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BEREZIN RADIUS INEQUALITIES VIA GEOMETRIC CONVEXITY

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Abstract For a bounded linear operator A on a reproducing kernel Hilbert space $\mathcal{H}(\Omega)$, with normalized reproducing kernel $\widehat{k}_\eta := \frac{k_\eta}{\|k_\eta\|_{\mathcal{H}}}$, the Berezin transform, Berezin radius and Berezin norm are defined respectively by $\widetilde{A}(\eta) := \left\langle A\widehat{k}_\eta, \widehat{k}_\eta \right\rangle_{\mathcal{H}}$, $\mathrm{ber}(A) := \sup_{\eta\in\Omega} \left|\widetilde{A}(\eta)\right|$ and $\|A\|_{\mathrm{Ber}} = \sup_{\eta\in\Omega} \left\|A\widehat{k}_\eta\right\|$. A straightforward comparison between these characteristics yields the inequalities $\mathrm{ber}(B^*A) \leq \frac{1}{2}\|(A^*A)+(B^*B)\|_{\mathrm{ber}}$. In this paper, we prove further inequalities relating them, and give some applications of geometrically convex functions to Berezin radius inequalities.

1 Introduction

In this article, we present some applications of geometrically convex functions to Berezin radius inequalities.

Let $\mathcal{L}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $(\mathcal{H}, \langle .,. \rangle)$ with the identity operator $1_{\mathcal{H}}$ in $\mathcal{L}(\mathcal{H})$. Throughout this paper we work in reproducing kernel Hilbert space (RKHS). These spaces are complete inner-product spaces comprised of complex-valued functions defined on a set Ω, where point evaluation is bounded. Formally, that is, if Ω is a set and $\mathcal{H} = \mathcal{H}(\Omega)$ is a subset of all functions $\Omega \to \mathbb{C}$, then $\mathcal{H}$ is an RKHS on Ω if it is a complete inner product space and point evaluation at each $\eta \in \Omega$ is a bounded linear functional on $\mathcal{H}$. Via the classical Riesz representation theorem, we know if $\mathcal{H}$ is an RKHS on Ω, there is a unique element $k_\eta \in \mathcal{H}$ such that $h(\eta) = \langle h, k_\eta \rangle_{\mathcal{H}}$ for every $\eta \in \Omega$ and all $h \in \mathcal{H}$. The element k_η is called the reproducing kernel at η. Further, we will denote the normalized reproducing kernel at η as $\widehat{k}_\eta := \frac{k_\eta}{\|k_\eta\|_{\mathcal{H}}}$.

The Berezin transform associates smooth functions with operators on Hilbert spaces of analytic functions.

Definition 1.1. Let $\mathcal{H}$ be an RKHS on a set Ω and let A be a bounded linear operator on $\mathcal{H}$.

(i) For $\eta \in \Omega$, the Berezin transform of the operator A at η (or Berezin symbol of A) is

$$\widetilde{A}(\eta) := \left\langle A\widehat{k}_\eta, \widehat{k}_\eta \right\rangle_{\mathcal{H}}.$$

(ii) The Berezin range of the operator A (or Berezin set of A) is

$$\mathrm{Ber}(A) := \mathrm{Range}(\widetilde{A}) = \left\{\widetilde{A}(\eta) : \eta \in \Omega\right\}.$$

(iii) The Berezin radius of the operator A (or Berezin number of A) is

$$\mathrm{ber}(A) := \sup_{\eta\in\Omega} \left|\widetilde{A}(\eta)\right|.$$

(iv) The Berezin norm of the operator A is

$$\|A\|_{\text{Ber}} := \sup_{\eta\in\Omega} \left\|A\widehat{k}_\eta\right\|.$$

For each bounded operator A on $\mathcal{H}$, the Berezin transform $\widetilde{A}$ is a bounded real-analytic function on Ω. Properties of the operator A are often reflected in properties of the Berezin transform $\widetilde{A}$. The Berezin transform itself was introduced by F. Berezin in [5] and has proven to be a critical tool in operator theory, as many foundational properties of important operators are encoded in their Berezin transforms. The Berezin set and number, also denoted by Ber(A) and ber(A), respectively, were purportedly first formally introduced by Karaev in [19].

It is clear that $\text{ber}\,(A) \le \|A\|_{\text{Ber}} \le \|A\|$, $\text{Ber}\,(A) \subset W\,(A)$ and $\text{ber}\,(A) \le w\,(A)$, where

$$W(A) := \{\langle Ax, x\rangle : x \in \mathcal{H} \text{ and } \|x\| = 1\}$$

is the numerical range of the operator A and

$$w(A) := \sup_{\|x\|=1} |\langle Ax, x\rangle|$$

is its numerical radius. The numerical range of an operator has some interesting properties. For example, it is well known that the spectrum of an operator is contained in the closure of its numerical range. For basic properties of the numerical radius, we refer to [7, 8, 22, 23, 24, 28].

Berezin range and Berezin radius of operators are new numerical characteristics of operators on the RKHS which are introduced by Karaev in [19]. For the basic properties and facts on these new concepts, see [1, 2, 4, 15, 20, 30, 31].

It is well-known that

$$\frac{1}{2}\|A\| \le w\,(A) \le \|A\| \tag{1.1}$$

and

$$\text{ber}\,(A) \le w\,(A) \le \|A\|. \tag{1.2}$$

for any $A \in \mathcal{L}\,(\mathcal{H}\,(\Omega))$.

In [16], Huban et al. obtained the following result,

$$\text{ber}\,(A) \le \frac{1}{2}\left\|(A^*A)^{\frac{1}{2}} + (AA^*)^{\frac{1}{2}}\right\|_{\text{ber}} \le \frac{1}{2}\left(\|A\|_{\text{ber}} + \left\|A^2\right\|_{\text{ber}}^{\frac{1}{2}}\right) \le \|A\|_{\text{ber}}. \tag{1.3}$$

Another refinement of this inequality has been shown in [17], that if $A, B \in \mathcal{L}\,(\mathcal{H}\,(\Omega))$, then

$$\text{ber}^{2r}\,(B^*A) \le \frac{1}{2}\left\|(A^*A)^{2r} + (B^*B)^{2r}\right\|_{\text{ber}}, \text{ for all } r \ge 1, \tag{1.4}$$

and

$$\text{ber}\,(B^*A) \le \frac{1}{2}\|(A^*A) + (B^*B)\|_{\text{ber}}. \tag{1.5}$$

In this paper, by using some ideas of [17, 18, 29], we present several applications including geometrically convex functions when applied to the Berezin radius and the Berezin norm of reproducing kernel Hilbert space operators.

