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EXISTNECE RESULTS FOR THE FRACTIONAL ORDER GENERALIZED CAUCHY PROBLEM WITH NON-INSTANTNEOUS IMPULSES ON BANACH SPACE

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Abstract In this article, we present the existence of a solution of the fractional-order semi-linear classical and non-local generalized Cauchy problem with non-instantaneous impulses on the Banach space. The existence results of the problem with classical conditions are established using operator semigroup theory and generalized Banach contraction principle. The problem with non-local conditions is established through operator semigroup theory and Krasnoselkii's fixed point theorem. This article is also derived uniqueness results for the problem with classical conditions. Finally, illustrations for the Cauchy problem with the classical and nonlocal problems are added to validate derived results.

1 Introduction

In the past few decades, fractional Calculus became one of the important branches of the applied mathematics. This is because fractional-order dynamical models give much better approximations in many physical situations like seepage flow in porous media, anomalous diffusion, the nonlinear oscillations of the earthquake, traffic flow, electromagnetism, dynamics of many infectious diseases. The details of applications are found in books of [1, 2] and articles of [3, 4, 5, 6, 7, 8, 12, 9, 10, 11, 37, 38, 39, 40, 41, 42, 54, 55]. Existence theory of fractional differential equations and evolution equations of Caputo type with classical conditions using different fixed point theory is found in the articles of [13, 14, 15, 57] and non-local condition is found in the articles of [17, 18, 19, 20, 21, 22, 23, 56].

Changes in state at a fixed moment or for a small interval of time in dynamical systems are model into impulsive dynamical systems. These impulses are instantaneous or non-instantaneous depending on the time at which impulses are applied. Existence results and applications of the instantaneous integer order impulsive dynamical or evolution systems are found in [24, 25, 26, 27] while, existence results for fractional instantaneous impulsive equation are found in [28, 29, 30, 31, 32, 33, 34, 35]. In some dynamic process changes in state are applied for small-time interval time rather than a fixed moment. Existence results of fractional order impulsive dynamical systems and evolution systems with non-instantaneous impulses with local and non-local conditions are studied by [36, 43, 44]

In this article, we established sufficient conditions for the existence of the fractional order generalized Cauchy problem:

$$\begin{aligned} {}^{c}D^{\lambda}s(\varsigma) &= As(\varsigma) + f_k\left(\varsigma, s(\varsigma), \int_0^{\varsigma} a_k(\varsigma,\zeta,s(\zeta))d\zeta\right),\ \varsigma \in [\zeta_{k-1}, \varsigma_k),\ k = 1,2,\cdots,p \\ u(\varsigma) &= g_k(k, s(\varsigma)),\ \varsigma \in [\varsigma_{k-1}, \zeta_k) \end{aligned}$$

with local condition $s(0) = s_0$ and non-local condition $s(0) = s_0 + h(s)$ over the interval $[0, T]$ in a Banach space $\mathcal{U}$. Here $A : \mathcal{U} \to \mathcal{U}$ is linear operator, $P_k s = \int_0^{\varsigma} a_k(\varsigma,\zeta,s(\zeta))d\zeta$ are nonlinear Volterra integral operator on $\mathcal{U}$, $f_k : [0,T] \times \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ are nonlinear functions applied in the intervals $[\zeta_{k-1}, \varsigma_k)$ and $g_k : [0, T] \times \mathcal{U}$ are set of nonlinear functions applied in the interval $[\varsigma_k, \zeta_k)$ for all $k = 1, 2, \cdots, p$.

The structure of the article is as follows:

1. Section (2) present some basic defination realted to fractional calculus and theories related to fixed points.
2. Section (3) discusses the existence and uniqueness of mild solution for the impulsive fractional integro-differential with classical condition.
3. Section (4) discusses the sufficient condition for existence of the impulsive fractional integro-differential equation with nonlocal condition.
4. Finally conclusion is found in the section (5)

2 Preliminaries

This section discussed preliminaries about fractional differential operators and some definitions and theorem from the functional analysis.

Definition 2.1. [46] "The Liouville-Caputo fractional derivative of order $\beta > 0$, $n - 1 < \beta < n$, $n \in \mathbb{N}$, is defined as

$$^{c}D^{\lambda}_{\varsigma_0+}h(\varsigma) = \frac{1}{\Gamma(n-\lambda)} \int_{\varsigma_0}^{\varsigma} (\varsigma - q)^{n-\lambda-1} \frac{d^n h(q)}{dq^n} dq$$

where, the function $h(\varsigma)$ has absolutely continuous derivatives up to order $(n - 1)$".

Theorem 2.2. *(Banach Fixed Point Theorem)[52] "Let F be closed subset of a Banach Space $(\mathcal{X}, ||\cdot||)$ and let $T : F \to F$ contraction then, T has unique fixed point in F".*

Theorem 2.3. *(Krasnoselskii's Fixed Point Theorem)[52] "Let E be closed convex nonempty subset of a Banach Space $(\mathcal{X}, ||\cdot||)$ and P and Q are two operators on E satisfying:*

(1) $Pv + Qs \in F$, whenever $v, s \in F$,

(2) P is contraction,

(3) Q is completely continuous

then, the equation $Pv + Qv = v$ has unique solution".

