Palestine Journal of Mathematics

Vol. 12(1)(2023) , 273–284

ON EDGE PRODUCT NUMBER OF HYPERGRAPH

Megha M. Jadhav and **Kishor F. Pawar**

Communicated by T. Tamizh Chelvam

this paper appeared already in **Vol. 11(4) (2022) , 183 -194**

MSC 2010 Classifications: Primary 05C65.

Keywords and phrases: edge product hypergraph, edge product number, chain hypergraph.

Abstract In this paper we introduced the notion of an edge product number of a hypergraph. The edge product number, $\mathcal{EP}_n(\mathcal{H})$ of a hypergraph $\mathcal{H}$ is the smallest number r such that $\mathcal{H} \cup rK_2$ is an edge product hypergraph. Later we obtained the edge product number of a star hypergraph, open chain hypergraph and closed chain hypergraph for different sizes.

1 Introduction

The idea of graph labeling was introduced by Rosa during 1960 in [7]. It is the concept of assigning labels to vertices, edges or both subject to certain conditions. Traditionally, the set of labels which we assign to vertices or edges is a subset of integers. Graph labelings provide us useful models for many applications such as astronomy, radar X-ray crystallography, circuit design, data base management, communication network addressing etc. Rosa in [7] introduced three types of labelings called α, β and ρ-labeling. Following this,many different types of labelings were introduced by many others. For more details about graph labeling reader may refer to an extensive dynamic survey of Gallian [4]. In labeling problems, we have to find the optimal way of labeling vertices or edges or both with distinct integers, k-tuples of integers, or group elements subject to certain conditions. Such problems often turn up in link with applications in circuit layout, network addressing or code designs.

Harary [5] introduced the notion of sum graphs of a graph G. A graph $G(V, E)$ is said to be a sum graph if there exists a bijection labeling f from the vertex set V to a set S of positive integers such that $xy \in E$ if and only if $f(x) + f(y) \in S$. The product analogue of sum graphs was first introduced by Thavamani in 2011. He introduced edge product graph and edge product number of a graph in [8] and [9] respectively. A graph G is said to be an edge product graph if the edges of G can be labeled with distinct positive integers such that the product of all the label of the edges incident on a vertex is again an edge label of G and if the product of any collection of edges is a label of an edge in G then they are incident on a vertex. For many applications, the edges or vertices are given labels that are meaningful in the associated domain. An enormous body of literature has grown around graph labeling in the last four decades. However much less is known about labeling of hypergraph. Hypergraph is a generalization of a graph in which any subset of a given set may be an edge rather than two element subsets. With graphs we are limited to describe and model only pairwise interactions while hypergraphs model any type of groupwise complex interactions. Hypergraphs have demonstrated their power as a tool for understanding problems in a extensive variety of scientific fields.

In [6] Jadhav and Pawar introduced the notion of an edge function and using this edge function an edge product hypergraph is defined. This paper [6] addresses the problem of hypergraph labeling, this labeling gives the notions of hypergraph called edge product hypergraph and unit edge product hypergraph. The present paper deals with one parameter called edge product number of a hypergraph by which one can verify that the labeling applied to given problem is optimal or not. We initiate to study this labeling on certain types of hypergraphs and proved some results on the edge product number of that hypergraphs.

2 Preliminaries and Edge Product Hypergraph

We begin with recalling some basic definitions from [1]-[2] and [6] required for our purpose. In this paper, we consider a simple hypergraph (n, m) without isolated vertices and of size $m > 1$.

Definition 2.1. A hypergraph $\mathcal{H}$ is a pair $\mathcal{H}(V, E)$ where V is a finite nonempty set and E is a collection of subsets of V. The elements of V are called vertices and the elements of E are called edges or hyperedges. And $\cup_{e_i \in E} e_i = V$ and $e_i \neq \phi$ are required, for all $e_i \in E$. The number of vertices in $\mathcal{H}$ is called the order of the hypergraph and is denoted by $|V|$. The number of edges in $\mathcal{H}$ is called the size of $\mathcal{H}$ and is denoted by $|E|$. A hypergraph of order n and size m is called a (n, m) hypergraph. The number $|e_i|$ is called the degree (cardinality) of the edges e_i. The rank of a hypergraph $\mathcal{H}$ is $r(\mathcal{H}) = max_{e_i \in E}|e_i|$.

Definition 2.2. For any vertex v in a hypergraph $\mathcal{H}(V, E)$, the set $N[v] = \{u \in V : u \text{ is adjacent to } v\} \cup \{v\}$ is called the closed neighborhood of v in $\mathcal{H}$ and each vertex in the set $N[v] - \{v\}$ is called neighbor of v. The open neighborhood of the vertex v is the set $N[v] \setminus \{v\}$. If $S \subseteq V$ then $N(S) = \cup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$.

Definition 2.3. A simple hypergraph (or sperner family) is a hypergraph $\mathcal{H}(V, E)$ where $E = \{e_1, e_2, \cdots, e_m\}$ such that $e_i \subset e_j$ implies $i = j$.

Definition 2.4. For any hypergraph $\mathcal{H}(V, E)$ two vertices v and u are said to be adjacent if there exists an edge $e \in E$ that contains both v and u and non adjacent otherwise.

Definition 2.5. For any hypergraph $\mathcal{H}(V, E)$ two edges are said to be adjacent if their intersection is nonempty. If a vertex $v_i \in V$ belongs to an edge $e_j \in E$ then we say that they are incident to each other.

Definition 2.6. A star hypergraph is an intersecting family of edges having a common element x. It is denoted by $\mathcal{H}(x)$ and the vertex x is called the center of $\mathcal{H}(x)$.

Definition 2.7. The vertex degree of a vertex v is the number of vertices adjacent to the vertex v in $\mathcal{H}$. It is denoted by $d(v)$. The maximum (minimum) vertex degree of a hypergraph is denoted by $\Delta(\mathcal{H})(\delta(\mathcal{H}))$.

Definition 2.8. The edge degree of a vertex v is the number of edges containing the vertex v. It is denoted by $d_E(v)$. The maximum (minimum) edge degree of a hypergraph is denoted by $\Delta_E(\mathcal{H})(\delta_E(\mathcal{H}))$. A vertex of a hypergraph which is incident to no edge is called an isolated vertex. The edge degree (or vertex degree) of an isolated vertex is trivially 0. An edge of cardinality one is called a singleton (loop), a vertex of edge degree one is called a pendant vertex.

Definition 2.9. The hypergraph $\mathcal{H}(V, E)$ is called connected if for any pair of its vertices, there is a path connecting them. If $\mathcal{H}$ is not connected then it consists of two or more connected components, each of which is a connected hypergraph.

Definition 2.10. [6] Let $\mathcal{H}(V, E)$ be a simple and connected hypergraph. Let $V(\mathcal{H})$ be the vertex set of $\mathcal{H}$ and $E(\mathcal{H})$ be the edge set of $\mathcal{H}$. Let P be a set of positive integers such that $|E| = |P|$. Then any bijection $f : E \to P$ is called an edge function of the hypergraph H.

Definition 2.11. [6] The function $F(v) = \prod\{f(e)| \text{ edge } e \text{ is incident to the vertex } v\}$ on $V(H)$ is called an edge product function of the edge function f.

