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On Coupon Coloring of Some Cayley Graphs

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Abstract A k-coupon coloring of a graph G without isolated vertices is an assignment of colors from $[k] = \{1, 2, \dots, k\}$ to the vertices of G such that the neighborhood of every vertex of G contains vertices of all colors from $[k]$. The maximum k for which a k-coupon coloring exists is called the coupon coloring number of G. The Cayley graph $Cay(G, C)$ of a group G is a graph with vertex set G and edge set $E(Cay(G, C)) = \{gh : hg^{-1} \in C\}$, where C is a subset of G that is closed under taking inverses and does not contain the identity. For a commutative ring R with unity, $Cay(R^+, Z(R)^*)$ is denoted by $\mathbb{CAY}(R)$, where R^+ is the additive group and $Z(R)^*$ is the nonzero zero-divisors of R. In this paper, we have obtained bounds for the coupon coloring number of $\mathbb{CAY}(\mathbb{Z}_n)$ and $\mathbb{CAY}(\mathbb{Z}_n \times \mathbb{Z}_m)$, where $\mathbb{Z}_n$ is the commutative ring of integers modulo n and $\mathbb{Z}_n \times \mathbb{Z}_m$ is the Cartesian product of $\mathbb{Z}_n$ and $\mathbb{Z}_m$. We also found that in some cases these upper bounds are sharp. We have found the coupon coloring number of $Cay(\mathbb{Z}_\mathrm{n}, C)$ when $C = \{1, -1, a = -a\}$ and $C = \{1, -1, 2, -2\}$.

1 Introduction

The concept of coupon coloring number was introduced by Chen et al. in [5]. Let $G = (V, E)$ be a graph. The open neighborhood of a vertex v is $\{u \in V(G) : uv \in E(G)\}$. Let G be a graph without isolated vertices. A k-coupon coloring of G is an assignment of colors from $[k] = \{1, 2, \dots, k\}$ to the vertices of G such that the open neighborhood of every vertex of G contains vertices of all colors from $[k]$. The maximum k for which a k-coupon coloring exists is called the coupon coloring number of G and it is denoted by $\chi_c(G)$.

Let $G = (V, E)$ be a graph. $D \subseteq V$ is a dominating set if every vertex in $V \setminus D$ is adjacent to at least one vertex in D. Let $G = (V, E)$ be a graph without isolated vertices. $D' \subseteq V$ is a total dominating set (TDS) if every vertex of G is adjacent to at least one vertex in D'. The minimum cardinality among all the total dominating sets in G is called the total domination number, $\gamma_t(G)$. The coupon coloring number is also referred to as the total domatic number, introduced in [3], which is the maximum number of disjoint total dominating sets. Coupon coloring is studied in [6, 7, 8]. In [8] Y Shi et al. determined coupon coloring number of complete graphs, complete k-partite graphs, wheels, cycles, unicyclic graphs and bicyclic graphs.

Coupon coloring is interesting, not only because of its theoretical value, but also for its applications in the network science and some other related fields. Imagine the colors as different types of coupons. Then the coupon coloring demands that every vertex collect coupons of all different types from its neighbors. If we imagine that a bit from a k-bit message is assigned to the users $v_1, v_2, \dots, v_m$ and that every user has contact with at least one other user, then every user can reconstruct the entire message from her contacts if and only if the graph of contacts has a k-coupon coloring. The coupon coloring number of the graph of contacts determines the maximum length of the message that can be transmitted. In addition, results on coupon colorings have concrete applications in network science [2].

2 Preliminaries

All graphs considered in this paper are simple, finite and undirected. As usual K_n denotes the complete graph with n vertices. For vertices x and y of a graph G, we define distance $d(x, y)$ to be the length of a shortest path from x to y. The minimum and maximum degrees of vertices in a graph G are denoted by $\delta(G)$ and $\Delta(G)$ respectively. The diameter of a graph G is $diam(G) = \sup\{d(x, y) : x$ and y are vertices of $G\}$. Let G be a graph without isolated vertices. A k-vertex coloring, or simply a k-coloring of G is a mapping c from the vertex set of G to $[k] = \{1, 2, ..., k\}$. A vertex v is said to be a *bad vertex* in a k-coloring c, if its neighborhood does not contain vertices of all colors from $[k]$ and obviously, there are no bad vertices in a coupon coloring. Clearly, coupon coloring is an improper coloring and $\chi_c(G) \leq \delta(G)$.