Definition 2.4. (Completely Continuous Operator)[53] "Let X and Y be Banach spaces. Then the operator $T : D \subset X \to Y$ is called completely continuous if it is continuous and maps any bounded subset of D to relatively compact subset of Y".

3 Equation with Local Conditions

Thus section derived sufficient conditions for the existence and uniqueness for the following Cauchy problem:

$$\begin{aligned} &^{c}D^{\lambda}s(\varsigma) = As(\varsigma) + f_k\Big(\varsigma, s(\varsigma), \int_0^{\varsigma} a_k(\varsigma, \zeta, s(\zeta))d\zeta\Big),\ \varsigma \in [\zeta_k, \varsigma_{k+1}),\ i = 1, 2, \cdots, p \\ &\quad s(\varsigma) = g_k(\varsigma, s(\varsigma)),\ \varsigma \in [\varsigma_k, \zeta_k) \\ &\quad s(0) = s_0 \end{aligned} \tag{3.1}$$

over the interval $[0, T]$ in the Banach space $\mathcal{U}$,

Definition 3.1. The function $s(\varsigma)$ is called mild solution of the impulsive fractional equation (3.1) over the interval if $s(\varsigma)$ satisfies the integral equation

$$s(\varsigma)=\begin{cases}U(\varsigma)s_0+\displaystyle\int_0^{\varsigma}(\varsigma-\zeta)^{\lambda-1}V(\varsigma-\zeta)f_1(\varsigma,s(\zeta),P_1s(\zeta))d\zeta, & \varsigma\in[0,\varsigma_1)\\ g_k(\varsigma,s(\varsigma)), & \varsigma\in[\varsigma_k,\zeta_k)\\ U(\varsigma-\zeta_k)g_k(\zeta_k,s(\zeta_k))+\displaystyle\int_{\zeta_k}^{\varsigma}(\varsigma-\zeta)^{\lambda-1}V(\varsigma-\zeta)f_{k+1}(\varsigma,s(\zeta),P_{k+1}s(\zeta))d\zeta, & \varsigma\in[\zeta_k,\varsigma_{k+1})\end{cases}\tag{3.2}$$

where,

$$P_ks(\varsigma)=\int_0^{\varsigma}a_k(\varsigma,\zeta,s(\zeta))d\zeta,\ U(\varsigma)=\int_0^{\infty}\zeta_\lambda(\theta)S(\varsigma^\lambda\theta)d\theta,\ V(\varsigma)=\lambda\int_0^{\infty}\theta\zeta_\lambda(\theta)S(\varsigma^\lambda\theta)d\theta$$

are the linear operators defined on $\mathcal{U}$. Here, $\zeta_\lambda(\theta)$ is probability density function over the interval $[0,\infty)$ defined by

$$\zeta_\lambda(\theta)=\frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-\lambda n-1}\frac{\Gamma(n\lambda+1)}{n!}sin(n\pi\lambda)$$

and the operator $S(\varsigma)$ is semi-group generated by evolution operator A.

Assumption 3.2. Assumptions for the existence and uniqueness of the mild solution of fractional evolution equation with non-instantaneous impulses.
(A1) The evolution operator A generates C_0 semigroup $S(\varsigma)$ for all $\varsigma\in[0,T]$.
(A2) The function $f_k:[0,T]\times\mathcal{U}\times\mathcal{U}\to\mathcal{U}$ is continuous with respect to ς and there exist a positive constants f^*_{1k} and f^*_{2k} such that $||f_k(\varsigma,v_1,s_1)-f_k(\varsigma,v_2s_2)||\le f^*_{1k}||v_1-v_2||+f^*_{2k}||s_1-s_2||$ for $v_1,s_1,v_2,s_2\in B_{r_0}=\{s\in\mathcal{U};||s||\le r_0\}$ for some r_0 and for all $k=1,2,\cdots,p+1$.
(A3) The operator $P_k:[0,T]\times\mathcal{U}\to\mathcal{U}$ is continuous and there exist a constant p^*_k such that $||P_kv-P_ks||\le p^*_k||v-s||$ for $v,s\in B_{r_0}$ for all $k=1,2,\cdots,p+1$.
(A4) The functions $g_k:[\varsigma_k,\zeta_k]\times\mathcal{U}$ are continuous and there exist a positive constants $0<g^*_k<1$ such that $||g_k(\varsigma,v(\varsigma))-g_k(\varsigma,s(\varsigma))||\le g^*_k||v-s||$.