Definition 2.12. [6] The hypergraph $\mathcal{H}(V, E)$ is said to be an edge product hypergraph if there exists an edge function $f : E \to P$ such that the edge function f and the corresponding edge product function F of f on $V(\mathcal{H})$ have the following two conditions:

(i) $F(v) \in P$, for every $v \in V$.

(ii) If $f(e_1) \times f(e_2) \times \ldots \times f(e_p) \in P$, for some edges $e_1, e_2, \ldots, e_p \in E$ then the edges $e_1, e_2, \ldots, e_p$ are all incident to a vertex $v \in V$.

Example 2.13. [6] Let $\mathcal{H}(V,E)$ be a hypergraph, where $V=\{v_1,v_2,\dots,v_{16}\}$ and $E=\{e_1,e_2,\dots,e_7\}$. In which the edges of H are defined as follows:

$$e_1=\{v_1,v_5,v_6,v_{13},v_{14}\} \qquad e_2=\{v_1,v_2\},$$
$$e_3=\{v_1,v_3,v_4\} \qquad e_4=\{v_5,v_6,v_7,v_8\},$$
$$e_5=\{v_5,v_6,v_9\} \qquad e_6=\{v_5,v_6,v_{10},v_{11},v_{12}\},$$
$$e_7=\{v_{15},v_{16}\}.$$

Now define the edge function $f:E\to P$ by

$$f(e_1)=11, \quad f(e_2)=4, \quad f(e_3)=30, \quad f(e_4)=3,$$
$$f(e_5)=2, \quad f(e_6)=20, \quad f(e_7)=1320.$$

Then edge product function F of f will be as follows:

$$F(v_1)=1320, \quad F(v_2)=4, \quad F(v_3)=F(v_4)=30,$$
$$F(v_5)=F(v_6)=1320, \quad F(v_7)=F(v_8)=3, \quad F(v_9)=2,$$
$$F(v_{10})=F(v_{11})=F(v_{12})=20, \quad F(v_{13})=F(v_{14})=11, \quad F(v_{15})=F(v_{16})=1320.$$

Hence the given hypergraph is an edge product hypergraph.

3 Edge Product Number of a Hypergraph

In this section we introduced the notion of an edge product number of a hypergraph with suitable examples. Later we obtained the edge product number of a star hypergraph, open chain hypergraphs and closed chain hypergraphs of different sizes.

Definition 3.1. The smallest number r such that $\mathcal{H}\cup rK_2$ is an edge product hypergraph is called the edge product number of a hypergraph $\mathcal{H}$. Which is denoted by $\mathcal{EP}_n(\mathcal{H})$.

Example 3.2. Let $\mathcal{H}(V,E)$ be a hypergraph, where $V(\mathcal{H})=\{v_1,v_2,v_3,v_4,v_5\}$ and $E(\mathcal{H})=\{e_1,e_2\}$. In which the edges of H are defined as follows: $e_1=\{v_1,v_2,v_3,v_4\}$, $e_2=\{v_1,v_2,v_5\}$. Now consider the hypergraph $\mathcal{H}\cup K_2$ with vertex set

$$V=\{v_1,v_2,v_3,v_4,v_5\}\cup\{w_1,w_2\}.$$

$E=\{e_1,e_2\}\cup\{e_3\}$ where $e_1=\{v_1,v_2,v_3,v_4\}$, $e_2=\{v_1,v_2,v_5\}$, $e_3=\{w_1,w_2\}$.

Define the edge function $f:E\to P$ by $f(e_1)=2$, $f(e_2)=7$, $f(e_3)=14$ The edge product function F of f is defined by, $F(v_1)=14$, $F(v_2)=14$, $F(v_3)=F(v_4)=2$, $F(v_5)=7$, $F(w_1)=F(w_2)=14$. Hence $\mathcal{H}\cup K_2$ is an edge product hypergraph and $\mathcal{EP}_n(\mathcal{H})=1$.

For any connected hypergraph $\mathcal{H},\mathcal{EP}_n(\mathcal{H})\geq 1$. Let $\mathcal{EP}_n(\mathcal{H})=r$. An edge function $f:E\to P$ and its corresponding edge product function F that makes $\mathcal{H}\cup rK_2$ an edge product hypergraph are called an optimal edge function and optimal edge product function of $\mathcal{H}$ respectively. It should be noted that there can be many optimal functions for a given hypergraph. Let V_{in} and E_{in} be the vertex set and edge set of $\mathcal{H}$ respectively and V_{out} and E_{out} be that of rK_2. Then the $\mathcal{EP}_n(\mathcal{H})$=Cardinality of the set $\{F(v)|v\in V_{in}$ and $F(v)\notin f(E_{in})\}$. If $F(V_{in})\cap f(E_{in})$ is empty then the function F is called an outer edge product function otherwise it is called an inner edge product function. Therefore the range of F has at least r elements.

Theorem 3.3. *The optimal edge product function F has exactly r elements if and only if it is outer edge product function.*

Observation: For a hypergraph $\mathcal{H}$, if $\mathcal{EP}_n(\mathcal{H}\setminus v)\leq\mathcal{EP}_n(\mathcal{H})$.

Theorem 3.4. *For a hypergraph $\mathcal{H}$, if $\mathcal{EP}_n(\mathcal{H}\setminus v)<\mathcal{EP}_n(\mathcal{H})$ then $F(v)\cap f(e)$ is singleton, for some $e\in E_{out}$.*

Proof. Let $\mathcal{H}$ be a hypergraph. Let $\mathcal{EP}_n(\mathcal{H} \setminus v) < \mathcal{EP}_n(\mathcal{H})$. Assume contrary that $F(v) \cap f(e)$ is not singleton, for all $e \in E_{out}$ Then for any $e \in E_{out}$, we have $F(v) \cap f(e) = \{v_1, v_2, \ldots v_q\}$, $q > 1$. It follows, $F(v_1) = F(v_2) = \ldots = F(v_q) = p \in P$. Thus removing any vertex from $v_1, v_2, \ldots, v_q$ does not affect the edge product number of H. Hence we get, $\mathcal{EP}_n(\mathcal{H} \setminus v) = \mathcal{EP}_n(\mathcal{H})$, for all $v \in V$, which is a contradiction. Therefore $F(v) \cap f(e)$ must be singleton, for some $e \in E_{out}$. □

Definition 3.5. A k-uniform connected hypergraph $\mathcal{H}$ is called a chain hypergraph if the edges in $\mathcal{H}$ are interesting in such way that every edge in $\mathcal{H}$ is adjacent to at most two edges of $\mathcal{H}$ and edge degree of each vertex is at most 2.

Definition 3.6. A chain hypergraph $\mathcal{H}(V, E)$ is called closed if every edge of $\mathcal{H}$ is adjacent to exact two edges in $\mathcal{H}$; otherwise it is open.

Example 3.7. Let $\mathcal{H}(V, E)$ be a hypergraph, where $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E = \{e_1, e_2, e_3, e_4\}$. In which the edges of H are defined as follows:

$$e_1 = \{v_1, v_2, v_3\}, \qquad e_2 = \{v_2, v_3, v_4\},$$
$$e_3 = \{v_4, v_5, v_6\}, \qquad e_4 = \{v_5, v_6, v_1\}.$$

$\mathcal{H}(V, E)$ is a chain hypergraph which is closed.