Let G be a group and let C be a subset of G that is closed under taking inverses and does not contain the identity. Then the Cayley graph $Cay(G, C)$ is a graph with vertex set G and edge set

$$E(Cay(G, C)) = \{gh : hg^{-1} \in C\}.$$

Let $\mathbb{Z}_n$ denote the additive group of integers modulo n. If C is a subset of $\mathbb{Z}_n \backslash \{0\}$, then construct a directed graph $Cay(\mathbb{Z}_n, C)$ as follows. The vertices of $Cay(\mathbb{Z}_n, C)$ are elements of $\mathbb{Z}_n$ and (i, j) is an arc of $Cay(\mathbb{Z}_n, C)$ if and only if $j - i \in C$. The graph $Cay(\mathbb{Z}_n, C)$ is called a *circulant graph* of order n, and C is called its connection set. If the set C is symmetric, that is $C = -C = \{-x : x \in C\}$, then X will be an undirected graph. Let R be a commutative ring with unity, $Z(R)$ be the set of zero-divisors of R and $Z(R)^*$ be the set of nonzero zero-divisors of R. Then the Cayley graph of R with respect to its nonzero zero-divisors is the graph $Cay(R^+, Z(R)^*)$ denoted by $\mathbb{CAY}(R)$. This is the Cayley graph whose vertices are all elements of the additive group R^+ and in which two distinct vertices x and y are joined by an edge if and only if $x - y \in Z(R)^*$.

Let $\phi(n)$ denote Euler's phi-function. The Cartesian product $R_1 \times R_2$ of two commutative rings R_1 and R_2 is also a commutative ring defined on the set consisting of all ordered pairs (a, b) for which $a \in R_1$ and $b \in R_2$ with respect to componentwise operations.

The following results will be useful for the upcoming sections.

Theorem 2.1. *[8]*

(i) Let G be a complete graph with n vertices. Then $\chi_c(G) = \lfloor \frac{n}{2} \rfloor$.

(ii) Let $G = K_{n_1,n_2,...,n_k}$ be a complete k-partite graph where $k \geq 3$ and $n_1 \leq n_2 \leq \cdots \leq n_k$ such that $s = \sum_{i=1}^{k-1} n_i$ and $n = \sum_{i=1}^{k} n_i$. Then

$$\chi_c(G) = \begin{cases} \lfloor \frac{n}{2} \rfloor & \text{if } s \geq \frac{n}{2}, \\ s & \text{otherwise.} \end{cases}$$

Theorem 2.2. *[1] Let R be a ring. Then the following statements hold:*

(i) $\mathbb{CAY}(R)$ has no edge if and only if R is an integral domain.

(ii) If (R, M) is an Artinian local ring, then $\mathbb{CAY}(R)$ is a disjoint union of $|\frac{R}{M}|$ copies of the complete graph $K_{|M|}$.

(iii) $\mathbb{CAY}(R)$ cannot be a complete graph.

(iv) $\mathbb{CAY}(R)$ is a regular graph of degree $|Z(R)| - 1$ with isomorphic components.

3 Coupon coloring of $\mathbb{CAY}(\mathbb{Z}_n)$

By Theorem 2.2, $\mathbb{CAY}(R)$ has no edge if and only if R is an integral domain. So in this section consider only $\mathbb{Z}_n$ with n composite.

Theorem 3.1. *If $Z(\mathbb{Z}_n)$ is an ideal of $\mathbb{Z}_n$, $\chi_c(\mathbb{CAY}(\mathbb{Z}_n)) = \left\lfloor \frac{|Z(\mathbb{Z}_n)|}{2} \right\rfloor$.*

Proof. Let $Z(\mathbb{Z}_n)$ be an ideal of $\mathbb{Z}_n$. Then $n = p^k$ for some prime p and $Z(\mathbb{Z}_n)$ is the maximal ideal of $\mathbb{Z}_n$. By Theorem 2.2, $\mathbb{CAY}(\mathbb{Z}_n)$ is the disjoint union of $\left|\frac{\mathbb{Z}_n}{Z(\mathbb{Z}_n)}\right| = p$ copies of the complete graph $K_{|Z(\mathbb{Z}_n)|} = K_{p^{k-1}}$ since, $|Z(\mathbb{Z}_n)| = p^{k-1}$. Therefore, using Theorem 2.1

$$\begin{aligned}\chi_c(\mathbb{CAY}(\mathbb{Z}_n)) &= \chi_c(K_{p^{k-1}}) \\ &= \left\lfloor \frac{p^{k-1}}{2} \right\rfloor \\ &= \left\lfloor \frac{|Z(\mathbb{Z}_n)|}{2} \right\rfloor.\end{aligned}$$