Lemma 3.3. *[13]If the evolution operator A generates C_0 semigroup $S(\varsigma)$ then the operators $U(\varsigma)$ and $V(\varsigma)$ are strongly continuous and bounded. This is, there exist positive constant M such that $||U(\varsigma)s||\le M||s||$ and $||V(\varsigma)s||\le\frac{M}{\Gamma(\lambda)}||s||$ for all $\varsigma\in[0,T]$.*

Theorem 3.4. *If assumptions (A1)-(A4) holds, then the generalized semilinear fractional integro-differential equation with non-instantaneous impulses (3.1) has unique mild solution.*

Proof. We convert the equation (3.2) into operator equation $s(\varsigma)=\mathcal{F}u(\varsigma)$ by defining the operator $\mathcal{F}$ on $\mathcal{U}$ by

$$\mathcal{F}s(\varsigma)=\begin{cases}\mathcal{F}_1s(\varsigma), & \varsigma\in[0,\varsigma_1)\\ \mathcal{F}_{2k}s(\varsigma), & \varsigma\in[\varsigma_k,\zeta_k)\\ \mathcal{F}_{3k}s(\varsigma), & \varsigma\in[\zeta_{k-1},\varsigma_k)\end{cases}$$

where, $\mathcal{F}_1$, $\mathcal{F}_{2k}$ and $\mathcal{F}_{3k}$ are

$$\mathcal{F}_1s(\varsigma)=U(\varsigma)s_0+\int_0^{\varsigma}(\varsigma-\zeta)^{\lambda-1}V(\varsigma-\zeta)f_1(\varsigma,s(\zeta),P_1s(\zeta))d\zeta,\qquad \varsigma\in[0,\varsigma_1)$$

$$\mathcal{F}_{2k}s(\varsigma)=g_k(\varsigma,s(\varsigma)),\qquad \varsigma\in[\varsigma_k,\zeta_k)$$

$$\mathcal{F}_{3k}s(\varsigma)=U(\varsigma-\zeta_k)g_k(\zeta_k,s(\zeta_k))+\int_{\zeta_k}^{\varsigma}(\varsigma-\zeta)^{\lambda-1}V(\varsigma-\zeta)f_k(\varsigma,s(\zeta),P_{k+1}s(\zeta))d\zeta,\qquad \varsigma\in[\zeta_{k-1},\varsigma_k)$$

for all $k=1,2,\cdots p$.
If the operator equations $s(\varsigma)=\mathcal{F}_1s(\varsigma)$, $s(\varsigma)=\mathcal{F}_{2k}s(\varsigma)$ and $s(\varsigma)=\mathcal{F}_{3k}s(\varsigma)$ has unique solution over the interval $[0,\varsigma_1)$, $[\varsigma_k,\zeta_k)$ and $[\zeta_k,\varsigma_{k+1})$ for all $k=1,2,\cdots,p$ respectively then there exists

$s_1(\varsigma), s_{2k}(\varsigma)$ and $s_{3k}(\varsigma)$ such that $s_1(\varsigma) = \mathcal{F}_1 s(\varsigma)$, $s_{2k}(\varsigma) = \mathcal{F}_{2k} s(\varsigma)$ and $s_{3k}(\varsigma) = \mathcal{F}_{3k} s(\varsigma)$, and if we define

$$s(\varsigma) = \begin{cases} s_1(\varsigma), & [0, \varsigma_1) \\ s_{2k}(\varsigma), & [\varsigma_k, \zeta_k) \\ s_{3k}(\varsigma), & [\zeta_k, \varsigma_{k+1}) \end{cases}$$

then it is solution of operator equation $s(\varsigma) = \mathcal{F}s(\varsigma)$.
For all $\varsigma \in [0, \varsigma_1)$ and $u, v \in B_{r_0}$, and assuming (A1), (A2) and (A3) and using lemma-(3.3),

$$||\mathcal{F}_1^{(n)} v(\varsigma) - \mathcal{F}_1^{(n)} s(\varsigma)|| \leq \frac{\varsigma_1^{n\lambda} M^n (f_{11}^* + f_{21}^* p_1^*)}{n!(\Gamma(\lambda))^n} ||v - s|| \leq c^* ||v - s||.$$

If considering $\varsigma \to \infty$ over the interval $[0, \varsigma_1)$, $||\mathcal{F}_1^{(n)} v - \mathcal{F}_1^{(n)} s|| \leq c^* ||v - s|| \to 0$ for fixed ς_1. then there exist m such that $\mathcal{F}_1^{(m)}$ is contraction on B_{r_0} and by Banch fixed point theorem the equation $s(\varsigma) = \mathcal{F}_1 s(\varsigma)$ has unique solution over the interval $[0, \varsigma_1)$.

By assuming (A4) and for all $k = 1, 2, \cdots, p$, $\varsigma \in [\varsigma_k, \zeta_k)$ and $v, s \in \mathcal{U}$.

$$||\mathcal{F}_{2k} v(\varsigma) - \mathcal{F}_{2k} s(\varsigma)|| = ||g_k(\varsigma, v(\varsigma)) - g_k(\varsigma, s(\varsigma))|| \leq g_k^* ||v - s||.$$

This implies, $\mathcal{F}_{2k}$ is contraction and by theorem (2.2) the equation $s(\varsigma) = \mathcal{F}_{2k} s(\varsigma)$ has unique solution for the interval $[\varsigma_k, \zeta_k)$ for all $k = 1, 2, \cdots, p$. That is for all $k = 1, 2, \cdots, p$, $s(\varsigma) = g_k(\varsigma, s(\varsigma))$ has unique solution for all $\varsigma \in [\varsigma_k, \zeta_k)$. Since, g_k is contraction therefore, it leads to uniqueness of the solution at point ζ_k.

