=2}^{\infty}\frac{(\mu+1)_{\eta-1}(2\eta-1)^\varrho}{(\eta-1)!}z^{\eta-1}$$

be defined such that

$$\lambda(z,\varrho)*\lambda^{(-1)}(z,\varrho)=\frac{1}{(1-z)^{\mu+1}},\ \mu>-1.$$

We now define $\left(z\lambda^{(-1)}(z,\varrho)\right)$ as the following

$$\begin{aligned}(z\lambda(z,\varrho))*\left(z\lambda^{(-1)}(z,\varrho)\right)&=\frac{z}{(1-z)^{\mu+1}}\\&=z+\sum_{\eta=2}^{\infty}\frac{(\mu+1)_{\eta-1}}{(\eta-1)!}z^\eta,\mu>-1\end{aligned}$$

where $u\in A, z\in U$ and

$$\left(z\lambda^{(-1)}(z,\varrho)\right)=z+\sum_{\eta=2}^{\infty}\frac{(\mu+1)_{\eta-1}(2\eta-1)^\varrho}{(\eta-1)!}z^\eta$$

and obtain the following linear operator

$$\mathscr{I}_{\mu,\varrho}u(z)=\left(z\lambda^{(-1)}(z,\varrho)\right)*u(z).$$

A simple computation gives us

$$\mathscr{I}_{\mu,\varrho}u(z)=z+\sum_{\eta=2}^{\infty}\phi(\mu,\varrho,\eta)a_\eta z^\eta \tag{1.6}$$

$$\text{where } \phi(\mu,\varrho,\eta)=\frac{(\mu+1)_{\eta-1}(2\eta-1)^\varrho}{(\eta-1)!} \tag{1.7}$$

where $(\mu)_\eta$ is the Pochhammer symbol defined in terms of the Gamma function by
$(\mu)_\eta=\frac{\Gamma(\mu+\eta)}{\Gamma(\mu)}=\begin{cases}1, & \text{if } \eta=0;\\ \mu(\mu+1)\cdots(\mu+\eta-1), & \text{if } \eta\in\mathbb{N}\end{cases}.$

In this paper, using the lambda operator $\mathscr{I}_{\mu,\varrho}u$, we define a new subclass of functions belonging to the class A, motivated by Spanier and Oldham [11] and Venkateswarlu et al. [13].

Definition 1.1. A function $u\in A$ is said to be in the class $k-UST_s(\mu,\varrho,\gamma,t)$ if for all $z\in U$

$$\Re\left\{\frac{(1-t)z\left(\mathscr{I}_{\mu,\varrho}u(z)\right)'}{\mathscr{I}_{\mu,\varrho}u(z)-\mathscr{I}_{\mu,\varrho}u(tz)}\right\}\geq k\left|\frac{(1-t)z\left(\mathscr{I}_{\mu,\varrho}u(z)\right)'}{\mathscr{I}_{\mu,\varrho}u(z)-\mathscr{I}_{\mu,\varrho}u(tz)}-1\right|+\gamma,$$

for $k\geq 0, |t|\leq 1, t\neq 1, 0\leq\gamma<1$.

Furthermore, we say that a function $u \in k{-}UST_s(\mu, \varrho, \gamma, t)$ is in the subclass $k{-}\widetilde{U}ST_s(\mu, \varrho, \gamma, t)$ if $u(z)$ is of the following form

$$u(z) = z - \sum_{\eta=2}^{\infty} a_\eta z^\eta, \quad (a_\eta \geq 0, \eta \in \mathbb{N},\ z \in U). \tag{1.8}$$

The aim of the present paper is to study the coefficient bounds, partial sums and certain neighborhood results of the class $k - \widetilde{U}ST_s(\mu, \varrho, \gamma, t)$.

Firstly, we shall need the following lemmas [2] .