2 Known Lemmas

In the present section, we collect some auxiliary lemmas including Kittaneh [21] inequality, Young inequality [25] inequality and Zuo et al. [10] inequality.

$\mathcal{L}\,(\mathcal{H})$, which we mentioned above, an important class of operators in $\mathcal{L}\,(\mathcal{H})$ is the cone $\mathcal{L}\,(\mathcal{H})^+$ of positive operators; where an operator A is said to be positive if $\langle Ax, x\rangle \ge 0$ for all $x \in \mathcal{H}$. If $\mathcal{L}\,(\mathcal{H})^+$, we simply write $A \ge 0$. If, in addition to being positive, A is invertible, it is said to be strictly positive and it is denoted as $A > 0$. Recall that if I is a sub-interval of $(0, \infty)$ and $f : I \to (0, \infty)$, then f is called geometrical convex [27] if

$$f\left(a^{1-v}b^v\right) = f^{1-v}\,(a)\,f^v\,(b)\,, v \in [0, 1]\,. \tag{2.1}$$

We recall the following Jensen's type inequality [9, Theorem 1.2],

$$f(\langle Ax, x\rangle) \leq \langle f(A)x, x\rangle \tag{2.2}$$

for any unit $x \in \mathcal{H}$.

We remind the reader of the following inequality (see, e.g. [26, Theorem 6])

$$f(\langle Ax, x\rangle) \leq k(m, M, f)\langle f(A)x, x\rangle, \tag{2.3}$$

valid for the concave function $f : [m, M] \to \mathbb{R}$, the unit vector $x \in \mathcal{H}$ and the positive operator A satisfying $m \leq A \leq M$, for some positive scalars m, M. Here $k(m, M, f)$ is so called generalized Kantrovich constant and is defined by

$$k(m, M, f) = \min\left\{\frac{1}{f(t)}\left(\frac{M-t}{M-m}f(m) + \frac{t-m}{M-m}f(M)\right) : t \in [m, M]\right\}. \tag{2.4}$$

For $f(t) = t^r$, $r \in (0, 1]$, the constant $k(m, M, t^r)$ is well known by the following formula [9, Definition 2.2]

$$k(m, M, t^r) = \frac{h-h^r}{(1-r)(h-1)}\left(\frac{1-r}{r}\frac{h^r-1}{h-h^r}\right)^r, h = \frac{M}{m}.$$

We recall the following useful inequality which is known in the literature as the generalized mixed Schwarz inequality (see, e.g., [21]):

Lemma 2.1. *Let $A \in \mathcal{L}(\mathcal{H})$ and for any $x, y \in \mathcal{H}$.*

(i) If $0 \leq \alpha \leq 1$, then

$$|\langle Ax, y\rangle| \leq \sqrt{\left\langle |A|^{2\alpha} x, x\right\rangle \left\langle |A^*|^{2(1-\alpha)} y, y\right\rangle}. \tag{2.5}$$

(ii) If f and g are nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$, then

$$|\langle Ax, y\rangle| \leq \sqrt{\|f(|A|)x\| \, \|g(|A^*|)y\|}.$$

The next lemma gives an additive refinement of the scalar Young inequality (see [25, Theorem 2.1]).

Lemma 2.2. *If $a, b > 0$ and $0 \leq \lambda \leq 1$, then*

$$a^\lambda b^{(1-\lambda)} + r\left(\sqrt{a} - \sqrt{b}\right)^2 \leq (1-\lambda)a + \lambda b \tag{2.6}$$

where $r = \min\{\lambda, 1-\lambda\}$.

The multiplicative refinement of the Young inequality with the Kantorovich constant was given by Zuo et al. [10] as following:

Lemma 2.3. *Let $a, b > 0$. Then*

$$(1-\lambda)a + \lambda b \leq K_1(h, 2)\, a^{(1-\lambda)} b^\lambda, \tag{2.7}$$

where $0 \leq \lambda \leq 1$, $r = \min\{\lambda, 1-\lambda\}$, $h = \frac{a}{b}$ is such that

$$K_1(h, 2) = \frac{(h+1)^2}{4h},\ h > 0,$$

which has the properties

$$K_1(h, 2) = K_1\left(\frac{1}{h}, 2\right) \geq 1,\ h > 0,$$

and $K_1(h, 2)$ is increasing on $[1, \infty)$ and is decreasing on $(0, 1)$.

3 Main Results

In this section, we apply lemmas in previous section to prove new inequalities for Berezin radius inequalities on operators on $\mathcal{H} = \mathcal{H}(\Omega)$ and present the general form of some known inequalities in the literature. This gives a new perspective to these inequalities.

We start our work with the following result.

Theorem 3.1. *Let $A, B \in \mathcal{L}(\mathcal{H}(\Omega))$ and f be an increasing geometrically convex function. If in addition f is a convex, then*

$$f(\mathrm{ber}(B^*A)) \le \frac{1}{2}\|f(A^*A) + f(B^*B)\|_{\mathrm{ber}}. \tag{3.1}$$

Proof. Let $\eta \in \Omega$ be an arbitrary. By using the Cauchy-Schwarz inequality and the arithmetic-geometric mean inequality, we get

$$\begin{aligned}
f\left(\left|\left\langle B^*A\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\right|\right) &= f\left(\left|\left\langle A\widehat{k}_\eta, B\widehat{k}_\eta\right\rangle\right|\right) \le f\left(\left\|A\widehat{k}_\eta\right\|\left\|B\widehat{k}_\eta\right\|\right) \\
&= f\left(\sqrt{\left\langle A\widehat{k}_\eta, A\widehat{k}_\eta\right\rangle\left\langle B\widehat{k}_\eta, B\widehat{k}_\eta\right\rangle}\right) \\
&= f\left(\sqrt{\left\langle A^*A\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\left\langle B^*B\widehat{k}_\eta, \widehat{k}_\eta\right\rangle}\right) \\
&\le \sqrt{f\left(\left\langle A^*A\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\right) f\left(\left\langle B^*B\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\right)} \\
&\quad \text{(by the inequality (2.1))} \\
&\le \sqrt{\left\langle f(A^*A)\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\left\langle f(B^*B)\widehat{k}_\eta, \widehat{k}_\eta\right\rangle} \\
&\quad \text{(by the inequality (2.2))} \\
&\le \frac{1}{2}\left(\left\langle f(A^*A)\widehat{k}_\eta, \widehat{k}_\eta\right\rangle + \left\langle f(B^*B)\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\right) \\
&= \frac{1}{2}\left\langle (f(A^*A) + f(B^*B))\widehat{k}_\eta, \widehat{k}_\eta\right\rangle.
\end{aligned}$$