For all $k = 1, 2, \cdots, p$, $\varsigma \in [\zeta_{k-1}, \varsigma_k)$ and $v, s \in B_{r_0}$ and by assumptions (A1), (A2) and (A3) and lemma (3.3),

$$||\mathcal{F}_{3k}^{(n)} v(\varsigma) - \mathcal{F}_{3k}^{(n)} s(\varsigma)|| \leq \frac{(\varsigma_{k+1} - \zeta_k)^{n\lambda} M^n (f_{1k}^* + f_{2k}^* p_k^*)}{n!(\Gamma(\lambda))^n} ||v - s|| \leq c^* ||v - s||.$$

Clearly, $||\mathcal{F}_{3k}^{(n)} v - \mathcal{F}_{3k}^{(n)} s| \leq c^* ||v - s|| \to 0$ as $n \to \infty$ over interval $[\zeta_{k-1}, \varsigma_k)$ for all $k = 1, 2, \cdots, p+1$. Therefore there exist m such that $\mathcal{F}_{3k}^{(m)}$ is contraction on B_{r_0}. Thus by general Banach contraction theorem the operator equation $s(\varsigma) = \mathcal{F}_{3k} s(\varsigma)$ has unique solution over the interval $[\zeta_{k-1}, \varsigma_k)$ for all $k = 1, 2, \cdots, p+1$.
Hence, the operator equation $s(\varsigma) = \mathcal{F}s(\varsigma)$ has unique solution over the interval $[0, T]$ which is mild solution of the equation (3.1). □

Example 3.5. The fractional order integro-differential equation:

$$\begin{aligned} {}^cD_\varsigma^\lambda s(\varsigma, \eta) &= s_{\eta\eta}(\varsigma, \eta) + s(\varsigma, \eta) s_\eta(\varsigma, \eta) + \int_0^\varsigma e^{-s(\zeta, \eta)} d\zeta, && \varsigma \in [0, \frac{1}{3}) \\ {}^cD_\varsigma^\lambda s(\varsigma, \eta) &= s_{\eta\eta}(\varsigma, \eta) + s(\varsigma, \eta) s_\eta(\varsigma, \eta) && \varsigma \in [\frac{2}{3}, 1] \\ s(\varsigma, \eta) &= \frac{s(\varsigma, \eta)}{2(1 + s(\varsigma, \eta))}, && \varsigma \in [\frac{1}{3}, \frac{2}{3}) \end{aligned} \tag{3.3}$$

over the interval $[0, 1]$ with initial condition $s(0, \eta) = s_0(\eta)$ and boundary conditions $s(\varsigma, 0) = s(\varsigma, 1) = 0$. The equation (3.3) can be reformulated as fractional order abstract equation in $\mathcal{U} = L^2([0, 1], \mathbb{R})$ as:

$$\begin{aligned} {}^cD^\lambda z(\varsigma) &= Az(\varsigma) + f_k(\varsigma, z(\varsigma), T_k z(\varsigma)), && \varsigma \in [0, \frac{1}{3}) \cup [\frac{2}{3}, 1] \\ z(\varsigma) &= g(\varsigma, z(\varsigma)) && \varsigma \in [\frac{1}{3}, \frac{2}{3}) \end{aligned} \tag{3.4}$$

over the interval $[0, 1]$ by defining $z(\varsigma) = u(\varsigma, \cdot)$, operator $Au = u''$ (second order derivative with respect to η).
The functions f_1, f_2 and g over respected domains are defined as $f_1(\varsigma, z(\varsigma), P_1 z(\varsigma)) = \frac{(z^2(\varsigma))'}{2} +$

$\int_0^\varsigma e^{-z(\zeta)}d\zeta$, $f_2(\varsigma, z(\varsigma), P_2z(\varsigma)) = \frac{(z^2(\varsigma))'}{2}$ and $g(\varsigma, z(\varsigma)) = \frac{z(\varsigma)}{2(1+z(\varsigma))}$ respectively.
(1) The linear operator A over the domain $D(A) = \{s \in \mathcal{U}; s''$ exist and continuous with $s(0) = s(1) = 0\}$ is self-adjoint, with compact resolvent and is the infinitesimal generator of C_0 semi-group $S(\varsigma)$ over the interval $[0,1]$ given by$S(\varsigma)u = \sum_{n=1}^{\infty} exp(-n^2\pi^2\varsigma) < s, \phi_n > \phi_n$ and $\phi_n(\zeta) = \sqrt{2}sin(n\pi\zeta)$ for all $n = 1, 2, \cdots$ is the orthogonal basis for the space $\mathcal{U}$.
(2) The function $T_1, T_2 : [0,1]\times[0,1]\times\mathcal{U} \to \mathcal{U}$ are continuous with respect to ς and differentiable with respect to z for all z and hence P_1, P_2 are Lipschitz continuous with respect to z. This means there exist positive constant h_1^* and $h_2^* = 0$ such that $||P_k(\varsigma, z_1) - P_k(\varsigma, z_2)|| \le p_k^*||z_1 - z_2||$ for $k = 1, 2$.
(3) The function $f_1, f_2 : [0,1] \times \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ are continuous with respect to ς and is differential with respect to argument z, P_1z and P_2z. Therefore there exist positive constants f_{11}^* and f_{2k}^* such that $||f_k(\varsigma, z_1, P_kz_1) - f_k(\varsigma, z_2, P_kz_2)|| \le f_{1k}^*||z_1 - z_2|| + f_{2k}^*||P_kz_1 - P_kz_2||$, $z_1, z_2 \in B_{r_0}$ for some r_0 and $k = 1, 2$.
(4) The impulse g is continuous with respect to ς and Lipchitz continuous with respect to z with Lipschitz constant $g^* = \frac{1}{2} < 1$.
Thus, by theorem-(3.4) the integro-differential equation (3.4) has unique solution over $[0,1]$. Hence, the equation (3.3) has unique solution over the interval $[0,1]$.