Theorem 3.8. *The edge product number of a star hypergraph $\mathcal{H}(x)$ is one.*

Proof. Let $\mathcal{H}(x)$ be a star hypergraph with center x and $v_i^1, v_i^2, \ldots, v_i^{s_i}$ be the pendant vertices in e_i, for $1 \le i \le m$, where $s_i \in \mathbb{N}$ denotes the number of pendant vertices in e_i. Let $p_1 \times p_2 \times p_3 \times \ldots \times p_m = b$. Consider the hypergraph $\mathcal{H} \cup K_2$ with vertex set

$$V = \{x, v_1^1, v_1^2, \ldots, v_1^{s_1}, v_2^1, v_2^2, \ldots, v_2^{s_2}, \ldots, v_m^1, v_m^2, \ldots, v_m^{s_m}\}$$
$$\cup \{w_1, w_2\}, s_i \in \mathbb{N}, \text{ for } i = 1, 2, \ldots, m.$$

and edge set

$$E = \{e_1, e_2, \ldots, e_m\} \cup \{e_{m+1}\}$$

where $e_i = \{x, v_i^1, v_i^2, \ldots, v_i^{s_i}\}$, for $1 \le i \le m$, $e_{m+1} = \{w_1, w_2\}$. The set of all elements of P are $\{p_1, p_2, \ldots, p_m, b\}$. Define the edge function $f : E \to P$ by $f(e_i) = p_i$, for $1 \le i \le m$ and $f(e_{m+1}) = b$. Then the edge product function F of f will be

$$F(v_i^1) = F(v_i^2) = \ldots = F(v_i^{s_i}) = p_i, \text{ for } 1 \le i \le m,$$

$s_i \in \mathbb{N}$ and

$$F(x) = b$$
$$F(w_1) = F(w_2) = b.$$

Clearly, the range of F is P and if $f(e_1) \times f(e_2) \times \ldots \times f(e_q) \in P$, for some edges $e_1, e_2, \ldots, e_q \in E$ then all the edges $e_1, e_2, \ldots, e_q$ are incident to a vertex $x \in V$. Hence $\mathcal{H} \cup K_2$ is an edge product hypergraph. Hence the proof. □

Here we use the notation v_i for the pendant vertex belonging to the edge e_i and $v_{i,j}$ for the non-pendant vertex belonging to the edge e_i and e_j.

Theorem 3.9. *If $\mathcal{H}(V, E)$ is an open chain hypergraph of size 5. Then $\mathcal{EP}_n(\mathcal{H}) = 2$.*

Proof. Let $\mathcal{H}(V, E)$ be an open chain hypergraph of size 5. Let $v_i^1, v_i^2, \ldots, v_i^{s_i}$ be the pendant vertices in e_i, for $1 \le i \le 5$, where $s_i \in \mathbb{N}$ denotes the number of pendant vertices in e_i and $v_{i,i+1}^1, v_{i,i+1}^2, \ldots, v_{i,i+1}^{r_i}$, for $1 \le i \le 4$ be the non-pendant vertices in $\mathcal{H}$, where $r_i \in \mathbb{N}$ denotes the number of non-pendant vertices in $e_i \cap e_{i+1}$. Now we assume that $\mathcal{EP}_n(\mathcal{H}) = 1$. Then the hypergraph $\mathcal{H} \cup K_2$ is an edge product hypergraph with vertex set

$$V = \{v_1^1, v_1^2, \ldots, v_1^{s_1}, v_2^1, v_2^2, \ldots, v_2^{s_2}, \ldots, v_5^1, v_5^2, \ldots, v_5^{s_5},$$
$$v_{1,2}^1, v_{1,2}^2, \ldots v_{1,2}^{r_1}, v_{2,3}^1, v_{2,3}^2, \ldots, v_{2,3}^{r_2}, \ldots, v_{4,5}^1, v_{4,5}^2, \ldots, v_{4,5}^{r_4}\} \cup \{w_1, w_2\}$$

and the edge set

$$E = \{e_1, e_2, e_3, e_4, e_5\} \cup \{e_6\},$$

where $e_1 = \{v_1^1, v_1^2, \ldots, v_1^{s_1}, v_{1,2}^1, v_{1,2}^2, \ldots, v_{1,2}^{r_1}\}$,
$e_i = \{v_i^1, v_i^2, \ldots, v_i^{s_i}, v_{i-1,i}^1, v_{i-1,i}^2, \ldots, v_{i-1,i}^{r_{i-1}}, v_{i,i+1}^1, v_{i,i+1}^2, \ldots, v_{i,i+1}^{r_i}\}$, for $1 \leq i \leq 4$, $e_5 = \{v_5^1, v_5^2, \ldots, v_5^{s_5}, v_{4,5}^1, v_{4,5}^2, \ldots, v_{4,5}^{r_4}\}$, $e_6 = \{w_1, w_2\}$. Let the elements of P are $\{p_1, p_2, p_3, p_4, p_5, b\}$. The mapping $f : E \to P$ is an optimal edge function and F is the optimal edge product function of f . Let the optimal edge function f is defined by $f(e_i) = p_i$, for $1 \leq i \leq 5$ and $f(e_6) = b$. Then the optimal edge product function F of f will be

$$F(v_i^1) = F(v_i^2) = \ldots = F(v_i^{s_i}) = p_i, \text{ for } 1 \leq i \leq 5$$

$$F(v_{i,i+1}^1) = F(v_{i,i+1}^2) = \ldots = F(v_{i,i+1}^{r_i}) = p_i p_{i+1} = t_i \text{ say , for } 1 \leq i \leq 4$$

$$F(w_1) = F(w_2) = b.$$

Since $p_i p_{i+1} \neq p_{i+1} p_{i+2}$. Hence we have, $t_1 \neq t_2, t_2 \neq t_3, t_3 \neq t_4$. Now the edges e_1 and e_5 are adjacent to only one edge in $\mathcal{H}$. So t_2 can be $f(e_1) = p_1$ and t_3 can be $f(e_5) = p_5$. Also the range of function F is in P, it follows $t_1 = t_4 = b, t_2 = p_1$ and $t_3 = p_5$. Therefore $p_1 p_2 = p_4 p_5, t_2 = p_1$ and $t_3 = p_5$ that is $t_2 p_2 = p_4 t_3, p_2 p_3 p_2 = p_4 p_3 p_4 \Rightarrow p_2 = p_4$. This contradicts the assumption that the elements of P are distinct. Hence $\mathcal{EP}_n(\mathcal{H}) > 1$. The edge function given in Example 3.10 shows that $\mathcal{EP}_n(\mathcal{H}) = 2$. □

Example 3.10. Let $\mathcal{H}(V, E)$ be an open chain hypergraph of size 5, Then $\mathcal{H} \cup 2K_2$ is an edge product hypergraph with the vertex set $V = \{v_1, v_2, \ldots, v_{13}\} \cup \{w_1, w_2, w_3, w_4\}$ and the edge set $E = \{e_1, e_2, e_3, e_4, e_5\} \cup \{e_6, e_7\}$. In which the edges of $\mathcal{H}$ are defined as follows:

$$e_1 = \{v_1, v_2, v_3, v_4\}, \quad e_2 = \{v_2, v_3, v_4, v_5\}, \quad e_3 = \{v_5, v_6, v_7, v_8\},$$
$$e_4 = \{v_6, v_7, v_9, v_{10}\}, \quad e_5 = \{v_{10}, v_{11}, v_{12}, v_{13}\}, \quad e_6 = \{w_1, w_2\}, \quad e_7 = \{w_3, w_4\}.$$

Define the edge function $f : E \to P$ by

$$f(e_1) = 2^8, \quad f(e_2) = 2^2, \quad f(e_3) = 2^7, \quad f(e_4) = 2^3,$$
$$f(e_5) = 2^6, \quad f(e_6) = 2^{10}, \quad f(e_7) = 2^9.$$

Example 3.11. Let $\mathcal{H}(V, E)$ be an open chain hypergraph of size 6, Then $\mathcal{H} \cup K_2$ is an edge product hypergraph.