Lemma 1.2. *Let w be a complex number. Then*

$$\Re\,(w) \geq \alpha \ \textit{ if and only if } \ |w - (1+\alpha)| \leq |w + (1-\alpha)|.$$

Lemma 1.3. *Let w be a complex number and α, γ be real numbers. Then*

$$\Re\,(w) > \alpha|w-1| + \gamma \ \textit{ if and only if } \ \Re\{w(1+\alpha e^{i\theta}) - \alpha e^{i\theta}\} > \gamma, -\pi < \theta \leq \pi.$$

2 Coefficient bounds

Theorem 2.1. *The function u defined by (1.8) is in the class $k - \widetilde{U}ST_s(\mu, \varrho, \gamma, t)$ if and only if*

$$\sum_{\eta=2}^{\infty} \phi(\mu, \varrho, \eta)|\eta(k+1) - u_\eta(k+\gamma)|a_\eta \leq 1 - \gamma, \tag{2.1}$$

where $k \geq 0, |t| \leq 1, t \neq 1, 0 \leq \gamma < 1$ and $u_\eta = 1 + t + \cdots + t^{\eta-1}$.

Proof. By Definition 1.1, we get

$$\Re\left\{\frac{(1-t)z\left(\mathscr{I}_{\mu,\varrho}u(z)\right)'}{\mathscr{I}_{\mu,\varrho}u(z) - \mathscr{I}_{\mu,\varrho}u(tz)}\right\} \geq k\left|\frac{(1-t)z\left(\mathscr{I}_{\mu,\varrho}u(z)\right)'}{\mathscr{I}_{\mu,\varrho}u(z) - \mathscr{I}_{\mu,\varrho}u(tz)} - 1\right| + \gamma.$$

Then by Lemma 1.3, we have

$$\Re\left\{\frac{(1-t)z\left(\mathscr{I}_{\mu,\varrho}u(z)\right)'}{\mathscr{I}_{\mu,\varrho}u(z) - \mathscr{I}_{\mu,\varrho}u(tz)}(1+ke^{i\theta}) - ke^{i\theta}\right\} \geq \gamma, \quad -\pi < \theta \leq \pi$$

or equivalently

$$\Re\left\{\frac{(1-t)z\left(\mathscr{I}_{\mu,\varrho}u(z)\right)'(1+ke^{i\theta})}{\mathscr{I}_{\mu,\varrho}u(z) - \mathscr{I}_{\mu,\varrho}u(tz)} - \frac{ke^{i\theta}\left[\mathscr{I}_{\mu,\varrho}u(z) - \mathscr{I}_{\mu,\varrho}u(tz)\right]}{\mathscr{I}_{\mu,\varrho}u(z) - \mathscr{I}_{\mu,\varrho}u(tz)}\right\} \geq \gamma. \tag{2.2}$$

Let $F(z) = (1-t)z\left(\mathscr{I}_{\mu,\varrho}u(z)\right)'(1+ke^{i\theta}) - ke^{i\theta}\left[\mathscr{I}_{\mu,\varrho}u(z) - \mathscr{I}_{\mu,\varrho}u(tz)\right]$
and $E(z) = \mathscr{I}_{\mu,\varrho}u(z) - \mathscr{I}_{\mu,\varrho}u(tz)$.
By Lemma 1.2, (2.2) is equivalent to

$$|F(z) + (1-\gamma)E(z)| \geq |F(z) - (1+\gamma)E(z)|, \quad \text{for } 0 \leq \gamma < 1.$$

But

$$\begin{aligned}
|F(z) + (1-\gamma)E(z)| = &\Big|(1-t)\Big\{(2-\gamma)z - \sum_{\eta=2}^{\infty}\phi(\mu,\varrho,\eta)(\eta + u_\eta(1-\gamma))a_\eta z^\eta \\
&- ke^{i\theta}\sum_{\eta=2}^{\infty}\phi(\mu,\varrho,\eta)(\eta - u_\eta)a_\eta z^\eta\Big\}\Big| \\
\geq &|1-t|\Big\{(2-\gamma)|z| - \sum_{\eta=2}^{\infty}\phi(\mu,\varrho,\eta)|\eta + u_\eta(1-\gamma)|a_\eta|z^\eta| \\
&- k\sum_{\eta=2}^{\infty}\phi(\mu,\varrho,\eta)|\eta - u_\eta|a_\eta|z^\eta|\Big\}.
\end{aligned}$$