4 Equation with Non-local Conditions

This section derived sufficient conditions for the existence of solution of following non-local Cauchy Problem

$$\begin{aligned} &{}^cD^\lambda s(\varsigma) = As(\varsigma) + f_{k+1}\left(\varsigma, s(\varsigma), \int_0^\varsigma a_{k+1}(\varsigma, \zeta, s(\zeta))d\zeta\right),\ \varsigma \in [\zeta_i, \varsigma_{i+1}),\ i = 1, 2, \cdots, p \\ &s(\varsigma) = g_i(\varsigma, s(\varsigma)),\ \varsigma \in [\varsigma_i, \zeta_i) \\ &s(0) = s_0 + h(s) \end{aligned} \tag{4.1}$$

in the Banach space $\mathcal{U}$.

Definition 4.1. The function $s(\varsigma)$ is called mild solution of the impulsive fractional integro-differentia equation (3.1) over the interval $[0, T]$ if, $s(\varsigma)$ satisfies the integral equation

$$s(\varsigma) = \begin{cases} U(\varsigma)(s_0 + h(s)) + \int_0^\varsigma (\varsigma-\zeta)^{\lambda-1}V(\varsigma-\zeta)f(\varsigma, s(\zeta), P_1s(\zeta))d\zeta, & \varsigma \in [0, \varsigma_1) \\ g_k(\varsigma, s(\varsigma)), & \varsigma \in [\varsigma_k, \zeta_k) \\ U(\varsigma-\zeta_k)g_k(\zeta_k, s(\zeta_k)) + \int_{\zeta_k}^\varsigma (\varsigma-\zeta)^{\lambda-1}V(\varsigma-\zeta)f_{k+1}(\varsigma, s(\zeta), P_{k+1}s(\zeta))d\zeta & \varsigma \in [\zeta_k, \varsigma_{k+1}) \end{cases} \tag{4.2}$$

where,

$$P_ks(\varsigma) = \int_0^\varsigma a_k(\varsigma, \zeta, s(\zeta))d\zeta,\ U(\varsigma) = \int_0^\infty \zeta_\lambda(\theta)S(\varsigma^\lambda\theta)d\theta,\ V(\varsigma) = \lambda\int_0^\infty \theta\zeta_\lambda(\theta)S(\varsigma^\lambda\theta)d\theta$$

are the linear operators defined on $\mathcal{U}$. Here, $\zeta_\lambda(\theta)$ is probability density function over the interval $[0, \infty)$ defined by

$$\zeta_\lambda(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-\lambda n-1}\frac{\Gamma(n\lambda+1)}{n!}sin(n\pi\lambda)$$

and the operator $S(\varsigma)$ is semi-group generated by evolution operator A.

Assumption 4.2. Assumptions for the existence of the mild solution of fractional evolution equation with non-instantaneous impulses.
(B1) The evolution operator A generates C_0 semigroup $S(\varsigma)$ for all $\varsigma \in [0, T]$.
(B2) The function $f_k(\varsigma, \cdot, \cdot)$ are continuous and $f_k(\cdot, u, v)$ is measurable on $[0, T]$. Also there

exist $\beta \in (0, \lambda)$ with $m_f \in L^{\frac{1}{\beta}}([0,T], \mathcal{R})$ such that $|f_k(\varsigma, v, s)| \leq m_f k(\varsigma)$ for all $u, v \in \mathcal{U}$. Also $m_f(\varsigma) = sup_{\varsigma \in [0,T]}\{f_{kt}(\varsigma); k = 1, 2, \cdots, k+1\}$ with $M_1 = ||m_f||_L^{1/\beta}$.
(B3) The operator $P_k : [0,T] \times \mathcal{U} \to \mathcal{U}$ are continuous and there exist a constants h_k^* such that $||P_k v - P_k s|| \leq p_k^* ||v - s||$ and let p^* be maximum of p_k^* for all $k = 1, 2, \cdots, k+1$.
(B4) The operator $h : \mathcal{U} \to \mathcal{U}$ is Lipschiz continuous with respect to u with Lipschitz constant $0 < h^* \leq 1$.
(B5) The functions $g_k : [\varsigma_k, \zeta_k) \times \mathcal{U}$ are Lipchitz continuous positive constants $0 < g_k^* < 1$ such that $||g_k(\varsigma, v(\varsigma)) - g_k(\varsigma, s(\varsigma))|| \leq g_k^* ||v - s||$.