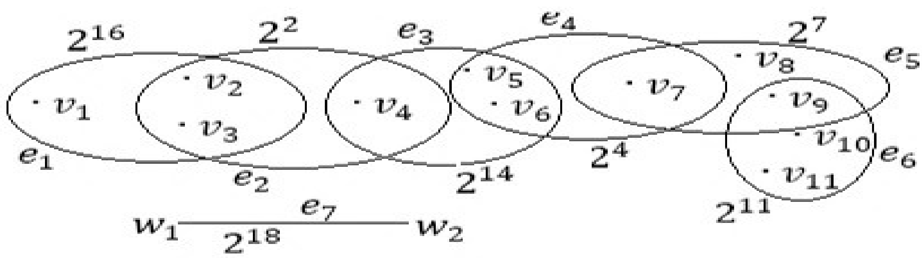

Figure 1. Example: 3.11

Theorem 3.12. *If $\mathcal{H}(V, E)$ is an open chain hypergraph of size $m \geq 7$. Then $\mathcal{EP}_n(\mathcal{H}) = 2$.*

Proof. Let $\mathcal{H}(V,E)$ be an open chain hypergraph of size $m \geq 7$. Let $v_i^1, v_i^2, \ldots, v_i^{s_i}$ be the pendant vertices in e_i, for $i = 1,2,\ldots,m$, where the number $s_i \in \mathbb{N}$ denotes the number of pendant vertices in e_i and $v_{i,i+1}^1, v_{i,i+1}^2, \ldots, v_{i,i+1}^{r_i}$, for $i = 1,2,\ldots,m-1$, be the non-pendant vertices in $\mathcal{H}$, where the number $r_i \in \mathbb{N}$ denotes the number of non-pendant vertices in $e_i \cap e_{i+1}$. Now let us assume that $\mathcal{EP}_n(\mathcal{H}) = 1$. Then the hypergraph $\mathcal{H} \cup K_2$ is an edge product hypergraph with the vertex set

$$V = \{v_1^1, v_1^2, \ldots, v_1^{s_1}, v_2^1, v_2^2, \ldots, v_2^{s_2}, \ldots, v_m^1, v_m^2, \ldots, v_m^{s_m},$$
$$v_{1,2}^1, v_{1,2}^2, \ldots, v_{1,2}^{r_1}, v_{2,3}^1, v_{2,3}^2, \ldots, v_{2,3}^{r_2}, \ldots, v_{m-1,m}^1, v_{m-1,m}^2, \ldots, v_{m-1,m}^{r_{m-1}}\} \cup \{w_1, w_2\}$$

and edge set

$$E = \{e_1, e_2, \ldots, e_m\} \cup \{e_{m+1}\},$$

where $e_1 = \{v_1^1, v_1^2, \ldots, v_1^{s_1}, v_{1,2}^1, v_{1,2}^2, \ldots, v_{1,2}^{r_1}\}$,
$e_i = \{v_i^1, v_i^2, \ldots, v_i^{s_i}, v_{i-1,i}^1, v_{i-1,i}^2, \ldots, v_{i-1,i}^{r_{i-1}}, v_{i,i+1}^1, v_{i,i+1}^2, \ldots, v_{i,i+1}^{r_i}\}$, for $2 \leq i \leq m-1$, $e_m = \{v_m^1, v_m^2, \ldots, v_m^{s_m}, v_{m-1,m}^1, v_{m-1,m}^2, \ldots, v_{m-1,m}^{r_{m-1}}\}$, $e_{m+1} = \{w_1, w_2\}$. The set of all elements of P are $\{p_1, p_2, \ldots, p_m, b\}$. The mapping $f : E \to P$ is an optimal edge function and F is the optimal edge product function of f. Let the optimal edge function f is defined by, $f(e_i) = p_i, 1 \leq i \leq m$ and $f(e_{m+1}) = b$. Then the optimal edge product function F of f will be

$$F(v_i^1) = F(v_i^2) = \ldots = F(v_i^{s_i}) = p_i, 1 \leq i \leq m$$
$$F(v_{i,i+1}^1) = F(v_{i,i+1}^2) = \ldots = F(v_{i,i+1}^{r_i}) = p_i p_{i+1} = t_i \text{ say }, 1 \leq i \leq m-1$$
$$F(w_1) = F(w_2) = b.$$

Since $p_i p_{i+1} \neq p_{i+1} p_{i+2}$, hence we have $t_1 \neq t_2, t_2 \neq t_3, \ldots, t_{m-2} \neq t_{m-1}$. Further according to the definition, the range of F is in P, so t_2 can be $f(e_1) = p_1$ and t_{m-2} can be $f(e_m) = p_m$. But we obtain $t_3 = t_4$ for some $m \geq 7$, which is a contradiction. Hence $\mathcal{EP}_n(\mathcal{H}) \geq 2$, for $m \geq 7$. Now consider the hypergraph $\mathcal{H} \cup 2K_2$ with the vertex set

$$V = \{v_1^1, v_1^2, \ldots, v_1^{s_1}, v_2^1, v_2^2, \ldots, v_2^{s_2}, \ldots, v_m^1, v_m^2, \ldots, v_m^{s_m},$$
$$v_{1,2}^1, v_{1,2}^2, \ldots, v_{1,2}^{r_1}, v_{2,3}^1, v_{2,3}^2, \ldots, v_{2,3}^{r_2}, \ldots, v_{m-1,m}^1, v_{m-1,m}^2, \ldots, v_{m-1,m}^{r_{m-1}}\} \cup \{w_1, w_2, w_3, w_4\}$$

and the edge set

$$E = \{e_1, e_2, \ldots, e_m\} \cup \{e_{m+1}, e_{m+2}\}.$$

Then there may arise two cases

Case-1: m is odd: Consider, $m = 2c+1$, for all $c \geq 3$. Let $A = c^2 + 1 + \frac{c(c+1)}{2}$ and the elements of $P = \{2^{c+j} : 1 \leq j \leq c\} \cup \{2^{A+k} : 0 \leq k \leq c\} \cup \{2^{A+2c}, 2^{A+2c+1}\}$. We define the edge function $f : E \to P$ as follows: $f(e_1) = 2^{A+c}$, $f(e_{2i}) = 2^{c+i}$, for $1 \leq i \leq c$, $f(e_{2i+1}) = 2^{A+c-i}$, for $1 \leq i \leq c$, $f(e_{m+1}) = 2^{A+2c}$, $f(e_{m+2}) = 2^{A+2c+1}$. Then edge product function F of f is