Therefore, taking the supremum over $\lambda \in \Omega$ we deduce

$$\begin{aligned}
f(\mathrm{ber}(B^*A)) &\le f\left(\sup_{\eta\in\Omega}\left|\left\langle B^*A\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\right|\right) \\
&= \sup_{\eta\in\Omega} f\left(\left|\left\langle B^*A\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\right|\right) \\
&\le \frac{1}{2}\sup_{\eta\in\Omega}\left\langle (f(A^*A) + f(B^*B))\widehat{k}_\eta, \widehat{k}_\eta\right\rangle \\
&\le \frac{1}{2}\|f(A^*A) + f(B^*B)\|_{\mathrm{ber}}.
\end{aligned}$$

This completes the proof. □

Next, we may define double convex functions (see [6, Example 2.12]). A function $f(t)$ is double convex if:

1. $f(t)$ is a non-negative continuous function defined on a positive interval $Q \subset [0, \infty)$,
2. $f(t)$ is convex,
3. $f(t)$ is geometrically convex, i.e., $f(\sqrt{xy}) \le \sqrt{f(x)f(y)}$ for all $x, y \in Q$.

Now, let the function $f(t) = t^r$ $(r \ge 1)$ be double convex functions. Then, the inequality (3.1) implies (1.5).

Corollary 3.2. *Let $A, B, S \in \mathcal{L}(\mathcal{H}(\Omega))$ and f be an increasing geometrically convex function. If in addition f is a convex, then*

$$f(\text{ber}(ASB)) \leq \frac{1}{2} \|f(A|S^*|A^*) + f(B^*|S|B)\|_{\text{ber}}.$$

Proof. Let $\widehat{k}_\eta$ be a normalized reproducing kernel and $X = U|S|$ be the polar decomposition of S. Then

$$f(\text{ber}(ASB)) = f(\text{ber}(AU|S|B)) = f\left(\text{ber}\left(\left(|S|^{1/2} U^* A^*\right)^* \left(|S|^{1/2} B\right)\right)\right).$$

By substituting $B = |S|^{1/2} U^* A^*$ and $A = |S|^{1/2} B$ in Theorem 3.1, we have the desired inequality, noting that when $S = U|S|$ is the polar decomposition of S, $|S^*| = U|S|U^*$. □

Another interesting inequality for $f(\text{ber}(ASB))$ maybe obtained as follows. First, notice that if f is a convex function and $\alpha \leq 1$, it follows that

$$f(\alpha t) \leq \alpha f(t) + (1-\alpha) f(0). \tag{3.2}$$

This follows by direct calculus computations for the function $g(t) = f(\alpha t) - \alpha f(t)$.

For the communing results, we will use the term norm-contractive to mean an operator S whose Berezin norm satisfies $\|S\|_{\text{ber}} \leq 1$.

Proposition 3.3. *Let $A, B, S \in \mathcal{L}(\mathcal{H}(\Omega))$ and f be an increasing geometrically convex function. For the norm-contractive S, we have the inequality*

(i) $f(\text{ber}(B^*SA)) \leq \|S\|_{\text{ber}} f(\|A\|_{\text{Ber}} \|B\|_{\text{Ber}}) + (1 - \|S\|_{\text{ber}}) f(0)$.

(ii) if $f(0) = 0$, then $f(\text{ber}(B^*SA)) \leq \|S\|_{\text{ber}} f(\|A\|_{\text{Ber}} \|B\|_{\text{Ber}})$.

Proof. Let $\widehat{k}_\eta$ be a normalized reproducing kernel. Proceeding as in Theorem 3.1 and noting (3.2), we have

$$\begin{aligned} f\left(\left|\left\langle B^*SA\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\right|\right) &= f\left(\left|\left\langle SA\widehat{k}_\eta, B\widehat{k}_\eta\right\rangle\right|\right) \\ &\leq f\left(\left\|SA\widehat{k}_\eta\right\|_{\text{ber}} \left\|B\widehat{k}_\eta\right\|_{\text{ber}}\right) \text{ (by the Cauchy-Schwarz inequality)} \\ &\leq f\left(\|S\|_{\text{ber}} \left\|A\widehat{k}_\eta\right\|_{\text{ber}} \left\|B\widehat{k}_\eta\right\|_{\text{ber}}\right) \\ &\leq \|S\|_{\text{ber}} f\left(\left\|A\widehat{k}_\eta\right\|_{\text{ber}} \left\|B\widehat{k}_\eta\right\|_{\text{ber}}\right) + (1 - \|S\|_{\text{ber}}) f(0) \end{aligned}$$

and

$$\sup_{\eta\in\Omega} f\left(\left|\left\langle B^*SA\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\right|\right) \leq \sup_{\eta\in\Omega} \left\{\|S\|_{\text{ber}} f\left(\left\|A\widehat{k}_\eta\right\|_{\text{ber}} \left\|B\widehat{k}_\eta\right\|_{\text{ber}}\right) + (1 - \|S\|_{\text{ber}}) f(0)\right\}$$

which is equivalent to

$$f(\text{ber}(B^*SA)) \leq \|S\|_{\text{ber}} f(\|A\|_{\text{Ber}} \|B\|_{\text{Ber}}) + (1 - \|S\|_{\text{ber}}) f(0).$$

and completes the proof of (i). The other (ii) inequalities follow by letting $f(0) = 0$. □

In particular, if $f(t) = t^r$, we obtain the following extension of (1.4).