Theorem 4.3. *(Existence Theorem) Under the assumptions (B1)-(B5), the nonlocal semi-linear fractional order integro-differential equation* (4.2) *has mild solution provided* $Mh^* < 1$ *and* $Mg^* < 1$.

Proof. From the lemma-(3.3) $||U(\varsigma)|| \leq M$ for all $s \in B_k = \{s \in \mathcal{U} : ||s|| \leq k\}$ for any positive constant k. Therefore,

$$|U(\varsigma)(s_0 + h(s))| \leq M(|s_0| + h^*||s|| + |h(0)|). \tag{4.3}$$

According to (B2) $f(\cdot, v, s)$ is measurable on $[0,T]$ and one can easily shows that $(\varsigma - \zeta)^{\lambda-1} \in L^{\frac{1}{1-\beta}}[0, \varsigma]$ for all $\varsigma \in [0,T]$ and $\beta \in (0, \lambda)$. Let $b = \frac{\lambda-1}{1-\beta} \in (-1, 0)$, $M_1 = ||m_f||_{L^{\frac{1}{\beta}}}$. By Holder's inequality and assumption (B2), for $\varsigma \in [0,T]$,

$$\begin{aligned}\int_0^\varsigma |(\varsigma - \zeta)^{\lambda-1} V(\varsigma - \zeta) f_k(\zeta, s(\zeta), P_k s(\zeta))| d\zeta \quad &\leq \quad \frac{M}{\Gamma(\lambda)} \left(\int_0^\varsigma (\varsigma - \zeta)^{\frac{\lambda-1}{1-\beta}} d\zeta \right)^{1-\beta} \\ M_1 \leq \frac{M M_1}{\Gamma(\lambda)(1+b)^{1-\beta}} T^{(1+b)(1-\beta)}.\end{aligned} \tag{4.4}$$

For $\varsigma \in [0, \varsigma_1)$ and for positive r we define F_1 and F_2 on B_r as, $F_1 s(\varsigma) = U(\varsigma)(s_0 + s(u))$ and $F_2 s(\varsigma) = \int_0^\varsigma (\varsigma - \zeta)^{\lambda-1} V(\varsigma - \zeta) f_1(\varsigma, s(\zeta), P_1 s(\zeta)) d\zeta$ respectively then, $s(\varsigma)$ is mild solution of the semilinear fractional integro-differential equation if and only if the operator equation $s = F_1 s + F_2 s$ has solution for $s \in B_r$ for some r. Therefore the existence of a mild solution of (3.1) over the interval $[0, \varsigma_1)$ is equivalent to determining a positive constant r_0, such that $F_1 + F_2$ has a fixed point on B_{r_0}.
Step:1 $||F_1 v + F_2 s|| \leq r_0$ for some positive r_0.
Let $v, s \in B_{r_0}$, and choose

$$r_0 = M \frac{|s_0| + |h(z)|}{1 - Mh^*} + \frac{M M_1}{(1 - Mh^*)\Gamma(\lambda)(1+b)^{1-\beta}} \varsigma_1^{(1+b)(1-\beta)},$$

and using inequalities (4.3) and (4.4)

$$|F_1 v(\varsigma) + F_2 s(\varsigma)| \leq M(|s_0| + h^*||v|| + |h(0)|) + \frac{M M_1}{\Gamma(\lambda)(1+b)^{1-\beta}} \varsigma_1^{(1+b)(1-\beta)} \leq r_0 \quad (\text{since, } Mh^* < 1)$$

Therefore, $||F_1 v + F_2 s|| \leq r_0$ for every pairs $v, s \in B_{r_0}$.
Step: 2 F_1 is contraction on B_{r_0}.
For any $v, s \in B_{r_0}$ and $\varsigma \in [0, \varsigma_1)$, we have $|F_1 v(\varsigma) - F_1 s(\varsigma)| \leq Mh^*||v - s||$. Taking supremum over $[0, \varsigma_1)$, $||F_1 v - F_1 s|| \leq Mh^*||v - s||$. Since, $Mh^* < 1$, F_1 is contraction.
Step: 3 F_2 is completely continuous operator on B_{r_0}.
Let $\{s_n\}$ be the sequence in B_{r_0} converging to $s \in B_{r_0}$ then,

$$\begin{aligned}|F_2 s_n(\varsigma) - F_2 s(\varsigma)| &\leq \int_0^\varsigma (\varsigma - \zeta)^{\lambda-1} |V(\varsigma - \zeta)||f_1(\zeta, s_n(\zeta), P_1 s_n(\zeta)) - f_1(\zeta, s(\zeta), P_1 s(\zeta))| d\zeta \\ &\leq \frac{M t_1^\lambda}{\Gamma(\lambda+1)} \sup_{\zeta \in [0, \varsigma_1)} |f_1(\zeta, s_n(\zeta), P_1 s_n(\zeta)) - f_1(\zeta, s(\zeta), P_1 s(\zeta))|\end{aligned}$$

Continuity of f and K leads to $||F_2 s_n - F_2 s|| \to 0$ as $n \to \infty$. Thus, F_2 is continuous.
To show $\{F_2 s(\varsigma), s \in B_{r_0}\}$ is relatively compact it is sufficient to show that the family of

functions $\{F_2 s, s \in B_{r_0}\}$ is uniformly bounded and equi-continuous, and for any $\varsigma \in [0,\varsigma_1)$, $\{F_2 s(\varsigma), s \in B_{r_0}\}$ is relatively compact in $\mathcal{U}$.
Clearly for any $s \in B_{r_0}$, $||F_2 s|| \leq r_0$, which means that the family $\{F_2 s(\varsigma), u \in B_{r_0}\}$ is uniformly bounded.