The proof is complete. □

The above theorem gives the coupon coloring number of $\mathbb{CAY}(\mathbb{Z}_n)$ when $Z(\mathbb{Z}_n)$ is an ideal of $\mathbb{Z}_n$. Next we consider the case that $Z(\mathbb{Z}_n)$ is not an ideal of $\mathbb{Z}_n$. We found the exact value of the coupon coloring number of $\mathbb{CAY}(\mathbb{Z}_n)$ in some cases when $Z(\mathbb{Z}_n)$ is not an ideal of $\mathbb{Z}_n$.

Theorem 3.2. *Suppose that $Z(\mathbb{Z}_n)$ is not an ideal of $\mathbb{Z}_n$. Then $\chi_c(\mathbb{CAY}(\mathbb{Z}_n)) \geq \frac{n}{p}$, where p is the least prime divisor of n.*

Proof. $Z(\mathbb{Z}_n)$ is not an ideal of $\mathbb{Z}_n$, so there exists at least two prime divisors for n. Let $n = p_1^{r_1} p_2^{r_2} \dots p_m^{r_m}$, where $p_1 < p_2 < \cdots < p_m$ and $H = \{0, p_2, 2p_2, \dots, (p_1 - 1)p_2\}$. Define the set $D_i = ip_1 + H, i = 1, 2, \dots, \frac{n}{p_1}$. Clearly, $D_{\frac{n}{p_1}} = H$. D_i's are disjoint, for, let $x \in D_i \cap D_j, i \neq j$. Then $x = ip_1 + h_1 = jp_1 + h_2$, $h_1, h_2 \in H$. So, $ip_1 + sp_2 = jp_1 + tp_2$ where $i, j \in \{1, 2, \dots, \frac{n}{p_1}\}$ and $s, t \in \{1, 2, \dots, p_1 - 1\}$. Since $i \neq j$, $t \neq s$ and p_1 divides $t - s$, a contradiction. Hence, D_i's are disjoint and so $\left\{D_i : i = 1, 2, \dots, \frac{n}{p_1}\right\}$ is a partition of $\mathbb{Z}_n$. Define the coloring $c : V(\mathbb{CAY}(\mathbb{Z}_n)) \to \left[\frac{n}{p_1}\right]$ by

$$c(x) = k, \text{ if } x \in D_k.$$

Let $x \in \mathbb{Z}_n$. Then $x \in D_k = kp_1 + H$ for some $k \in \left\{D_i : i = 1, 2, \dots, \frac{n}{p_1}\right\}$. So, $x = kp_1 + tp_2, t \in \{0, 1, 2, \dots, p_1 - 1\}$. Then x is adjacent to the vertices $p_1 + tp_2, 2p_1 + tp_2, \dots, (k-1)p_1 + tp_2, (k+1)p_1 + tp_2, \dots, \frac{n}{p_1}p_1 + tp_2$ with colors $1, 2, \dots, k-1, k+1, \dots, \frac{n}{p_1}$. Also, x is adjacent to $kp_1 + (t+1)p_2$ with color k. Thus, c is a coupon coloring of $\mathbb{CAY}(\mathbb{Z}_n)$ and so $\chi_c(\mathbb{CAY}(\mathbb{Z}_n)) \geq \frac{n}{p}$. □

Theorem 3.3. *If n is even and $Z(\mathbb{Z}_n)$ is not an ideal of $\mathbb{Z}_n$, then*

$$\chi_c(\mathbb{CAY}(\mathbb{Z}_n)) = \frac{n}{2}.$$

Proof. Each color in a coupon coloring must appear at least twice and so there can be at most $\lfloor \frac{n}{2} \rfloor$ colors in a coupon coloring of $\mathbb{CAY}(\mathbb{Z}_n)$. Therefore, $\chi_c(\mathbb{CAY}(\mathbb{Z}_n)) \leq \lfloor \frac{n}{2} \rfloor = \frac{n}{2}$, since n is even.