$$F(v_i^1) = F(v_i^2) = \ldots = F(v_i^{s_i}) = f(e_i), \text{ for } 1 \leq i \leq m$$
$$F(v_{2i-1,2i}^1) = F(v_{2i-1,2i}^2) = \ldots = F(v_{2i-1,2i}^{r_{2i-1}}) = f(e_{2i-1}) \times f(e_{2i})$$
$$= 2^{A+c-i+1} \times 2^{c+i} = 2^{A+2c+1} = f(e_{m+2}), \text{ for } 1 \leq i \leq c$$
$$F(v_{2i,2i+1}^1) = F(v_{2i,2i+1}^2) = \ldots = F(v_{2i,2i+1}^{r_{2i}}) = f(e_{2i}) \times f(e_{2i+1})$$
$$= 2^{c+i} \times 2^{A+c-i} = 2^{A+2c} = f(e_{m+1}), \text{ for } 1 \leq i \leq c$$
$$F(w_1) = F(w_2) = 2^{A+2c}$$
$$F(w_3) = F(w_4) = 2^{A+2c+1}.$$

Clearly the $F(v) = P$. Now the set P of labels which we have chosen is the union of three sets namely $P_1 = \{2^{c+1}, 2^{c+2}, \ldots, 2^{c+c}\}$, $P_2 = \{2^A, 2^{A+1}, \ldots, 2^{A+c}\}$ and $P_3 = \{2^{A+2c}, 2^{A+2c+1}\}$ such that $P = P_1 \cup P_2 \cup P3$ and we observe following properties:

a. $2^{c+1} \times 2^{c+2} = 2^{2c+3} > 2^{2c}$

b. product of all elements of $P_1 = 2^{c+1} \times 2^{c+2} \times \ldots \times 2^{2c} = 2^{c^2+\frac{c(c+1)}{2}} < 2^A < 2^{A+2c} < 2^{A+2c+1}$

c. $2^{c+1} \times 2^A > 2^{A+c}$.

d. $2^{c+1} \times 2^{c+2} \times 2^A > 2^{A+2c+1}$ that is product of two smallest elements of P_1 and the smallest element of P_2 is greater than largest element of P_3

e. $2^A \times 2^{A+1} > 2^{A+2c+1}$ that is the product of two smallest elements of P_2 is greater than largest element of P_3

f. $2^{c+1} \times 2^{A+2c} > 2^{A+2c+1}$ that is product of the smallest element of P_1 and the smallest element of P_3 is greater than the largest element of P_3.

Hence from above we conclude that, If the product of a collection of more than one element of P is in P then the collection consists of exactly two elements with one from P_1 and other from P_2 with the product being in P_3. And for that collection of two elements, we have the corresponding edges incident to a vertex $v \in V$. Hence $\mathcal{H} \cup 2K_2$ is an edge product hypergraph and its edge product number is 2.

Case-2: m is even:
consider $m = 2c$, for all $c \geq 4$. Let $A = c^2 + \frac{c(c+1)}{2}$.
The elements of $P = \{2^{c-1+j} : 1 \leq j \leq c\} \cup \{2^{A+j} : 1 \leq j \leq c\} \cup \{2^{A+2c-1}, 2^{A+2c}\}$.
Define the edge function $f : E \to P$ by, $f(e_{2i}) = 2^{c-1+i}$, for $1 \leq i \leq c$, $f(e_{2i-1}) = 2^{A+c+1-i}$, for $1 \leq i \leq c$. $f(e_{m+1}) = 2^{A+2c-1}$, $f(e_{m+2}) = 2^{A+2c}$. Then edge product function F of f will be

$$F(v_i^1) = F(v_i^2) = \ldots = F(v_i^{s_i}) = f(e_i), \text{ for } 1 \leq i \leq m$$

$$\begin{aligned} F(v_{2i-1,2i}^1) = F(v_{2i-1,2i}^2) = \ldots = F(v_{2i-1,2i}^{r_{2i-1}}) &= f(e_{2i-1}) \times f(e_{2i}) \\ &= 2^{A+c+1-i} \times 2^{c-1+i} = 2^{A+2c} = f(e_{m+2}), \text{ for } 1 \leq i \leq c \\ F(v_{2i,2i+1}^1) = F(v_{2i,2i+1}^2) = \ldots = F(v_{2i,2i+1}^{r_{2i}}) &= f(e_{2i}) \times f(e_{2i+1}) \\ &= 2^{c-1+i} \times 2^{A+c-i} = 2^{A+2c-1} = f(e_{m+1}), \text{ for } 1 \leq i \leq c \end{aligned}$$

$$F(w_1) = F(w_2) = f(e_{m+1}) = 2^{A+2c-1}$$

$$F(w_3) = F(w_4) = f(e_{m+2}) = 2^{A+2c}.$$

Clearly the range of an edge product function F is the elements of P and the rest of the proof is similar to the previous case.

□

Example 3.13. Let $\mathcal{H}(V, E)$ is an open chain hypergraph of size 7. Then $\mathcal{H} \cup 2K_2$ is an edge product hypergraph.

Theorem 3.14. *If $\mathcal{H}(V, E)$ is a closed chain hypergraph. Then $\mathcal{EP}_n(\mathcal{H}) \geq 2$.*

Proof. Without loss of generality, let us consider the closed chain hypergraph of smallest size. Let $\mathcal{H}(V, E)$ be a closed chain hypergraph of size 3. Suppose $\mathcal{EP}_n(\mathcal{H}) = 1$. Then the hypergraph $\mathcal{H} \cup K_2$ is an edge product hypergraph with vertex set

$$\begin{aligned} V = \{&v_1^1, v_1^2, \ldots, v_1^{s_1}, v_2^1, v_2^2, \ldots, v_2^{s_2}, v_3^1, v_3^2, \ldots, v_3^{s_3}, \\ &v_{1,2}^1, v_{1,2}^2, \ldots, v_{1,2}^{r_1}, v_{2,3}^1, v_{2,3}^2 \ldots, v_{2,3}^{r_2}, v_{3,1}^1, v_{3,1}^2, \ldots, v_{3,1}^{r_3}\} \cup \{w_1, w_2\} \end{aligned}$$