For any $u \in B_{r_0}$ and $0 \leq \tau_1 < \tau_2 < \varsigma_1$,

$$
\begin{aligned}
&|F_2 s(\tau_2) - F_2 s(\tau_1)| \\
&\leq \left| \int_{\tau_1}^{\tau_2} (\tau_2 - \zeta)^{\lambda-1} V(\tau_2 - \zeta) f_1(\zeta, s(\zeta), P_1 s(\zeta)) d\zeta \right| + \left| \int_0^{\tau_1} \left[(\tau_2 - \zeta)^{\lambda-1} - (\tau_1 - \zeta)^{\lambda-1}\right] V(\tau_2 - \zeta) \right. \\
&\quad \left. f_1(\zeta, s(\zeta), P_1 s(\zeta)) d\zeta \right| + \left| \int_0^{\tau_1} (\tau_1 - \zeta)^{\lambda-1} \left[V(\tau_2 - \zeta) - V(\tau_1 - \zeta)\right] f(\zeta, s(\zeta), Kw(\zeta)) d\zeta \right| \\
&\leq I_1 + I_2 + I_3
\end{aligned}
$$

and assuming (B1), (B2), (B3) and Holder inequality the integrals $I_1 \leq \frac{MM_1}{\Gamma(\lambda)(1+b)^{1-\beta}}(\tau_2 - \tau_1)^{(1+b)(1-\beta)}$, $I_2 \leq \frac{MM_1}{\Gamma(\lambda)(1+b)^{1-\beta}}(\tau_2 - \tau_1)^{(1+b)(1-\beta)}$ and $I_3 \leq \frac{M_1}{(1+b)^{1-\beta}} \varsigma^{(1+b)(1-\beta)} \sup_{\zeta \in [\tau_1, \tau_2]} |V(\tau_2 - \zeta) - V(\tau_1 - \zeta)|$.
The integrals I_1 and I_2 are vanishes if $\tau_1 \to \tau_2$ as they contain term $(\tau_2 - \tau_1)$. By assumption (B1), the integral I_3 also vanishes as $\tau_1 \to \tau_2$. Therefore $|F_2 s(\tau_2) - F_2 s(\tau_1)|$ tends to zero as $\tau_1 \to \tau_2$ for independent choice of $s \in B_{r_0}$. Hence, the family $\{F_2 s, s \in B_{r_0}\}$ is equicontinuous. Consider $\mathcal{X}(\varsigma) = \{F_2 s(\varsigma), s \in B_{r_0}\}$ for all $\varsigma \in [0, \varsigma_1)$ It is obvious that $\mathcal{X}(0)$ is relatively compact.
Let $\varsigma \in [0, \varsigma_1)$ be fixed and for each $\epsilon \in [0, \varsigma_1)$, define an operator F_ϵ on B_{r_0} by formula $F_\epsilon s(\varsigma) = \int_0^{\varsigma-\epsilon} (\varsigma - \zeta)^{\lambda-1} V(\varsigma - \zeta) f_1(\varsigma, s(\zeta), P_1 s(\zeta)) d\zeta$. Compactness of the operator $V(\varsigma)$ leads to relative compactness of the set $X_\epsilon(\varsigma) = F_\epsilon s(\varsigma), s \in B_{r_0}$ in $\mathcal{U}$.
Moreover, from inequality (4.4)

$$
|F_2 s(\varsigma) - F_\epsilon s(\varsigma)| \leq \int_\epsilon^\varsigma |(\varsigma - \zeta)^{\lambda-1} V(\varsigma - \zeta) f_1(\varsigma, s(\zeta), P_1 s(\zeta))| d\zeta \leq \frac{MM_1}{\Gamma(\lambda)(1+b)^{1-\beta}} (\varsigma - \epsilon)^{(1+b)(1-\beta)}
$$

Therefore, $\mathcal{X}(\varsigma)$ is relatively compact as it is very closed to relatively compact set $X_\epsilon(\varsigma)$. Thus, by Ascoli-Arzela theorem the operator F_2 is completely continuous on B_{r_0}. Hence,by theorem (2.3), $F_1 + F_2$ has fixed point on B_{r_0} which is mild solution of the equation (4.1) over the interval $[0, \varsigma_1)$.