Suppose that $Z(\mathbb{Z}_n)$ is not an ideal of $\mathbb{Z}_n$. Since n is even, 2 is the smallest prime divisor of n. So, by Theorem 3.2, $\chi_c(\mathbb{CAY}(\mathbb{Z}_n)) \geq \frac{n}{2}$. Hence, $\chi_c(\mathbb{CAY}(\mathbb{Z}_n)) = \frac{n}{2}$. □

Theorem 3.4. *Suppose that $Z(\mathbb{Z}_n)$ is not an ideal of $\mathbb{Z}_n$. If n is odd and $3 \in Z(\mathbb{Z}_n)$, then*

$$\chi_c(\mathbb{CAY}(\mathbb{Z}_n)) = \frac{n}{3}.$$

Proof. By Theorem 3.2, $\chi_c(\mathbb{CAY}(\mathbb{Z}_n)) \geq \frac{n}{3}$, since n is odd and $3 \in Z(\mathbb{Z}_n)$, the smallest prime divisor of n is 3.

Note that $2 \notin Z(\mathbb{Z}_n)$, since n is odd. Suppose that $\mathbb{CAY}(\mathbb{Z}_n)$ has a total dominating set D with two vertices. Then $D = \{x, x+z\}$ for some $x \in \mathbb{Z}_n, z \in Z(\mathbb{Z}_n)$. But $x + 2 \in \mathbb{Z}_n$ is not adjacent to both x and $x + z$, since $2 \notin Z(\mathbb{Z}_n)$. So D cannot be a total dominating set. Thus, a total dominating set should contain at least 3 vertices. This implies that there are at most $\frac{n}{3}$ disjoint total dominating sets. Hence, the total domatic number is at most $\frac{n}{3}$. That is, $\chi_c(\mathbb{CAY}(\mathbb{Z}_n)) \leq \frac{n}{3}$. □

4 Coupon coloring of $\mathbb{CAY}(\mathbb{Z}_n \times \mathbb{Z}_m)$

If R_1 and R_2 are two finite commutative rings, then $Reg(R_1 \times R_2) = Reg(R_1) \times Reg(R_2)$. So, $(a, b) \in Z(R_1 \times R_2)$ if and only if either $a \in Z(R_1)$ or $b \in Z(R_2)$. Therefore, in $\mathbb{CAY}(\mathbb{Z}_n \times \mathbb{Z}_m)$, (x, y) is adjacent to (a, b) if and only if either $x - a \in Z(\mathbb{Z}_n)$ or $y - b \in Z(\mathbb{Z}_m)$.

Theorem 4.1. *$\mathbb{CAY}(\mathbb{Z}_n \times \mathbb{Z}_m)$ is a $(nm - \phi(n)\phi(m) - 1)$-regular graph.*

Proof. By Theorem 2.2, $\mathbb{CAY}(\mathbb{Z}_n \times \mathbb{Z}_m)$ is an $(|Z(\mathbb{Z}_n \times \mathbb{Z}_m)| - 1)$-regular graph. Here $|Z(\mathbb{Z}_n \times \mathbb{Z}_m)| - 1 = (nm - \phi(n)\phi(m) - 1)$, so $\mathbb{CAY}(\mathbb{Z}_n \times \mathbb{Z}_m)$ is an $(nm - \phi(n)\phi(m) - 1)$-regular graph. □

Lemma 4.2. *Let $(x, y) \in \mathbb{CAY}(\mathbb{Z}_n \times \mathbb{Z}_m)$. Then (x, y) is adjacent only to the vertices*

$$\left[\bigcup_{x-i\in Z(\mathbb{Z}_n)} H_i\right] \bigcup \left[\bigcup_{x-i\notin Z(\mathbb{Z}_n)} [(0, y) + J_i]\right] \setminus \{(x, y)\}$$

where $H_i = \{(i, b) : b \in \mathbb{Z}_m\}$, $J_i = \{(i, z) : z \in Z(\mathbb{Z}_m)\}$ and $i \in \mathbb{Z}_n$.