Also

$$
\begin{aligned}
|F(z)-(1+\gamma)E(z)| = \Big|(1-t)\{-\gamma z - \sum_{\eta=2}^{\infty}\phi(\mu,\varrho,\eta)(\eta-u_\eta(1+\gamma))a_\eta z^\eta \\
- ke^{i\theta}\sum_{\eta=2}^{\infty}\phi(\mu,\varrho,\eta)(\eta-u_\eta)a_\eta z^\eta\}\Big| \\
\leq |1-t|\Big\{\gamma|z| + \sum_{\eta=2}^{\infty}\phi(\mu,\varrho,\eta)|\eta-u_\eta(1+\gamma)||a_\eta||z^\eta| \\
+ k\sum_{\eta=2}^{\infty}\phi(\mu,\varrho,\eta)|\eta-u_\eta|a_\eta|z^\eta|\Big\}.
\end{aligned}
$$

So

$$
\begin{aligned}
&|F(z)+(1-\gamma)E(z)| - |F(z)-(1+\gamma)E(z)| \\
&\geq |1-t|\Big\{2(1-\gamma)|z| - \sum_{\eta=2}^{\infty}\phi(\mu,\varrho,\eta)\left[|\eta+u_\eta(1-\gamma)| + |\eta-u_\eta(1+\gamma)| + 2k|\eta-u_\eta|\,\right]a_\eta|z^\eta|\Big\} \\
&\geq 2(1-\gamma)|z| - \sum_{\eta=2}^{\infty}2\phi(\mu,\varrho,\eta)\big|\eta(k+1)-u_\eta(k+\gamma)\big|a_\eta|z^\eta| \geq 0
\end{aligned}
$$

or

$$\sum_{\eta=2}^{\infty}\phi(\mu,\varrho,\eta)\big|\eta(k+1)-u_\eta(k+\gamma)\big|a_\eta \leq 1-\gamma.$$

Conversely, suppose that (2.1) holds. Then we must show

$$\Re\left\{\frac{(1-t)z\left(\mathscr{I}_{\mu,\varrho}u(z)\right)'(1+ke^{i\theta}) - ke^{i\theta}\left[\mathscr{I}_{\mu,\varrho}u(z)-\mathscr{I}_{\mu,\varrho}u(tz)\right]}{\mathscr{I}_{\mu,\varrho}u(z)-\mathscr{I}_{\mu,\varrho}u(tz)}\right\} \geq \gamma.$$

Upon choosing the values of z on the positive real axis where $0 \leq |z| = r < 1$, the above inequality reduces to

$$\Re\left\{\frac{(1-\gamma)-\sum_{\eta=2}^{\infty}\phi(\mu,\varrho,\eta)[\eta(1+ke^{i\theta})-u_\eta(\gamma+ke^{i\theta})]a_\eta z^{\eta-1}}{1-\sum_{\eta=2}^{\infty}\phi(\mu,\varrho,\eta)u_\eta a_\eta z^{\eta-1}}\right\} \geq 0.$$

Since $\Re(-e^{i\theta}) \geq -|e^{i\theta}| = -1$, the above inequality reduces to

$$\Re\left\{\frac{(1-\gamma)-\sum_{\eta=2}^{\infty}\phi(\mu,\varrho,\eta)[\eta(1+k)-u_\eta(\gamma+k]a_\eta r^{\eta-1}}{1-\sum_{\eta=2}^{\infty}\phi(\mu,\varrho,\eta)u_\eta a_\eta r^{\eta-1}}\right\} \geq 0.$$