and the edge set

$$E = \{e_1, e_2, e_3\} \cup \{e_4\}$$

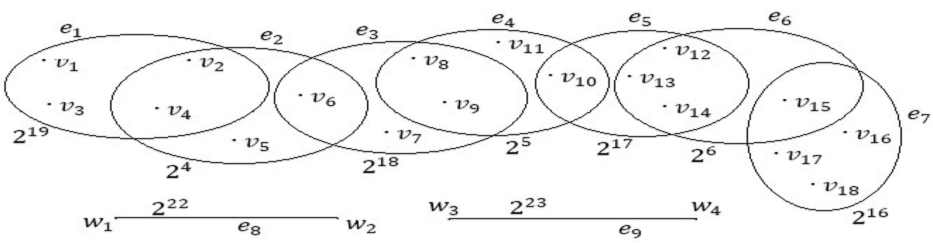

Figure 2. Example: 3.13

where

$$e_1 = \{v_1^1, v_1^2, \dots, v_1^{s_1}, v_{1,2}^1, v_{1,2}^2, \dots, v_{1,2}^{r_1}, v_{3,1}^1, v_{3,1}^2, \dots, v_{3,1}^{r_3}\},$$
$$e_2 = \{v_2^1, v_2^2, \dots, v_2^{s_2}, v_{1,2}^1, v_{1,2}^2, \dots, v_{1,2}^{r_1}, v_{2,3}^1, v_{2,3}^2, \dots, v_{2,3}^{r_2}\},$$
$$e_3 = \{v_3^1, v_3^2, \dots, v_3^{s_3}, v_{2,3}^1, v_{2,3}^2, \dots, v_{2,3}^{r_2}, v_{3,1}^1, v_{3,1}^2, \dots, v_{3,1}^{r_3}\},$$
$$e_4 = \{w_1, w_2\}.$$

Let the optimal edge function $f : E \to P$ defined by $f(e_i) = p_i$, for $1 \le i \le 3$ and $f(e_4) = b$. where $P = \{p_1, p_2, p_3, b\}$. Then optimal edge product function F of f will be

$$F(v_i^1) = F(v_i^2) = \dots = F(v_i^{s_i}) = p_i, \text{ for } 1 \le i \le 3$$
$$F(v_{i,i+1}^1) = F(v_{i,i+1}^2) = \dots = F(v_{i,i+1}^{r_i}) = p_i p_{i+1} = t_i \text{ say, for } 1 \le i \le 2$$
$$F(v_{3,1}^1) = F(v_{3,1}^2) = \dots = F(v_{3,1}^{r_3}) = p_3 p_1 = t_3,$$
$$F(w_1) = F(w_2) = b.$$

Since $p_i p_{i+1} \neq p_{i+1} p_{i+2}$, we have $t_1 \neq t_2, t_2 \neq t_3$. Now t_1 can be $f(e_3) = p_3$. Since range of F is in P, it follows $t_1 = p_3$, $t_2 = t_3 = b$, a contradiction. Hence $\mathcal{EP}_n(\mathcal{H}) \ge 2$. □

Example 3.15. Let $\mathcal{H}(V, E)$ be a closed chain hypergraph of size 3. Then $\mathcal{EP}_n(\mathcal{H}) = 2$.

Theorem 3.16. *If $\mathcal{H}(V, E)$ is a closed chain hypergraph of size $m \ge 4$. Then $\mathcal{EP}_n(\mathcal{H}) = 3$.*

Proof. Let $\mathcal{H}(V, E)$ be a closed chain hypergraph of size $m \ge 4$. By Theorem 3.14, $\mathcal{EP}_n(\mathcal{H}) \ge 2$. Now let us assume that $\mathcal{EP}_n(\mathcal{H}) = 2$. Then the hypergraph $\mathcal{H} \cup 2K_2$ is an edge product hypergraph. Consider the hypergraph $\mathcal{H} \cup 2K_2$ with vertex set

$$V = \{v_1^1, v_1^2, \dots, v_1^{s_1}, v_2^1, v_2^2, \dots, v_2^{s_2}, \dots, v_m^1, v_m^2, \dots, v_m^{s_m},$$
$$v_{1,2}^1, v_{1,2}^2, \dots, v_{1,2}^{r_1}, v_{2,3}^1, v_{2,3}^2, \dots, v_{2,3}^{r_2}, \dots, v_{m-1,m}^1, v_{m-1,m}^2, \dots,$$
$$v_{m-1,m}^{r_{m-1}}, v_{m,1}^1, v_{m,1}^2, \dots, v_{m,1}^{r_m}\} \cup \{w_1, w_2, w_3, w_4\}$$

and edge set

$$E = \{e_1, e_2, \dots, e_m\} \cup \{e_{m+1}, e_{m+2}\},$$

where $e_1 = \{v_1^1, v_1^2, \dots, v_1^{s_1}, v_{1,2}^1, v_{1,2}^2, \dots, v_{1,2}^{r_1}, v_{m,1}^1, v_{m,1}^2, \dots, v_{m,1}^{r_m}\}$,
$e_i = \{v_i^1, v_i^2, \dots, v_i^{s_i}, v_{i-1,i}^1, v_{i-1,i}^2, \dots, v_{i-1,i}^{r_{i-1}}, v_{i,i+1}^1, v_{i,i+1}^2, \dots, v_{i,i+1}^{r_i}\}$, for $2 \le i \le m-1$,

$$e_m = \{v_m^1, v_m^2, \dots, v_m^{s_m}, v_{m-1,m}^1, v_{m-1,m}^2, \dots, v_{m-1,m}^{r_{m-1}}, v_{m,1}^1, v_{m,1}^2, \dots, v_{m,1}^{r_m}\},$$

$e_{m+1} = \{w_1, w_2\}$, $e_{m+3} = \{w_3, w_4\}$.We prove two cases of theorem separately.

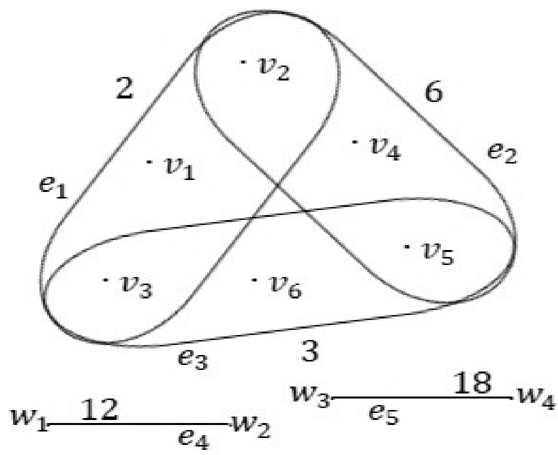

Figure 3. Example: 3.15

Case-1 m is even: Consider $m = 2c$, for all $c \geq 2$. Let the optimal edge function is defined by, $f(e_i) = p_i$, for $1 \leq i \leq m$ and $f(e_{m+1}) = b_1$, $f(e_{m+3}) = b_2$, where $P = \{p_1, p_2, p_3 \ldots, p_m, b_1, b_2\}$. Then the optimal edge product function will be

$$F(v_i^1) = F(v_i^2) = \ldots = F(v_i^{s_i}) = p_i, for 1 \leq i \leq m$$

$$F(v_{i,i+1}^1) = F(v_{i,i+1}^2) = \ldots = F(v_{i,i+1}^{r_i}) = p_i p_{i+1} = t_i \text{ say, for } 1 \leq i \leq m-1$$

$$F(v_{2c,1}^1) = F(v_{2c,1}^2) = \ldots = F(v_{2c,1}^{r_{2c}}) = p_{2c} \times p_1 = t_{2c}$$

$$F(w_1) = F(w_2) = b_1$$

$$F(w_3) = F(w_4) = b_2.$$

Since $p_i p_{i+1} \neq p_{i+1} p_{i+2}$, we have $t_i \neq t_{i+1}$, for $1 = 1, 2, \ldots, m-1$ and $t_{2c} \neq t_1$. It follows,

$$t_{2c} = t_2 = t_4 = \ldots = t_{2c-2}$$

and

$$t_1 = t_3 = t_5 = \ldots = t_{2c-1}$$

that is

$$p_{2c} \times p_1 = p_2 \times p_3 = p_4 \times p_5 = \ldots = p_{2c-2} \times p_{2c-1},$$

$$p_1 \times p_2 = p_3 \times p_4 = p_5 \times p_6 = \ldots = p_{2c-1} \times p_{2c}$$

$$\frac{p_{2c}}{p_2} = \frac{p_2}{p_4} = \frac{p_4}{p_6} = \ldots = \frac{p_{2c-2}}{p_{2c}}.$$