Proof. Let $(x, y), (a, b) \in \mathbb{CAY}(\mathbb{Z}_n \times \mathbb{Z}_m)$ and $(a, b) \neq (x, y)$. If $(a, b) \in \bigcup_{x-i\in Z(\mathbb{Z}_n)} H_i$, then $(a, b) \in H_i$ for some i such that $x - i \in Z(\mathbb{Z}_n)$. So, $x - i = z, z \in Z(\mathbb{Z}_n)$ and this implies that $i = z + x$. Thus, $(a, b) \in H_{z+x}$, since $(a, b) \in H_i$. Therefore, $(a, b) = (z + x, s)$ for some $s \in \mathbb{Z}_m, z \in Z(\mathbb{Z}_n)$ and so (a, b) is adjacent to (x, y), since,

$$(x, y) - (a, b) = (x, y) - (z + x, s) = (-z, y - s) \in Z(\mathbb{Z}_n \times \mathbb{Z}_m).$$

If $(a, b) \in \bigcup_{x-i\notin Z(\mathbb{Z}_n)} [(0, y) + J_i]$, then $(a, b) \in (0, y) + J_i$ for some i such that $x - i \notin Z(\mathbb{Z}_n)$. In this case, $(a, b) = (0, y) + (i, w), w \in Z(\mathbb{Z}_m)$. So,

$$(x, y) - (a, b) = (x, y) - [(0, y) + (i, w)] = (x - i, -w) \in Z(\mathbb{Z}_n \times \mathbb{Z}_m).$$

Therefore, (a, b) is adjacent to (x, y). Thus (x, y) is adjacent to the vertices of $\bigcup_{x-i\in Z(\mathbb{Z}_n)} H_i$ and $\bigcup_{x-i\notin Z(\mathbb{Z}_n)} [(0, y) + J_i]$ except (x, y). Clearly, all these $[m - \phi(m)]n + \phi(m)[n - \phi(n)] - 1 = nm - \phi(n)\phi(m) - 1$ vertices are distinct and by Theorem 4.1 these are the only vertices adjacent to (x, y). □

Theorem 4.3. *$\mathbb{CAY}(\mathbb{Z}_n \times \mathbb{Z}_m)$ is a connected graph with $diam(\mathbb{CAY}(\mathbb{Z}_n \times \mathbb{Z}_m)) = 2$.*

Proof. Any $(a, b) \in \mathbb{Z}_n \times \mathbb{Z}_m$ with either $a = 0$ or $b = 0$ is in $Z(\mathbb{Z}_n \times \mathbb{Z}_m)$. Let $(a, b), (x, y) \in \mathbb{Z}_n \times \mathbb{Z}_m = V(\mathbb{CAY}(\mathbb{Z}_n \times \mathbb{Z}_m))$ such that $(a, b) \neq (x, y)$. If $a = x$ or $b = y$, then (a, b) is adjacent to (x, y). Suppose that neither $a = x$ nor $b = y$, then $(a, b) - (a, y) - (x, y)$ is a path in $\mathbb{CAY}(\mathbb{Z}_n \times \mathbb{Z}_m)$, since

$$\begin{aligned}(a, b) - (a, y) &= (0, b - y) \in Z(\mathbb{Z}_n \times \mathbb{Z}_m)^* \\ (a, y) - (x, y) &= (a - x, 0) \in Z(\mathbb{Z}_n \times \mathbb{Z}_m)^*.\end{aligned}$$

Moreover, there are non-adjacent vertices, $(a, b), (a + 1, b + 1)$. □

Theorem 4.4. *$\chi_c(\mathbb{CAY}(\mathbb{Z}_n \times \mathbb{Z}_m)) \geq \frac{nm}{p}$, where p is the minimum of the least prime divisors of n and m.*

Proof. Let $n = p_1^{r_1} p_2^{r_2} \dots p_u^{r_u}$ and $m = q_1^{s_1} q_2^{s_2} \dots q_v^{s_v}$ with $p_1 < p_2 < \cdots < p_u$ and $q_1 < q_2 < \cdots < q_v$. Without loss of generality assume that $\min\{p_1, q_1\} = p_1 = p$. Define for all $i = 1, \dots, \frac{n}{p}, j = 0, 1, \dots, m - 1$,

$$D_{i,j} = \left\{ \left([i - 1]p + k, j \right) : k = 0, 1, \dots, p - 1 \right\}.$$

Then $\left\{D_{i,j} : i = 1, \dots, \frac{n}{p}, j = 0, 1, \dots, m-1\right\}$ is a partition of elements of $\mathbb{Z}_n \times \mathbb{Z}_m$ with $\frac{nm}{p}$ parts. For, let $(x, y) \in D_{i,j} \cap D_{s,t}$. Then

$$(x, y) = \Big([i-1]p + k_1, j\Big) = \Big([s-1]p + k_2, t\Big).$$

So, $[i-1]p + k_1 = [s-1]p + k_2$ and $j = t$. $[i-1]p + k_1 = [s-1]p + k_2$ implies that $\big([i-1] - [s-1]\big)p = k_2 - k_1$. Since $k_1, k_2 \in \{0, 1, \dots, p-1\}$, $k_2 - k_1 \leq p$. Thus, $k_2 - k_1 = 0$ and so $[i-1] - [s-1] = 0$ that is $i = s$. Therefore, $D_{i,j} = D_{s,t}$.