Corollary 3.4. *Let $A, B, S \in \mathcal{L}(\mathcal{H}(\Omega))$. If S is norm-contractive and $r \geq 1$, then*

$$\text{ber}^r(B^*SA) \leq \frac{\|S\|_{\text{ber}}^r}{2} \left\|(A^*A)^r + (B^*B)^r\right\|_{\text{ber}} \tag{3.3}$$

Proof. Let $\eta \in \Omega$ be an arbitrary. By using the Cauchy-Schwarz inequality for the function $f(t) = t^r$, we have

$$\begin{aligned}
f\left(\left|\left\langle B^* S A \widehat{k}_\eta, \widehat{k}_\eta \right\rangle\right|\right) &= f\left(\left|\left\langle S A \widehat{k}_\eta, B \widehat{k}_\eta \right\rangle\right|\right) \\
&\leq f\left(\left\|S A \widehat{k}_\eta\right\|_{\text{ber}} \left\|B \widehat{k}_\eta\right\|_{\text{ber}}\right) \\
&\leq f\left(\|S\|_{\text{ber}} \left\|A \widehat{k}_\eta\right\|_{\text{ber}} \left\|B \widehat{k}_\eta\right\|_{\text{ber}}\right) \\
&\leq f\left(\|S\|_{\text{ber}}\right) f\left(\left\|A \widehat{k}_\eta\right\|_{\text{ber}} \left\|B \widehat{k}_\eta\right\|_{\text{ber}}\right) \\
&= f\left(\|S\|_{\text{ber}}\right) f\left(\sqrt{\left\langle A \widehat{k}_\eta, A \widehat{k}_\eta \right\rangle \left\langle B \widehat{k}_\eta, B \widehat{k}_\eta \right\rangle}\right) \\
&\leq f\left(\|S\|_{\text{ber}}\right) \sqrt{f\left(\left\langle A^* A \widehat{k}_\eta, \widehat{k}_\eta \right\rangle\right) f\left(\left\langle B^* B \widehat{k}_\eta, \widehat{k}_\eta \right\rangle\right)} \\
&\leq f\left(\|S\|_{\text{ber}}\right) \sqrt{\left\langle f\left(A^* A\right) \widehat{k}_\eta, \widehat{k}_\eta \right\rangle \left\langle f\left(B^* B\right) \widehat{k}_\eta, \widehat{k}_\eta \right\rangle} \\
&\quad \text{(by the inequality (2.2))} \\
&\leq f\left(\|S\|_{\text{ber}}\right) \frac{1}{2}\left(\left\langle f\left(A^* A\right) \widehat{k}_\eta, \widehat{k}_\eta \right\rangle + \left\langle f\left(B^* B\right) \widehat{k}_\eta, \widehat{k}_\eta \right\rangle\right) \\
&= f\left(\|S\|_{\text{ber}}\right) \frac{1}{2}\left\langle \left(f\left(A^* A\right) + f\left(B^* B\right)\right) \widehat{k}_\eta, \widehat{k}_\eta \right\rangle .
\end{aligned}$$

Hence

$$\left|\left\langle B^* S A \widehat{k}_\eta, \widehat{k}_\eta \right\rangle\right|^r \leq \frac{1}{2} \|S\|_{\text{ber}}^r \left\langle \left(A^* A\right)^r + \left(B^* B\right)^r \widehat{k}_\eta, \widehat{k}_\eta \right\rangle .$$

By taking supremum over $\eta \in \Omega$, we have

$$\text{ber}^r\left(B^* S A\right) \leq \frac{\|S\|_{\text{ber}}^r}{2} \left\|\left(A^* A\right)^r + \left(B^* B\right)^r\right\|_{\text{ber}} .$$

Thus the desired result has been obtained. □

Our next target is to show similar inequalities for geometrically convex functions which are concave, instead of convex.

Theorem 3.5. *Let $A, B \in \mathcal{L}(\mathcal{H}(\Omega))$ be such that $0 \leq m \leq A, B \leq M$ and f be an increasing geometrically convex function. If in addition f is a concave, then for any $\eta \in \Omega$, we have*

$$f\left(\text{ber}\left(A^{1/2} S B^{1/2}\right)\right) \leq \frac{k(m, M, f)}{2} \|S\|_{\text{ber}} \|f(A) + f(B)\|_{\text{ber}}, \tag{3.4}$$

for the norm-expansive S (i.e., $\|S\| \geq 1$).

Proof. Let $\eta \in \Omega$ be the arbitrary. Proceeding as in Proposition 3.3 and noting (2.2) and the inequality $f(\alpha t) \leq \alpha f(t)$ when f is a concave and $\alpha \geq 1$, we obtain the desired inequality. □

Remark 3.6. In particular, the function $f(t) = t^r$, $0 < r \leq 1$ satisfies the conditions of Theorem 3.5. Further, noting that

$$f\left(\|S\|_{\text{ber}} \|A\|_{\text{Ber}} \|B\|_{\text{Ber}}\right) = f\left(\|S\|_{\text{ber}}\right) f\left(\|A\|_{\text{Ber}} \|B\|_{\text{Ber}}\right),$$

we obtain the inequality

$$\text{ber}\left(A^{1/2} S B^{1/2}\right) \leq \left(\frac{k(m, M, f)}{2}\right)^{1/r} \|S\|_{\text{ber}} \|A^r + B^r\|_{\text{Ber}}^{1/2},$$

for the positive operators A, B satisfying $0 \leq m \leq A$, $B \leq M$ and the norm-expansive S.

It follows from Theorem 3.7 in [18] that if $A, B \in \mathcal{L}(\mathcal{H}(\Omega))$, $0 < \alpha < 1$ and $r \geq 1$, then

$$\mathrm{ber}^r(A+B) \leq 2^{r-2}\left(\left\||A|^{2\alpha r}+|B|^{2\alpha r}\right\|_{\mathrm{ber}}+\left\||A^*|^{2(1-\alpha)r}+|B^*|^{2(1-\alpha)r}\right\|_{\mathrm{ber}}\right). \tag{3.5}$$

Our next result is the generalization of (3.5).

Theorem 3.7. *Let $A, B \in \mathcal{L}(\mathcal{H}(\Omega))$ and f be an increasing geometrically convex function. If in addition f is convex, then for any $\alpha \in [0,1]$,*

$$f\left(\left\|\frac{A+B}{2}\right\|_{\mathrm{Ber}}\right) \leq \frac{1}{4}\left(\left\|f\left(|A|^{2\alpha}\right)+f\left(|B|^{2\alpha}\right)\right\|_{\mathrm{ber}}+\left\|f\left(|A^*|^{2(1-\alpha)}\right)+f\left(|B^*|^{2(1-\alpha)}\right)\right\|_{\mathrm{ber}}\right) \tag{3.6}$$

and

$$f\left(\mathrm{ber}\left(\frac{A+B}{2}\right)\right) \leq \frac{1}{4}\left(\|f(|A|)+f(|A^*|)+f(|B|)+f(|B^*|)\|_{\mathrm{ber}}\right). \tag{3.7}$$