Letting $r \to 1^-$, we have desired conclusion. □

Corollary 2.2. *If* $u(z) \in k-\widetilde{U}ST_s(\mu,\varrho,\gamma,t)$ *then*

$$a_\eta \leq \frac{1-\gamma}{\phi(\mu,\varrho,\eta)|\eta(k+1)-u_\eta(k+\gamma)|}$$

where $k \geq 0, |t| \leq 1, t \neq 1, 0 \leq \gamma < 1$, $u_\eta = 1+t+\cdots+t^{\eta-1}$. *and* $\phi(\mu,\varrho,\eta)$ *is given by* (1.7). *Equality holds for the function*

$$u(z) = z - \frac{1-\gamma}{\phi(\mu,\varrho,\eta)|\eta(k+1)-u_\eta(k+\gamma)|}z^\eta.$$

3 Neighborhood results

The notion of $\alpha-$neighborhood was first introduced and studied by Goodman [3] and Ruscheweyh [7],

Definition 3.1. Let $k \geq 0, |t| \leq 1, t \neq 1, 0 \leq \gamma < 1, \alpha \geq 0$ and $u_\eta = 1 + t + \cdots + t^{\eta-1}$. We define the $\alpha-$neighborhood of a function $u \in A$ and denote by $N_\alpha(u)$ consisting of all functions $g(z) = z - \sum\limits_{\eta=2}^{\infty} b_\eta z^\eta \in S, \ \ (b_\eta \geq 0, \eta \in \mathbb{N})$ satisfying

$$\sum_{\eta=2}^{\infty} \frac{\phi(\mu,\varrho,\eta)|\eta(k+1) - u_\eta(k+\gamma)|}{1-\gamma}|a_\eta - b_\eta| \leq 1-\alpha.$$

Theorem 3.2. *Let $u(z) \in k - \widetilde{U}ST_s(\mu,\varrho,\gamma,t)$ and for all real θ we have $\gamma(e^{i\theta}-1) - 2e^{i\theta} \neq 0$. For any complex number ϵ with $|\epsilon| < \alpha(\alpha \geq 0)$, if f satisfies the following condition:*

$$\frac{u(z)+\epsilon z}{1+\epsilon} \in k - \widetilde{U}ST_s(\mu,\varrho,\gamma,t)$$

then $N_\alpha(u) \subset k - \widetilde{U}ST_s(\mu,\varrho,\gamma,t)$.

Proof. It is obvious that $u \in k - \widetilde{U}ST_s(\mu,\varrho,\gamma,t)$ if and only if

$$\left|\frac{(1-t)z\left(\mathscr{I}_{\mu,\varrho}u(z)\right)'(1+ke^{i\theta}) - (ke^{i\theta}+1+\gamma)\left(\mathscr{I}_{\mu,\varrho}u(z) - \mathscr{I}_{\mu,\varrho}u(tz)\right)}{(1-t)z\left(\mathscr{I}_{\mu,\varrho}u(z)\right)'(1+ke^{i\theta}) + (1-ke^{i\theta}-\gamma)\left(\mathscr{I}_{\mu,\varrho}u(z) - \mathscr{I}_{\mu,\varrho}u(tz)\right)}\right| < 1, \ (-\pi < \theta \leq \pi),$$

for any complex number ξ with $|\xi| = 1$, we have

$$\frac{(1-t)z\left(\mathscr{I}_{\mu,\varrho}u(z)\right)'(1+ke^{i\theta}) - (ke^{i\theta}+1+\gamma)\left(\mathscr{I}_{\mu,\varrho}u(z) - \mathscr{I}_{\mu,\varrho}u(tz)\right)}{(1-t)z\left(\mathscr{I}_{\mu,\varrho}u(z)\right)'(1+ke^{i\theta}) + (1-ke^{i\theta}-\gamma)\left(\mathscr{I}_{\mu,\varrho}u(z) - \mathscr{I}_{\mu,\varrho}u(tz)\right)} \neq \xi.$$

In other words, we must have

$$(1-\xi)(1-t)z\left(\mathscr{I}_{\mu,\varrho}u(z)\right)'(1+ke^{i\theta}) - (ke^{i\theta}+1+\gamma+\xi(-1+ke^{i\theta}+\gamma))\left(\mathscr{I}_{\mu,\varrho}u(z) - \mathscr{I}_{\mu,\varrho}u(tz)\right) \neq 0$$

which is equivalent to

$$z - \sum_{\eta=2}^{\infty} \frac{\phi(\mu,\varrho,\eta)\left((\eta-u_\eta)(1+ke^{i\theta}-\xi ke^{i\theta}) - \xi(\eta+u_\eta) - u_\eta\gamma(1-\xi)\right)}{\gamma(\xi-1)-2\xi} z^\eta \neq 0.$$