Define a coloring c by

$$c(x, y) = j\frac{n}{p} + i, \text{ for } (x, y) \in D_{i,j}.$$

Let $(x, y) \in D_{s,t}$ for some $s \in \left\{1, \dots, \frac{n}{p}\right\}$ and $t \in \{0, 1, \dots, m-1\}$. Then $(x, y) = \Big([s-1]p + k, t\Big)$ for some $k \in \{0, 1, \dots, p-1\}$ and so in $\mathbb{CAY}(\mathbb{Z}_n \times \mathbb{Z}_m)$, (x, y) is adjacent to the $\frac{nm}{p} - 1$ vertices of the set

$$\left\{\Big([i-1]p + k, j\Big) : i = 1, \dots, \frac{n}{p}, j = 0, 1, \dots, m-1\right\} \setminus \{(x, y)\},$$

since

$$\begin{aligned}(x, y) - \Big([i-1]p + k, j\Big) &= \Big([s-1]p + k, t\Big) - \Big([i-1]p + k, j\Big) \\ &= \Big(([s-1] - [i-1])p, t - j\Big) \in Z(\mathbb{Z}_n \times \mathbb{Z}_m)^*.\end{aligned}$$

So, (x, y) is adjacent to the vertices of colors $1, 2, \dots, \frac{nm}{p}$ except the color $t\frac{n}{p} + s = c(x, y)$. But (x, y) is also adjacent to the vertex $\Big([s-1]p + k \pm 1, t\Big)$ which is distinct from the above vertices and with color $t\frac{n}{p} + s$. Thus c is a coupon coloring of $\mathbb{CAY}(\mathbb{Z}_n \times \mathbb{Z}_m)$. Therefore, $\chi_c(\mathbb{CAY}(\mathbb{Z}_n \times \mathbb{Z}_m)) \geq \frac{nm}{p}$. □

Theorem 4.5. *Assume that either n or m is even. Then*

$$\chi_c(\mathbb{CAY}(\mathbb{Z}_n \times \mathbb{Z}_m)) = \frac{nm}{2}.$$

Proof. Any color in a coupon coloring must appear at least twice. Here we have nm vertices and so there can be at most $\lfloor\frac{nm}{2}\rfloor$ colors in a coupon coloring of $\mathbb{CAY}(\mathbb{Z}_n \times \mathbb{Z}_m)$. Therefore, $\chi_c(\mathbb{CAY}(\mathbb{Z}_n \times \mathbb{Z}_m)) \leq \lfloor\frac{nm}{2}\rfloor = \frac{nm}{2}$, since n or m is even.

If n or m is even, then 2 will be the minimum of the smallest prime divisors of n and m. So by Theorem 4.4, $\chi_c(\mathbb{CAY}(\mathbb{Z}_n \times \mathbb{Z}_m)) \geq \frac{nm}{2}$. Hence, $\chi_c(\mathbb{CAY}(\mathbb{Z}_n \times \mathbb{Z}_m)) = \frac{nm}{2}$. □

Lemma 4.6. *Suppose that p and q are prime numbers with $p \leq q$. Then a dominating set of $\mathbb{CAY}(\mathbb{Z}_p \times \mathbb{Z}_q)$ contains at least p vertices.*

Proof. Suppose that D is a dominating set of $\mathbb{CAY}(\mathbb{Z}_p \times \mathbb{Z}_q)$ with $p - 1$ vertices. Define $H_i = \{(i, b) : b \in \mathbb{Z}_q\}$ for all $i \in \mathbb{Z}_p$. Then $\{H_i : i \in \mathbb{Z}_p\}$ is a partition of the vertices of $\mathbb{CAY}(\mathbb{Z}_p \times \mathbb{Z}_q)$. Since D contains only $p - 1$ vertices, there must exist at least one $s \in \mathbb{Z}_p$ such that no vertex of H_s is in D. Consider the vertices of $H_s = \{(s, 0), (s, 1), \dots, (s, q-1)\}$. By