Proof. Let $\eta, \mu \in \Omega$ be an arbitrary. We have

$$\begin{aligned}
f\left(\frac{1}{2}\left|\left\langle (A+B)\widehat{k}_\eta, \widehat{k}_\mu\right\rangle\right|\right) &\leq f\left(\frac{1}{2}\left(\left|\left\langle A\widehat{k}_\eta, \widehat{k}_\mu\right\rangle\right|+\left|\left\langle B\widehat{k}_\eta, \widehat{k}_\mu\right\rangle\right|\right)\right)\\
&\leq \frac{1}{2}\left(f\left(\left|\left\langle A\widehat{k}_\eta, \widehat{k}_\mu\right\rangle\right|\right)+f\left(\left|\left\langle B\widehat{k}_\eta, \widehat{k}_\mu\right\rangle\right|\right)\right)\\
&\leq \frac{1}{2}f\left(\sqrt{\left\langle |A|^{2\alpha}\widehat{k}_\eta, \widehat{k}_\mu\right\rangle\left\langle |A^*|^{2(1-\alpha)}\widehat{k}_\eta, \widehat{k}_\mu\right\rangle}\right)\\
&\quad+\frac{1}{2}f\left(\sqrt{\left\langle |B|^{2\alpha}\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\left\langle |B^*|^{2(1-\alpha)}\widehat{k}_\mu, \widehat{k}_\mu\right\rangle}\right)\\
&\quad\text{(by the inequality (2.5))}\\
&\leq \frac{1}{2}\sqrt{f\left(\left\langle |A|^{2\alpha}\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\right)f\left(\left\langle |A^*|^{2(1-\alpha)}\widehat{k}_\mu, \widehat{k}_\mu\right\rangle\right)}\\
&\quad+\frac{1}{2}\sqrt{f\left(\left\langle |B|^{2\alpha}\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\right)f\left(\left\langle |B^*|^{2(1-\alpha)}\widehat{k}_\mu, \widehat{k}_v\right\rangle\right)}\\
&\quad\text{(by the geometrically convex)}\\
&\leq \frac{1}{2}\sqrt{\left\langle f\left(|A|^{2\alpha}\right)\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\left\langle f\left(|A^*|^{2(1-\alpha)}\right)\widehat{k}_\mu, \widehat{k}_\mu\right\rangle}\\
&\quad+\frac{1}{2}\sqrt{\left\langle f\left(|B|^{2\alpha}\right)\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\left\langle f\left(|B^*|^{2(1-\alpha)}\right)\widehat{k}_\mu, \widehat{k}_\mu\right\rangle}\\
&\quad\text{(by the inequality (2.2))}
\end{aligned}$$

$$\leq \frac{1}{4}\left\langle\left(f\left(|A|^{2\alpha}\right)+f\left(|B|^{2\alpha}\right)\right)\widehat{k}_\eta, \widehat{k}_\eta\right\rangle+\frac{1}{4}\left\langle\left(f\left(|A^*|^{2(1-\alpha)}\right)+f\left(|B^*|^{2(1-\alpha)}\right)\right)\widehat{k}_\mu, \widehat{k}_\mu\right\rangle$$

(by the AM-GM inequality).

In particular, for $\eta = \mu$, we obtain

$$\begin{aligned}
f\left(\left\|\frac{A+B}{2}\widehat{k}_\eta\right\|\right) &\leq \frac{1}{4}\left\langle\left(f\left(|A|^{2\alpha}\right)+f\left(|B|^{2\alpha}\right)\right)\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\\
&\quad+\frac{1}{4}\left\langle\left(f\left(|A^*|^{2(1-\alpha)}\right)+f\left(|B^*|^{2(1-\alpha)}\right)\right)\widehat{k}_\eta, \widehat{k}_\eta\right\rangle.
\end{aligned}$$

Taking the supremum over $\eta \in \Omega$, we have desired inequality

$$f\left(\left\|\frac{A+B}{2}\right\|_{\mathrm{Ber}}\right) \leq \frac{1}{4}\left(\left\|f\left(|A|^{2\alpha}\right)+f\left(|B|^{2\alpha}\right)\right\|_{\mathrm{ber}}+\left\|f\left(|A^*|^{2(1-\alpha)}\right)+f\left(|B^*|^{2(1-\alpha)}\right)\right\|_{\mathrm{ber}}\right).$$

By applying same procedure for $\eta = \mu$, it follows that

$$f\left(\frac{1}{2}\left|\left\langle (A+B)\widehat{k}_\eta, \widehat{k}_\eta \right\rangle\right|\right) \le \frac{1}{4}\left\langle \left(f\left(|A| + f\left(|A^*|\right) + f\left(|B|\right)\widehat{k} + f\left(|B^*|\right)\right)\right)\widehat{k}_\eta, \widehat{k}_\eta \right\rangle$$

and

$$\sup_{\eta\in\Omega} f\left(\frac{1}{2}\left|\left\langle (A+B)\widehat{k}_\eta, \widehat{k}_\eta \right\rangle\right|\right) \le \sup_{\eta\in\Omega}\frac{1}{4}\left\langle \left(f\left(|A| + f\left(|A^*|\right) + f\left(|B|\right)\widehat{k} + f\left(|B^*|\right)\right)\right)\widehat{k}_\eta, \widehat{k}_\eta \right\rangle$$

which is equivalent to

$$f\left(\text{ber}\left(\frac{A+B}{2}\right)\right) \le \frac{1}{4}\left(\left\|f\left(|A|\right) + f\left(|A^*|\right) + f\left(|B|\right) + f\left(|B^*|\right)\right\|_{\text{ber}}\right).$$

Hence, we get the required inequality. □

Remark 3.8. This shows that letting $A = B$, the above Berezin radius inequality reduces to (3.5).

Next, we show the concave version of Theorem 3.7.

Theorem 3.9. *Let $A, B \in \mathcal{L}(\mathcal{H}(\Omega))$, $\alpha \in [0,1]$ and f be an increasing geometrically convex function. Assume that for positive scalar m, M,*

$$m \le |A|^{2\alpha}, |A^*|^{2(1-\alpha)}, |B|^{2\alpha}, |B^*|^{2(1-\alpha)} \le M.$$

If f is concave, then

$$f\left(\text{ber}(A+B)\right) \le \frac{K}{2}\left(\left\|f\left(|A|^{2\alpha}\right) + f\left(|B|^{2\alpha}\right)\right\|_{\text{ber}} + \left\|f\left(|A^*|^{2(1-\alpha)}\right) + f\left(|B^*|^{2(1-\alpha)}\right)\right\|_{\text{ber}}\right) \tag{3.8}$$

and

$$f\left(\text{ber}(A+B)\right) \le \frac{K}{2}\left(\left\|f\left(|A|\right) + f\left(|A^*|\right) + f\left(|B|\right) + f\left(|B^*|\right)\right\|_{\text{ber}}\right), \tag{3.9}$$

where $K = k(m, M, f)$.