However, $u \in k - \widetilde{U}ST_s(\mu,\varrho,\gamma,t)$ if and only if $\frac{(u*h)}{z} \neq 0, z \in U - \{0\}$, where $h(z) = z - \sum\limits_{\eta=2}^{\infty} c_\eta z^\eta$ and

$$c_\eta = \frac{\phi(\mu,\varrho,\eta)\left((\eta-u_\eta)(1+ke^{i\theta}-\xi ke^{i\theta}) - \xi(\eta+u_\eta) - u_\eta\gamma(1-\xi)\right)}{\gamma(\xi-1)-2\xi}$$

we note that

$$|c_\eta| \leq \frac{\phi(\mu,\varrho,\eta)\,|\eta(1+k) - u_\eta(k+\gamma)|}{1-\gamma}$$

since $\frac{u(z)+\epsilon z}{1+\epsilon} \in k - \widetilde{U}ST_s(\mu,\varrho,\gamma,t)$, therefore $z^{-1}\left(\frac{u(z)+\epsilon z}{1+\epsilon} * h(z)\right) \neq 0$, which is equivalent to

$$\frac{(u*h)(z)}{(1+\epsilon)z} + \frac{\epsilon}{1+\epsilon} \neq 0. \tag{3.1}$$

Now suppose that $\left|\frac{(u*h)(z)}{z}\right| < \alpha$. Then by (3.1), we must have

$$\left|\frac{(u*h)(z)}{(1+\epsilon)z} + \frac{\epsilon}{1+\epsilon}\right| \geq \frac{|\epsilon|}{|1+\epsilon|} - \frac{1}{|1+\epsilon|}\left|\frac{(u*h)(z)}{z}\right|$$
$$> \frac{|\epsilon|-\alpha}{|1+\epsilon|} \geq 0,$$

this is a contradiction by $|\epsilon| < \alpha$ and however, we have $\left|\frac{(u*h)(z)}{z}\right| \geq \alpha$.
If $g(z) = z - \sum\limits_{\eta=2}^{\infty} b_\eta z^\eta \in N_\alpha(u)$, then

$$\alpha - \left|\frac{(g*h)(z)}{z}\right| \leq \left|\frac{((u-g)*h)(z)}{z}\right| \leq \sum_{\eta=2}^{\infty} |a_\eta - b_\eta||c_\eta||z^\eta|$$
$$< \sum_{\eta=2}^{\infty} \frac{\phi(\mu,\varrho,\eta)|\eta(1+k) - u_\eta(k+\gamma)|}{1-\gamma}|a_\eta - b_\eta| \leq \alpha.$$

□

4 Partial sums

In this section, applying methods used by Silverman [9] and Silvia [10], we investigate the ratio of a function of the form (1.8) to its sequence of partial sums $u_m(z) = z + \sum\limits_{\eta=2}^{m} a_\eta z^\eta$.

Theorem 4.1. *If u of the form (1.1) satisfies the condition (2.1) then*

$$\Re\left\{\frac{u(z)}{u_m(z)}\right\} \geq 1 - \frac{1}{\delta_{m+1}} \tag{4.1}$$

and

$$\delta_\eta = \begin{cases} 1, & \text{if } \eta = 2, 3, \cdots, m; \\ \delta_{m+1}, & \text{if } \eta = m+1, m+2, \cdots. \end{cases} \tag{4.2}$$

where

$$\delta_\eta = \frac{\phi(\mu,\varrho,\eta)|\eta(1+k) - u_\eta(k+\gamma)|}{1-\gamma}. \tag{4.3}$$