Lemma 4.2, for all $j \in \mathbb{Z}_q$, (s, j) is adjacent only to the vertices

$$\begin{aligned} & \left[\bigcup_{s-i \in Z(\mathbb{Z}_p)} H_i\right] \bigcup \left[\bigcup_{s-i \notin Z(\mathbb{Z}_p)} [(0,j) + \{(i,0)\}]\right] \setminus \{(s,j)\} \\ = & \left[\bigcup_{s-i=0} H_i\right] \bigcup \left[\bigcup_{s-i \in \{1,2,\dots,p-1\}} \{(i,j)\}\right] \setminus \{(s,j)\} \\ = & \ H_s \setminus \{(s,j)\} \cup \{(s-1,j),(s-2,j),\dots(s-(p-1),j)\} \\ = & \ H_s \setminus \{(s,j)\} \cup \big\{(s-k,j) : k \in \{1,2,\dots,p-1\}\big\}. \end{aligned}$$

Since D is a dominating set and no vertex of H_s is in D, there should exist $k \in \{1, 2, \dots, p-1\}$ such that $(s-k, j) \in D$ for all $j \in \mathbb{Z}_q$. That is,

$$\{(s-k_0, 0), (s-k_1, 1), \dots, (s-k_{q-1}, q-1)\} \subseteq D$$

where, $k_t \in \{1, 2, \dots, p-1\}, t \in \mathbb{Z}_q$. Then D should contain at least q vertices. But this is not possible, since $p \leq q$ and D contains only $p-1$ vertices. □

Theorem 4.7. *Suppose that p and q are prime numbers. Then*

$$\chi_c(\mathbb{CAY}(\mathbb{Z}_p \times \mathbb{Z}_q)) = \max\{p, q\}.$$

Proof. Without loss of generality, assume that $p \leq q$. By Theorem 4.4, $\chi_c(\mathbb{CAY}(\mathbb{Z}_p \times \mathbb{Z}_q)) \geq q$. From Lemma 4.6, a dominating set of $\mathbb{CAY}(\mathbb{Z}_p \times \mathbb{Z}_q)$ should contain at least p vertices. So any total dominating set of $\mathbb{CAY}(\mathbb{Z}_p \times \mathbb{Z}_q)$ contains at least p vertices. Hence, there can be at most q disjoint total dominating sets. That is $\chi_c(\mathbb{CAY}(\mathbb{Z}_p \times \mathbb{Z}_q)) \leq q$. □

5 Coupon coloring of some circulant graphs

Theorem 5.1. *Let $C = \{1, -1, a = -a\}$. Then*

$$\chi_c(Cay(\mathbb{Z}_n, C)) = \begin{cases} 3, & \text{if } n \equiv 0 \ (mod\ 3) \\ 2, & \text{otherwise.} \end{cases}$$

Proof. Clearly $Cay(\mathbb{Z}_n, C)$ is a 3-regular graph. Let $a \in \mathbb{Z}_n$ and let $a = -a$. Then n must be even, since $2a = a + a = 0 = n$.
Case 1: $n \equiv 0$ (mod 3) Define $c : V(Cay(\mathbb{Z}_n, C)) \to [3]$ by

$$c(i) = \begin{cases} 1, & \text{if } i \equiv 0 (\text{mod } 3) \\ 2, & \text{if } i \equiv 1 (\text{mod } 3) \\ 3, & \text{if } i \equiv 2 (\text{mod } 3). \end{cases}$$

Then c is a coupon coloring on $Cay(\mathbb{Z}_n, C)$. For, let $i \in V(Cay(\mathbb{Z}_n, C))$ such that $i \equiv 0$ (mod 3). Then $c(i) = 1$ and the neighbors of i are $i-1, i+1$ and $i+a$. Since $i \equiv 0$ (mod 3) and $a = \frac{n}{2} \equiv 0$ (mod 3), $i + a \equiv a \equiv 0$ (mod 3) , $i - 1 \equiv -1 \equiv 2$ (mod 3) and $i + 1 \equiv 1$ (mod 3). So $c(i + a) = 1$, $c(i - 1) = 3$ and $c(i + 1) = 2$. All other possibilities can be proved similarly.
Case 2: $n \equiv 1$ (mod 3)

Suppose that $Cay(\mathbb{Z}_n, C)$ has a 3-coupon coloring. Then at least one color should be given to at most $\lfloor \frac{n}{3} \rfloor$ vertices. This color class D must be a total dominating set. But $\lfloor \frac{n}{3} \rfloor = \frac{n-1}{3}$ and $Cay(\mathbb{Z}_n, C)$ is a 3-regular graph. So, D an dominate at most $n-1$ vertices and D cannot be a TDS.