Suppose the last term $\frac{p_{2c-1}}{p_{2c}} > 2^0$, then we get,

$$p_{2c} > p_2 > p_4 \ldots > p_{2c-2} > p_{2c},$$

a contradiction. Hence the hypergraph $\mathcal{H} \cup 2K_2$ is not an edge product hypergraph. Therefore $\mathcal{EP}_n(\mathcal{H}) \geq 3$.

Now consider the hypergraph $\mathcal{H} \cup 3K_2$. Let

$$V = \{v_1^1, v_1^2, \ldots, v_1^{s_1}, v_2^1, v_2^2, \ldots, v_2^{s_2}, \ldots, v_m^1, v_m^2, \ldots, v_m^{s_m},$$
$$v_{1,2}^1, v_{1,2}^2, \ldots, v_{1,2}^{r_1}, v_{2,3}^1, v_{2,3}^2, \ldots, v_{2,3}^{r_2}, \ldots,$$
$$v_{m-1,m}^1, v_{m-1,m}^2, \ldots, v_{m-1,m}^{r_{m-1}}, v_{m,1}^1, v_{m,1}^2, \ldots, v_{m,1}^{r_m}\} \cup \{w_1, w_2, w_3, w_4, w_5, w_6\}$$

and $E = \{e_1, e_2, \ldots, e_m,\} \cup \{e_{m+1}, e_{m+2}, e_{m+3}\}$.
Let $A = c^2 + \frac{c(c+1)}{2}$. The elements of

$$P = \{2^{c+i} : 1 \leq i \leq c\} \cup \{2^{A+i} : 1 \leq i \leq c\} \cup \{2^{A+c+2}, 2^{A+2c+1}, 2^{A+2c+2}\}$$

Define the edge function $f : E \to P$ by

$$\begin{aligned} f(e_{2i-1}) &= 2^{c+i}, \text{ for } 1 \leq i \leq c, \\ f(e_{2i}) &= 2^{A+c+1-i}, \text{ for } 1 \leq i \leq c-1. \\ f(e_{2c}) &= 2^{A+1}, \\ f(e_{m+1}) &= 2^{A+c+2}, \\ f(e_{m+2}) &= 2^{A+2c+1}, \\ f(e_{m+3}) &= 2^{A+2c+2}. \end{aligned}$$

Then the edge product function F of f will be

$$\begin{aligned} F(v_i^1) = F(v_i^2) = \ldots = F(v_i^{s_i}) &= f(e_i), \text{for } 1 \leq i \leq m \\ F(v_{2i,2i+1}^1) = F(v_{2i,2i+1}^2) = \ldots = F(v_{2i,2i+1}^{r_{2i}}) &= f(e_{2i}) \times f(e_{2i+1}) \\ &= 2^{A+c+1-i} \times 2^{c+i+1} = 2^{A+2c+2} = f(e_{m+2}), \text{for } 1 \leq i \leq c-1 \\ F(v_{2i-1,2i}^1) = F(v_{2i-1,2i}^2) = \ldots = F(v_{2i-1,2i}^{r_{2i-1}}) &= f(e_{2i-1}) \times f(e_{2i}) \\ &= 2^{c+i} \times 2^{A+c+1-i} = 2^{A+2c+1} = f(e_{m+2}), \text{for } 1 \leq i \leq c-1 \\ F(v_{2c,1}^1) = F(v_{2c,1}^2) = \ldots = F(v_{2c,1}^{r_{2c}}) &= f(e_{2c}) \times f(e_1) \\ &= 2^{A+1} \times 2^{c+1} = 2^{A+c+2} = f(e_{m+1}) \\ F(v_{2c-1,2c}^1) = F(v_{2c-1,2i}^2) = \ldots = F(v_{2c-1,2i}^{r_{2c}}) &= f(e_{2c-1}) \times f(e_{2c}) \\ &= 2^{2c} \times 2^{A+1} = 2^{A+2c+1} = f(e_{m+2}) \end{aligned}$$

$$\begin{aligned} F(w_1) = F(w_2) = f(e_{m+1}) &= 2^{A+c+2} \\ F(w_3) = F(w_4) = f(e_{m+2}) &= 2^{A+2c+1} \\ F(w_5) = F(w_6) = f(e_{m+3}) &= 2^{A+2c+2}. \end{aligned}$$

Clearly, the range of F is P. Now the set P of labels which we have chosen is the union of three sets namely $P_1 = \{2^{c+1}, 2^{c+2}, \ldots, 2^{c+c}\}$, $P_2 = \{2^{A+1}, 2^{A+2}, \ldots, 2^{A+c}\}$ and $P_3 = \{2^{A+c+2}, 2^{A+2c+1}, 2^{A+2c+2}\}$ such that $P = P_1 \cup P_2 \cup P_3$ and we have the following observations:

a. $2^{c+1} \times 2^{c+2} > 2^{2c}$
b. $2^{c+1} \times 2^{c+2} \times \ldots \times 2^{c+c} < 2^{A+1}$
c. $2^{c+1} \times 2^{A+1} = 2^{A+c+2}$.
d. $2^{c+1} \times 2^{c+2} \times 2^{A+1} > 2^{A+2c+2}$ that is product of two smallest elements of P_1 and one smallest element of P_2 is greater than largest element of P_3.
e. $2^{A+1} \times 2^{A+2} > 2^{A+2c+2}$ that is the product of two smallest elements of P_2 is greater than largest element of P_3.
f. $2^{c+1} \times 2^{A+c+2} > 2^{A+2c+2}$ that is product of smallest element of P_1 and the smallest element of P_3 is greater than the largest element of P_3.
g. $2^{c+1} \times 2^{A+c} = 2^{A+2c+1}$.

Hence from above observations and after verifying all the possibilities of product of two or more labels of edges is in P, we conclude that, if the product of a collection of more than one element of P is in P then the collection consists of exactly two elements with one from P_1 and other from P_2 with the product being in P_3 . And for that collection of two elements, we have the corresponding edges incident to a vertex $v \in V$. Hence $\mathcal{H} \cup 3K_2$ is an edge product hypergraph and $\mathcal{EP}_n(\mathcal{H}) = 3$

Case-2 m is odd: Consider $m = 2c + 1$, for all $c \geq 2$.