Proof. Let $\eta, \mu \in \Omega$ be the arbitrary. We have

$$\begin{aligned} f\left(\left|\left\langle (A+B)\widehat{k}_\eta, \widehat{k}_\mu \right\rangle\right|\right) &\le f\left(\left|\left\langle A\widehat{k}_\eta, \widehat{k}_\mu \right\rangle\right| + \left|\left\langle B\widehat{k}_\eta, \widehat{k}_\mu \right\rangle\right|\right) \\ &\le \left(f\left(\left|\left\langle A\widehat{k}_\eta, \widehat{k}_\mu \right\rangle\right|\right) + f\left(\left|\left\langle B\widehat{k}_\eta, \widehat{k}_\mu \right\rangle\right|\right)\right) \\ &\le f\left(\sqrt{\left\langle |A|^{2\alpha}\widehat{k}_\eta, \widehat{k}_\eta \right\rangle \left\langle |A^*|^{2(1-\alpha)}\widehat{k}_\mu, \widehat{k}_\mu \right\rangle}\right) \\ &+ f\left(\sqrt{\left\langle |B|^{2\alpha}\widehat{k}_\eta, \widehat{k}_\eta \right\rangle \left\langle |B^*|^{2(1-\alpha)}\widehat{k}_\mu, \widehat{k}_\mu \right\rangle}\right) \\ &\text{(by the inequality (2.5))} \end{aligned}$$

$$\leq \left(\sqrt{f\left(\left\langle |A|^{2\alpha}\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle\right) f\left(\left\langle |A^*|^{2(1-\alpha)}\widehat{k}_{\mu},\widehat{k}_{\mu}\right\rangle\right)}\right)$$
$$+\left(\sqrt{f\left(\left\langle |B|^{2\alpha}\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle\right) f\left(\left\langle |B^*|^{2(1-\alpha)}\widehat{k}_{\mu},\widehat{k}_{\mu}\right\rangle\right)}\right)$$
$$\leq K\left(\sqrt{\left\langle f\left(|A|^{2\alpha}\right)\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle \left\langle f\left(|A^*|^{2(1-\alpha)}\right)\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle}\right)$$
$$+K\left(\sqrt{\left\langle f\left(|B|^{2\alpha}\right)\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle \left\langle f\left(|B^*|^{2(1-\alpha)}\right)\widehat{k}_{\mu},\widehat{k}_{\mu}\right\rangle}\right)$$

(by the inequality (2.3))

$$\leq \frac{K}{2}\left\langle\left(f\left(|A|^{2\alpha}\right)+f\left(|B|^{2\alpha}\right)\right)\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle$$
$$+\frac{K}{2}\left\langle\left(f\left(|A^*|^{2(1-\alpha)}\right)+f\left(|B^*|^{2(1-\alpha)}\right)\right)\widehat{k}_{\mu},\widehat{k}_{\mu}\right\rangle$$

(by the AM-GM inequality).

Now, by taking the supremum over $\eta\in\Omega$ with $\eta=\mu$,

$$\sup_{\eta\in\Omega} f\left(\left|\left\langle (A+B)\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle\right|\right) \leq \sup_{\eta\in\Omega}\frac{K}{2}\left\langle\left(f\left(|A|^{2\alpha}\right)+f\left(|B|^{2\alpha}\right)\right)\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle + \sup_{\eta\in\Omega}\frac{K}{2}\left\langle\left(f\left(|A^*|^{2(1-\alpha)}\right)+f\left(|B^*|^{2(1-\alpha)}\right)\right)\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle,$$

we then conclude that

$$f\left(\text{ber}\left(A+B\right)\right) \leq \frac{K}{2}\left(\left\|f\left(|A|^{2\alpha}\right)+f\left(|B|^{2\alpha}\right)\right\|_{\text{ber}} + \left\|f\left(|A^*|^{2(1-\alpha)}\right)+f\left(|B^*|^{2(1-\alpha)}\right)\right\|_{\text{ber}}\right).$$

The theorem is proved. □

In particular, if $f(t)=t^r$, $0\leq r\leq 1$, we get

$$\text{ber}^r\left(A+B\right) \leq \frac{h-h^r}{2\left(1-r\right)\left(h-1\right)}\left(\frac{1-r}{r}\frac{h^r-1}{h-h^r}\right)^r\left(\left\||A|^{2\alpha r}+|B|^{2\alpha r}\right\|_{\text{ber}}+\left\||A^*|^{2(1-\alpha)r}+|B^*|^{2(1-\alpha)r}\right\|_{\text{ber}}\right),$$

where $h=\frac{m}{M}$.

It follows from Theorem 3.3 in [18] that if $A\in\mathcal{L}\left(\mathcal{H}\left(\Omega\right)\right)$, $0<\alpha<1$ and $r\geq 1$, then

$$\left(\text{ber}\left(A\right)\right)^{2r} \leq \left\|\alpha\left|A\right|^{2r}+\left(1-\alpha\right)\left|A^*\right|^{2r}\right\|_{\text{ber}}. \tag{3.10}$$

Now, we obtain some refinements of inequality (3.10) by applying refinements of the Young inequality. In this result, we will use the concave-version of the inequality (2.2) when $\eta\in\Omega$, $f:I\to\mathbb{R}$ is a concave function and A is self adjoint with spectrum in I.