The result in (4.1) is sharp for every m, with the extremal function

$$u(z) = z + \frac{z^{m+1}}{\delta_{m+1}}. \tag{4.4}$$

Proof. Define the function w, we may write

$$\frac{1+w(z)}{1-w(z)} = \delta_{m+1}\left\{\frac{u(z)}{u_m(z)} - \left(1 - \frac{1}{\delta_{m+1}}\right)\right\} \tag{4.5}$$
$$= \left\{\frac{1 + \sum\limits_{\eta=2}^{m} a_\eta z^{\eta-1} + \delta_{m+1} \sum\limits_{\eta=m+1}^{\infty} a_\eta z^{\eta-1}}{1 + \sum\limits_{\eta=2}^{m} a_\eta z^{\eta-1}}\right\}.$$

Then, from (4.5), we can obtain

$$w(z) = \frac{\delta_{m+1} \sum\limits_{\eta=m+1}^{\infty} a_\eta z^{\eta-1}}{2 + 2\sum\limits_{\eta=2}^{m} a_\eta z^{\eta-1} + \delta_{m+1} \sum\limits_{\eta=m+1}^{\infty} a_\eta z^{\eta-1}}$$

and

$$|w(z)| \leq \frac{\delta_{m+1} \sum\limits_{\eta=m+1}^{\infty} a_\eta}{2 - 2\sum\limits_{\eta=2}^{m} a_\eta - \delta_{m+1} \sum\limits_{\eta=m+1}^{\infty} a_\eta}.$$

Now $|w(z)| \leq 1$ if

$$2\delta_{m+1}\sum_{\eta=m+1}^{\infty} a_\eta \leq 2-2\sum_{\eta=2}^{m} a_\eta,$$

which is equivalent to

$$\sum_{\eta=2}^{m} a_\eta + \delta_{m+1}\sum_{\eta=m+1}^{\infty} a_\eta \leq 1. \tag{4.6}$$

It is suffices to show that the left hand side of (4.6) is bounded above by $\sum_{\eta=2}^{\infty}\delta_\eta a_n$, which is equivalent to

$$\sum_{\eta=2}^{m}(\delta_\eta - 1)a_\eta + \sum_{\eta=m+1}^{\infty}(\delta_\eta - \delta_{m+1})a_\eta \geq 0.$$

To see that the function given by (4.4) gives the sharp result, we observe that for $z = re^{i\pi/\eta}$,

$$\frac{u(z)}{u_m(z)} = 1 + \frac{z^m}{\delta_{m+1}}. \tag{4.7}$$

Taking $z \to 1^-$, we have

$$\frac{u(z)}{u_m(z)} = 1 - \frac{1}{\delta_{m+1}}.$$

This completes the proof of Theorem 4.1. □

We next determine bounds for $\frac{u_m(z)}{u(z)}$.

Theorem 4.2. *If u of the form (1.1) satisfies the condition (2.1) then*

$$\Re\left\{\frac{u_m(z)}{u(z)}\right\} \geq \frac{\delta_{m+1}}{1+\delta_{m+1}}. \tag{4.8}$$

The result is sharp with the function given by (4.4).

Proof. We may write

$$\begin{aligned}\frac{1+w(z)}{1-w(z)} =& (1+\delta_{m+1})\left\{\frac{u_m(z)}{u(z)} - \frac{\delta_{m+1}}{1+\delta_{m+1}}\right\}\\ =& \left\{\frac{1+\sum_{\eta=2}^{m} a_\eta z^{\eta-1} - \delta_{m+1}\sum_{\eta=m+1}^{\infty} a_\eta z^{\eta-1}}{1+\sum_{\eta=2}^{\infty} a_\eta z^{\eta-1}}\right\},\end{aligned} \tag{4.9}$$

where

$$w(z) = \frac{(1+\delta_{m+1})\sum_{\eta=m+1}^{\infty} a_\eta z^{\eta-1}}{-\left(2+2\sum_{\eta=2}^{m} a_\eta z^{\eta-1} - (1-\delta_{m+1})\sum_{\eta=m+1}^{\infty} a_\eta z^{\eta-1}\right)}$$

and

$$|w(z)| \leq \frac{(1+\delta_{m+1})\sum_{\eta=m+1}^{\infty} a_\eta}{2-2\sum_{\eta=2}^{m} a_\eta + (1-\delta_{m+1})\sum_{\eta=m+1}^{\infty} a_\eta} \leq 1. \tag{4.10}$$