So $\chi_c(Cay(\mathbb{Z}_n, C)) \leq 2$. Now define $c : V(Cay(\mathbb{Z}_n, C)) \to [2]$ by

$$c(i) = \begin{cases} 1, & \text{if } i \equiv 0 \ (\text{mod } 2) \\ 2, & \text{if } i \equiv 1 \ (\text{mod } 2). \end{cases}$$

Then clearly c is a 2-coupon coloring of $Cay(\mathbb{Z}_{\rm n}, C)$, and so $\chi_c(Cay(\mathbb{Z}_{\rm n}, C)) \geq 2$. Hence $\chi_c(Cay(\mathbb{Z}_{\rm n}, C)) = 2$.
Case 3: $n \equiv 2 \pmod 3$

Proof is similar as in Case 2. □

Theorem 5.2. *Let $Cay(\mathbb{Z}_n, C)$ be the circulant graph with $C = \{1, -1, 2, -2\}$, $2 \neq -2$. Then $\chi_c(Cay(\mathbb{Z}_n, C)) \leq 3$ and equality holds if $n \equiv 0$ (mod 6).*

Proof. Since $2 \neq -2$, $Cay(\mathbb{Z}_{\rm n}, C)$ is a 4-regular graph. Let a be any vertex of $Cay(\mathbb{Z}_{\rm n}, C)$. Then neighbors of a are $a-1, a-2, a+1$ and $a+2$. Since $(a+1)-(a-1) = 2 \in C$, there is an edge between $a-1$ and $a+1$. Similarly, $(a-1)-(a-2) = 1 \in C$; $(a+1)-(a+2) = -1 \in C$ and so $a-1$ and $a-2$ are adjacent; $a+1$ and $a+2$ are adjacent.

If c is a 4-coupon coloring of $Cay(\mathbb{Z}_{\rm n}, C)$, then without loss of generality we may assume that $c(a-1) = 1$, $c(a+1) = 2$, $c(a-2) = 3$ and $c(a+2) = 4$. Then $c(a)$ cannot be 1,2,3 or 4.

(i) If $c(a) = 1$, then the vertex $a+1$ will have two neighbors with color 1. Since $\Delta(Cay(\mathbb{Z}_{\rm n}, C)) = 4$, this will make the vertex $a+1$, a bad vertex.

(ii) If $c(a) = 2$, then the vertex $a-1$ will have two neighbors with color 2 and so $a-1$ is a bad vertex.

(iii) If $c(a) = 3$, then the vertex $a-1$ will have two neighbors with color 3 and $a-1$ will be a bad vertex.

(iv) If $c(a) = 4$, then the vertex $a+1$ will have two neighbors with color 4 and so $a+1$ is a bad vertex.

Hence 4-coupon coloring is not possible and so $\chi_c(Cay(\mathbb{Z}_{\rm n}, C)) \leq 3$.
Claim : $\chi_c(Cay(\mathbb{Z}_{\rm n}, C)) = 3$ if $n \equiv 0 \pmod 6$

Define $c : V(Cay(\mathbb{Z}_{\rm n}, C)) \to [3]$ by

$$c(i) = \begin{cases} 1, & \text{if } i \equiv 0, 1 \pmod 6 \\ 2, & \text{if } i \equiv 2, 3 \pmod 6 \\ 3, & \text{if } i \equiv 4, 5 \pmod 6. \end{cases}$$

Then c is a coupon coloring of $Cay(\mathbb{Z}_{\rm n}, C)$. For, let $i \in V(Cay(\mathbb{Z}_{\rm n}, C))$ such that $i \equiv 0 \pmod 6$. Then neighbors of i are $i-1, i-2, i+1$ and $i+2$.

Since $i-1 \equiv -1 \equiv 5 \pmod 6$, so $c(i-1) = 3$. Similarly since $i-2 \equiv -2 \equiv 4 \pmod 6$, so $c(i-2) = 3$, since $i+1 \equiv 1 \pmod 6$, so $c(i+1) = 1$ and since $i+2 \equiv 2 \pmod 6$, so $c(i+2) = 2$. Therefore, the four neighbors of i colored with all the three colors. Other cases can be proved similarly. Hence the claim holds. □

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