Theorem 3.10. *Let $\mathcal{H}=\mathcal{H}\left(\Omega\right)$ be a RKHS and $A\in\mathcal{L}\left(\mathcal{H}\right)$, $0\leq\alpha\leq 1$ and f be as in Theorem 1. Then*

$$f\left(\text{ber}^2\left(A\right)\right) \leq \left\|\alpha f\left(|A|^2\right)+\left(1-\alpha\right)f\left(|A^*|^2\right)\right\|_{\text{ber}}. \tag{3.11}$$

Proof. Let $\eta \in \Omega$ be the arbitrary. Then by the inequality (2.5) and monotony of f, we get

$$\begin{aligned} f\left(\left|\left\langle A\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle\right|^{2}\right) &\leq f\left(\left\langle |A|^{2\alpha}\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle\left\langle |A^{*}|^{2(1-\alpha)}\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle\right) \\ &\text{(by the inequality (2.5))} \\ &\leq f\left(\left\langle |A|^{2}\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle^{\alpha}\left\langle |A^{*}|^{2}\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle^{(1-\alpha)}\right) \\ &\leq f^{\alpha}\left(\left\langle |A|^{2}\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle\right)f^{(1-\alpha)}\left(\left\langle |A^{*}|^{2}\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle\right) \\ &\text{(by the inequality (2.1))} \\ &\leq \alpha f\left(\left\langle |A|^{2}\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle\right)+(1-\alpha)f\left(\left\langle |A^{*}|^{2}\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle\right) \\ &\text{(by the Young inequality)} \\ &\leq \alpha\left\langle\left(f|A|^{2}\right)\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle+(1-\alpha)\left\langle f\left(|A^{*}|^{2}\right)\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle \\ &\text{(by the inequality (2.2)).} \end{aligned}$$

Whence

$$\sup_{\eta\in\Omega} f\left(\left|\widetilde{A}(\eta)\right|^{2}\right) \leq \sup_{\eta\in\Omega}\left(\alpha\left\langle\left(f|A|^{2}\right)\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle+(1-\alpha)\left\langle f\left(|A^{*}|^{2}\right)\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle\right)$$

and

$$f\left(\mathrm{ber}^{2}(A)\right) \leq \left\|\alpha f\left(|A|^{2}\right)+(1-\alpha)f\left(|A^{*}|^{2}\right)\right\|_{\mathrm{ber}}$$

for all $\eta \in \Omega$, which implies the desired inequality (3.11). □

Now, we obtain some refinements of inequality (1.2) by applying refinements of the Young inequality.

Theorem 3.11. *Let $A \in \mathcal{L}(\mathcal{H}(\Omega))$ and f be an increasing geometrically convex function and $r = \min\{\lambda, 1-\lambda\}$, where $0 \leq \lambda \leq 1$. If f is a convex function, then*

$$f(\mathrm{ber}(A)) \leq \frac{1-2r}{2}\left\|f(|A|)+f(|A^{*}|)\right\|_{\mathrm{ber}}+2r\left\|f(|A|)\right\|_{\mathrm{ber}}. \tag{3.12}$$

Proof. Let $\eta \in \Omega$ be the arbitrary. Then

$$\begin{aligned} &f\left(\left|\left\langle A\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle\right|\right) \\ &\leq f\left(\sqrt{\left\langle |A|\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle\left\langle |A^{*}|\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle}\right) \\ &\text{(by the inequality (2.5))} \\ &\leq f\left(\left(\left\langle |A|\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle^{1-\alpha}\left\langle |A^{*}|\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle^{\alpha}\right)^{1/2}\left(\left\langle |A^{*}|\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle^{1-\alpha}\left\langle |A|\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle^{\alpha}\right)^{1/2}\right) \\ &\leq f\left(\frac{1}{2}\left(\left\langle |A|\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle^{1-\alpha}\left\langle |A^{*}|\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle^{\alpha}+\left\langle |A^{*}|\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle^{1-\alpha}\left\langle |A|\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle^{\alpha}\right)\right) \\ &\text{(by the AM-GM inequality)} \end{aligned}$$

$$\leq \frac{1}{2}\left(f\left(\left\langle |A|\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle^{1-\alpha}\left\langle |A^*|\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle^{\alpha}\right)+\frac{1}{2}f\left(\left\langle |A^*|\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle^{1-\alpha}\left\langle |A|\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle^{\alpha}\right)\right)$$

$$\leq \frac{1}{2}f^{1-\alpha}\left(\left\langle |A|\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle\right)f^{\alpha}\left(\left\langle |A^*|\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle\right)+\frac{1}{2}f^{1-\alpha}\left(\left\langle |A^*|\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle\right)f^{\alpha}\left(\left\langle |A|\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle\right)$$

(by the inequality (2.1))

$$\leq \frac{1-\alpha}{2}f\left(\left\langle |A|\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle\right)+\frac{\alpha}{2}f\left(\left\langle |A^*|\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle\right)$$

$$-\frac{r}{2}\left(\sqrt{f\left(\left\langle |A|\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle\right)}-\sqrt{f\left(\left\langle |A^*|\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle\right)}\right)^2$$

$$+\frac{1-\alpha}{2}f\left(\left\langle |A^*|\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle\right)+\frac{\alpha}{2}f\left(\left\langle |A|\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle\right)$$

$$-\frac{r}{2}\left(\sqrt{f\left(\left\langle |A^*|\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle\right)}-\sqrt{f\left(\left\langle |A|\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle\right)}\right)^2$$

(by the inequality (2.6))

$$=\frac{1}{2}\left\langle \left(f\left(|A|\right)+f\left(|A^*|\right)\right)\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle - r\left\langle \left(f\left(|A|\right)+f\left(|A^*|\right)\right)\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle$$

$$+2r\sqrt{\left\langle f\left(|A|\right)\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle\left\langle f\left(|A^*|\right)\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle}.$$

From this, it is immediate that

$$f\left(\left|\widetilde{A}\left(\eta\right)\right|\right)\leq \frac{1-2r}{2}\left\langle \left(f\left(|A|\right)+f\left(|A^*|\right)\right)\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle + 2r\left\langle f\left(|A|\right)\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle.$$

By taking the supremum over $\eta\in\Omega$ in above inequality, we deduce

$$\sup_{\eta\in\Omega} f\left(\left|\widetilde{A}\left(\eta\right)\right|\right)\leq \frac{1-2r}{2}\sup_{\eta\in\Omega}\left\langle \left(f\left(|A|\right)+f\left(|A^*|\right)\right)\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle + 2r\sup_{\eta\in\Omega}\left\langle f\left(|A|\right)\widehat{k}_{\eta},\widehat{k}_{\eta}\right\rangle.$$

Therefore, we get

$$f\left(\text{ber}\left(A\right)\right)\leq \frac{1-2r}{2}\left\|f\left(|A|\right)+f\left(|A^*|\right)\right\|_{\text{ber}}+2r\left\|f\left(|A|\right)\right\|_{\text{ber}},$$

which proves inequality (3.12). □

Remark 3.12. Letting, $f(t)=t$ in Theorem 3.11 implies

$$\text{ber}\left(A\right)\leq \frac{1-2r}{2}\left\||A|+|A^*|\right\|_{\text{ber}}+2r\left\||A|\right\|_{\text{ber}}$$

which is the result of [36, Theorem 2.6].