This last inequality is equivalent to

$$\sum_{\eta=2}^{m} a_\eta + \delta_{m+1}\sum_{\eta=m+1}^{\infty} a_\eta \leq 1. \tag{4.11}$$

It is suffices to show that the left hand side of (4.11) is bounded above by $\sum_{\eta=2}^{\infty} \delta_\eta a_\eta$, which is equivalent to

$$\sum_{\eta=2}^{m}(\delta_\eta - 1)a_\eta + \sum_{\eta=m+1}^{\infty}(\delta_\eta - \delta_{m+1})a_\eta \geq 0.$$

This completes the proof of Theorem 4.2. □

We next turn to ratios involving derivatives.

Theorem 4.3. *If u of the form* (1.1) *satisfies the condition* (2.1) *then*

$$\Re\left\{\frac{u'(z)}{u'_m(z)}\right\} \geq 1 - \frac{m+1}{\delta_{m+1}}, \tag{4.12}$$

$$\Re\left\{\frac{u'_m(z)}{u'(z)}\right\} \geq \frac{\delta_{m+1}}{1+m+\delta_{m+1}} \tag{4.13}$$

where

$$\delta_\eta \geq \begin{cases} 1, & \text{if } \eta = 1, 2, 3, \cdots, m; \\ \eta\frac{\delta_{m+1}}{m+1}, & \text{if } \eta = m+1, m+2, \cdots \end{cases}$$

and δ_η is defined by (4.3). *The estimates in* (4.12) *and* (4.13) *are sharp with the extremal function given by* (4.4).

Proof. Firstly, we will give proof of (4.12). We write

$$\frac{1+w(z)}{1-w(z)} = \delta_{m+1}\left\{\frac{u'(z)}{u'_m(z)} - \left(1 - \frac{m+1}{\delta_{m+1}}\right)\right\}$$

$$= \left\{\frac{1 + \sum_{\eta=2}^{m} \eta a_\eta z^{\eta-1} + \frac{\delta_{m+1}}{m+1}\sum_{\eta=m+1}^{\infty} \eta a_\eta z^{\eta-1}}{1 + \sum_{\eta=2}^{m} a_\eta z^{\eta-1}}\right\},$$

where

$$w(z) = \frac{\frac{\delta_{m+1}}{m+1}\sum_{\eta=m+1}^{\infty} \eta a_\eta z^{\eta-1}}{2 + 2\sum_{\eta=2}^{m} \eta a_\eta z^{\eta-1} + \frac{\delta_{m+1}}{m+1}\sum_{\eta=m+1}^{\infty} \eta a_\eta z^{\eta-1}}$$

and

$$|w(z)| \leq \frac{\frac{\delta_{m+1}}{m+1}\sum_{\eta=m+1}^{\infty} \eta a_\eta}{2 - 2\sum_{\eta=2}^{m} \eta a_\eta + \frac{\delta_{m+1}}{m+1}\sum_{\eta=m+1}^{\infty} \eta a_\eta}.$$

Now $|w(z)| \leq 1$ if and only if

$$\sum_{\eta=2}^{m} \eta a_\eta + \frac{\delta_{m+1}}{m+1}\sum_{\eta=m+1}^{\infty} \eta a_\eta \leq 1, \tag{4.14}$$

since the left hand side of (4.14) is bounded above by $\sum_{\eta=2}^{\infty} \delta_\eta a_\eta$.

The proof of (4.13) follows the pattern of that in Theorem (4.2).
This completes the proof of Theorem 4.3. □

Conclusion

This research has introduced a new Lambda operator related to analytic function and studied some basic properties of geometric function theory.Accordingly, some results to coefficient estimates,neighborhood result and partial sums have been considered,inviting further research for this field of study.

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