On the interval $[\varsigma_k, \zeta_k)$ for all $k = 1, 2, \cdots, p$ and for positive r we define F_1 and F_2 on B_r as, $F_1 s(\varsigma) = g_k(\varsigma, s(\varsigma))$ and $F_2 s(\varsigma) = 0$ then, $s(\varsigma)$ is mild solution of the semilinear fractional integro-differential equation if and only if the operator equation $s = F_1 s + F_2 s$ has solution for $u \in B_r$ for some r. Therefore the existence of a mild solution of (3.1) over the interval $[\varsigma_k, \zeta_k)$ is equivalent to determining a positive constant r_0, such that $F_1 + F_2$ has a fixed point on B_{r_0}. In fact, it is obvious due to assumption (B5).

On the interval $[\zeta_k, \varsigma_{k+1})$ for all $k = 1, 2, \cdots, p$ and for positive r we define F_1 and F_2 on B_r as, $F_1 s(\varsigma) = U(\varsigma - \zeta_k) g_k(\zeta_k, s(\zeta_k))$ and $F_2 s(\varsigma) = \int_{\zeta_k}^{\varsigma} (\varsigma - \zeta)^{\lambda-1} V(\varsigma - \zeta) f_{k+1}(\varsigma, s(\zeta), P_{k+1} s(\zeta)) d\zeta$ respectively then, $s(\varsigma)$ is mild solution of the semi-linear fractional integro-differential equation if and only if the operator equation $s = F_1 s + F_2 s$ has solution for $u \in B_r$ for some r. Therefore the existence of a mild solution of (3.1) over the interval $[\zeta_k, \varsigma_{k+1})$ is equivalent to determining a positive constant r_0, such that $F_1 + F_2$ has a fixed point on B_{r_0}.
Selecting,

$$
r_0 = M \frac{|s_0| + |g(\cdot, z)|}{1 - Mg^*} + \frac{MM_1}{(1 - Mg^*)\Gamma(\lambda)(1+b)^{1-\beta}} (\varsigma - \zeta_k)^{(1+b)(1-\beta)}
$$

and using similar arguments for interval $[0, \varsigma_1)$ and by theorem (2.3), $F_1 + F_2$ has fixed point on B_{r_0} which is mild solution of the equation (4.1) over the interval $[\zeta_k, \varsigma_{k+1})$. $\square$

Example 4.4. Fractional partial integro-differential system with nonlocal conditions:

$$\begin{aligned}
{}^cD^{\frac{1}{2}}s(\varsigma,\eta) &= s_{\eta\eta}(\varsigma,\eta) + \frac{1}{50}\int_0^{\varsigma} e^{-s(\zeta,\eta)}d\zeta, && \varsigma\in[0,\frac{1}{3}) \\
{}^cD^{\frac{1}{2}}s(\varsigma,\eta) &= s_{\eta\eta}(\varsigma,\eta) + \frac{1}{60}\int_0^{\varsigma} e^{-s(\zeta,\eta)}d\zeta, && \varsigma\in[\frac{2}{3},1] \\
s(\varsigma,\eta) &= \frac{s(\varsigma,\eta)}{10(1+s(\varsigma,\eta))}, && \varsigma\in[\frac{1}{3},\frac{2}{3})
\end{aligned} \tag{4.5}$$

over the interval $[0,1]$ with initial condition $s(0,\eta) = s_0(\eta) + \sum_{i=1}^{2}\frac{1}{3^i}s(1/i,\eta)$ and boundary conditions $s(\varsigma,0) = s(\varsigma,1) = 0$.
The equation (4.5) can be reformulated as fractional order abstract equation in $\mathcal{U} = L^2([0,1],\mathbb{R})$ as:

$$\begin{aligned}
{}^cD^{\lambda}z(\varsigma) &= Az(\varsigma) + f_k(\varsigma,z(\varsigma),P_kz(\varsigma)), && \varsigma\in[0,\frac{1}{3})\cup[\frac{2}{3},1] \\
z(\varsigma) &= g(\varsigma,z(\varsigma)) && \varsigma\in[\frac{1}{3},\frac{2}{3})
\end{aligned} \tag{4.6}$$

over the interval $[0,1]$ by defining $z(\varsigma) = s(\varsigma,\cdot)$, operator $As = s''$ (second order derivative with respect to η).
The functions f_1, f_2 and g over respected domains are defined as $f_1(\varsigma,z(\varsigma),P_1z(\varsigma)) = \frac{1}{50}\int_0^{\varsigma} e^{-z(\zeta)}d\zeta$, $f_2(\varsigma,z(\varsigma),P_2z(\varsigma)) = \frac{1}{60}\int_0^{\varsigma} e^{-z(\zeta)}d\zeta$ and $g(\varsigma,z(\varsigma)) = \frac{z(\varsigma)}{10(1+z(\varsigma))}$ respectively.
The equation (4.6) satisfies the conditions (B1-B5) of the hypothesis with $Mh^* < 1$ and $Mg^* < 1$. Hence the equation (4.6) has a mild solution over the interval $[0,1]$.

5 Conclusion

In this work, we have derived sufficient conditions for the existence of generalized non-instantaneous semilinear Cauchy problem. Sufficient conditions for the existence of mild solution for generalized non-instantaneous semilinear fractional evolution Cauchy problem with classical and non-local conditions derived using Banach fixed point theorem and Krasnoselskii's fixed point theorem respectively are weaker conditions.

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