In the following theorem, we improve inequality (3.10) for hyponormal operators. Recall that an operator $A\in\mathcal{L}(\mathcal{H})$ is said to be hyponormal, if $A^*A-AA^*\geq 0$, or equivalenty, if $\|A^*x\|\leq\|Ax\|$ for every $x\in\mathcal{H}$.

Theorem 3.13. *Let f be an increasing geometrically convex function. If $A\in\mathcal{L}(\mathcal{H}(\Omega))$ is hyponormal, $r=\min\{\alpha,1-\alpha\}$, where $0\leq\alpha\leq 1$, then*

$$f\left(\text{ber}\left(A\right)\right)\leq \frac{1}{\inf_{\eta\in\Omega}\xi\left(\eta\right)}\frac{\left\|f\left(|A|\right)+f\left(|A^*|\right)\right\|_{\text{ber}}}{2}$$

where $\xi\left(\eta\right)=K_1\left(\frac{f\left(\widetilde{|A|}(\eta)\right)}{f\left(\widetilde{|A^*|}(\eta)\right)},2\right)^r.$

Proof. Let $\eta \in \Omega$ be the arbitrary. We have

$$
\begin{aligned}
& f\left(\left|\left\langle A\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\right|\right) \\
& \leq f\left(\sqrt{\left\langle |A|\,\widehat{k}_\eta, \widehat{k}_\eta\right\rangle \left\langle |A^*|\,\widehat{k}_\eta, \widehat{k}_\eta\right\rangle}\right)
\end{aligned}
$$

(by the inequality (2.5))

$$
\begin{aligned}
& \leq f\left(\left(\left\langle |A^*|\,\widehat{k}_\eta, \widehat{k}_\eta\right\rangle^{1-\alpha} \left\langle |A|\,\widehat{k}_\eta, \widehat{k}_\eta\right\rangle^{\alpha}\right)^{1/2} \left(\left\langle |A|\,\widehat{k}_\eta, \widehat{k}_\eta\right\rangle^{1-\alpha} \left\langle |A^*|\,\widehat{k}_\eta, \widehat{k}_\eta\right\rangle^{\alpha}\right)^{1/2}\right) \\
& \leq f\left(\frac{1}{2}\left(\left\langle |A^*|\,\widehat{k}_\eta, \widehat{k}_\eta\right\rangle^{1-\alpha} \left\langle |A|\,\widehat{k}_\eta, \widehat{k}_\eta\right\rangle^{\alpha} + \left\langle |A|\,\widehat{k}_\eta, \widehat{k}_\eta\right\rangle^{1-\alpha} \left\langle |A^*|\,\widehat{k}_\eta, \widehat{k}_\eta\right\rangle^{\alpha}\right)\right)
\end{aligned}
$$

(by the AM-GM inequality)

$$
\begin{aligned}
& \leq \frac{1}{2}\left(f\left(\left\langle |A^*|\,\widehat{k}_\eta, \widehat{k}_\eta\right\rangle^{1-\alpha} \left\langle |A|\,\widehat{k}_\eta, \widehat{k}_\eta\right\rangle^{\alpha}\right) + \frac{1}{2} f\left(\left\langle |A|\,\widehat{k}_\eta, \widehat{k}_\eta\right\rangle^{1-\alpha} \left\langle |A^*|\,\widehat{k}_\eta, \widehat{k}_\eta\right\rangle^{\alpha}\right)\right) \\
& \leq \frac{1}{2} f^{1-\alpha}\left(\left\langle |A^*|\,\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\right) f^{\alpha}\left(\left\langle |A|\,\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\right) + \frac{1}{2} f^{1-\alpha}\left(\left\langle |A|\,\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\right) f^{\alpha}\left(\left\langle |A^*|\,\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\right)
\end{aligned}
$$

(by the inequality (2.1))

$$
\begin{aligned}
& \leq \frac{1}{2}\left[\frac{1}{K_1\left(\frac{f\left(\left\langle |A|\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\right)}{f\left(\left\langle |A^*|\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\right)}, 2\right)^r}\left((1-\alpha) f\left(\left\langle |A^*|\,\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\right) + \alpha f\left(\left\langle |A|\,\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\right)\right)\right] \\
& + \frac{1}{2}\left[\frac{1}{K_1\left(\frac{f\left(\left\langle A^*\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\right)}{f\left(\left\langle ||A||\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\right)}, 2\right)^r}\left((1-\alpha) f\left(\left\langle |A|\,\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\right) + \alpha f\left(\left\langle |A^*|\,\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\right)\right)\right]
\end{aligned}
$$

(by the inequality (2.7))

$$
= \frac{1}{2}\left[\frac{1}{K_1\left(\frac{f\left(\left\langle |A|\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\right)}{f\left(\left\langle |A^*|\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\right)}, 2\right)^r}\left\langle \left(f\left(|A^*|\right) + f\left(|A|\right)\right)\widehat{k}_\eta, \widehat{k}_\eta\right\rangle\right]
$$

for all $\eta \in \Omega$. By taking supremum over $\eta \in \Omega$, we have

$$
\sup_{\eta\in\Omega} f\left(\left|\widetilde{|A|}\,(\eta)\right|\right) \leq \frac{1}{2}\left[\frac{1}{\inf_{\eta\in\Omega} \xi\,(\eta)} \sup_{\eta\in\Omega}\left(\left(\widetilde{f\left(|A|\right) + f\left(|A^*|\right)}\right)(\eta)\right)\right].
$$

which implies that

$$
f\left(\operatorname{ber}(A)\right) \leq \frac{1}{\inf_{\eta\in\Omega} \xi\,(\eta)} \frac{\left\|f\left(|A|\right) + f\left(|A^*|\right)\right\|_{\text{ber}}}{2}
$$

where $\xi\,(\eta) = K_1\left(\frac{f\left(\widetilde{|A|}(\eta)\right)}{f\left(\widetilde{|A^*|}(\eta)\right)}, 2\right)^r$.

This completes the proof. □

Remark 3.14. Letting $f\,(t) = t$ in Theorem 3.13 implies

$$
\operatorname{ber}(A) \leq \frac{1}{\inf_{\eta\in\Omega} K_1\left(\left(\frac{\widetilde{|A|}(\eta)}{\widetilde{|A^*|}(\eta)}\right)^n, 2\right)^r} \frac{\left\||A| + |A^*|\right\|_{\text{ber}}}{2}
$$

which is the result of [36, Theorem 2.7].

For more recent results concerning Berezin radius inequalities for operators and other related results, we suggest [3, 11, 12, 13, 14, 15, 30, 32, 33, 34, 35, 36].

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