Consider the hypergraph $\mathcal{H} \cup 2K_2$. Let the optimal edge function f is defined by, $f(e_i) = p_i, 1 \leq i \leq m$ and $f(e_{m+1}) = b_1$, $f(e_{m+3}) = b_2$, where $P = \{p_1, p_2, \ldots, p_m, b_1, b_2\}$. Then the optimal edge product function will be

$$F(v_i^1) = F(v_i^2) = \ldots = F(v_i^{s_i}) = p_i, \text{ for } 1 \leq i \leq m$$

$$F(v_{i,i+1}^1) = F(v_{i,i+1}^2) = \ldots = F(v_{i,i+1}^{r_i}) = p_i p_{i+1} = t_i \text{ say, for } 1 \leq i \leq m-1$$

$$F(v_{2c+1,1}^1) = F(v_{2c+1,1}^2) = \ldots = F(v_{2c+1,1}^{r_{2c+1}}) = p_{2c+1} \times p_1 = t_{2c+1}$$

$$F(w_1) = F(w_2) = b_1.$$
$$F(w_3) = F(w_4) = b_2.$$

Since $p_i p_{i+1} \neq p_{i+1} p_{i+2}$, we get $t_i \neq t_{i+1}$, for $1 = 1, 2, \ldots m-1$ and $t_{2c+1} \neq t_1$. It follows,

$$t_1 = t_3 = t_5 = \ldots = t_{2c+1}$$
$$t_2 = t_4 = t_6 = \ldots = t_{2c}.$$

But $t_{2c+1} = t_1 \Rightarrow p_{2c+1} \times p_1 = p_1 \times p_2$, which is a contradiction. Thus the hypergraph $\mathcal{H} \cup 2K_2$ is not an edge product hypergraph. Hence $\mathcal{EP}_n(\mathcal{H}) \geq 3$. Now we consider the hypergraph $\mathcal{H} \cup 3K_2$. Let

$$V(\mathcal{H}) = \{v_1^1, v_1^2, \ldots, v_1^{s_1}, v_2^1, v_2^2, \ldots, v_2^{s_2}, \ldots, v_m^1, v_m^2, \ldots, v_m^{s_m},$$
$$v_{1,2}^1, v_{1,2}^2, \ldots, v_{1,2}^{r_1}, v_{2,3}^1, v_{2,3}^2, \ldots, v_{2,3}^{r_2}, \ldots, v_{m-1,m}^1, v_{m-1,m}^2, \ldots,$$
$$v_{m-1,m}^{r_{m-1}}, v_{m,1}^1, v_{m,1}^2, \ldots, v_{m,1}^{r_m}\} \cup \{w_1, w_2, w_3, w_4, w_5, w_6\}$$

and $E(\mathcal{H}) = \{e_1, e_2, \ldots, e_m\} \cup \{e_{m+1}, e_{m+2}, e_{m+3}\}$. The set of all elements of

$$P = \{2^{a-1+i} : 1 \leq i \leq c\} \cup \{2^{b+c-1-i} : 1 \leq i \leq c-1\} \cup$$
$$\{2^{\frac{a+b}{2}}, 2^{\frac{a+b}{2}+c-1}\} \cup \{2^{a+\frac{a+b}{2}+c-1}, 2^{a+b+c-2}, 2^{a+b+c-1}\},$$

where $a = c + 2$, $b = 3c^2 + 8c + 8$. Define the edge function $f : E \to P$ by

$$f(e_{2i-1}) = 2^{a-1+i}, \text{ for } 1 \leq i \leq c,$$
$$f(e_{2i}) = 2^{b+c-1-i}, \text{ for } 1 \leq i \leq c-1,$$
$$f(e_{2c}) = 2^{\frac{a+b}{2}},$$
$$f(e_{2c+1}) = 2^{\frac{a+b}{2}+c-1},$$
$$f(e_{m+1}) = 2^{a+\frac{a+b}{2}+c-1},$$
$$f(e_{m+2}) = 2^{a+b+c-2},$$
$$f(e_{m+3}) = 2^{a+b+c-1}.$$

The rest of the proof is similar to the previous case. Hence $\mathcal{H} \cup 3K_2$ is an edge product hypergraph and $\mathcal{EP}_n(\mathcal{H}) = 3$ for all $m \geq 3$.

□

Example 3.17. Let $\mathcal{H}(V, E)$ be a closed chain hypergraph of size 6. Then $\mathcal{EP}_n(\mathcal{H}) = 3$.

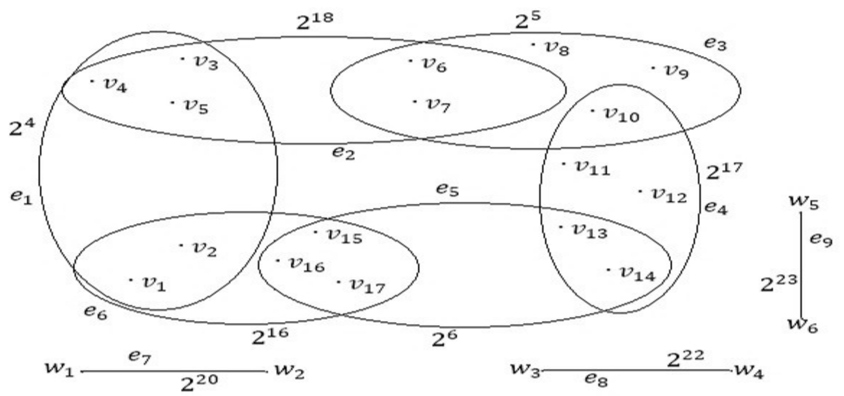

Figure 4. Example: 3.17

References

[1] C. Berge, *Graphs and Hypergraphs*, North-Holland, Amsterdam, 1973.

[2] C. Berge, *Hypergraphs, Combinatorics of Finite Sets*, North-Holland, Amsterdam, 1989.

[3] Bibin K. Jose, Zs. Tuza, *Hypergraph Domination and Strong Independence*, Appl. Anal. Discrete Math., **3** (2009), 347-–358.

[4] Gallian Joseph A, *A dynamic survey of graph labelling*, Electronic J. Combin., **16** (2000), 39–48.

[5] F. Harary, *Sum graphs and difference graphs*, Congress. Numer., **72** (1990), 101–108.

[6] Megha M. Jadhav and Kishor F. Pawar, *On Edge Product Hypergraphs*, Journal of Hyperstructures, **Accepted** (2021).

[7] A. Rosa, *On certain valuations of the vertices of a graph*, Theory of Graphs Inter Nat. Symposium, Rome, July 1966, Gordon and N.Y.Breach and Dunod, Paris, (1967) 349–355.

[8] Thavamani J. P. and Ramesh D.S.T., *Edge product graphs and its properties*, The IUP Journal of Computational Mathematics, **4** (2011), 30–38.

[9] J.P. Thavamani and D.S.T. Ramesh, *Edge product number of graphs in paths*, Journal of Mathematical Sciences & Computer Applications, **3** (2011), 78–84.

Author information

Megha M. Jadhav, Department of Mathematics, School of Mathematical Sciences, Kavayitri Bahinabai Chaudhari North Maharashtra University, Jalgaon - 425 001, (M.S.), INDIA.
E-mail: meghachalisgaon@gmail.com

Kishor F. Pawar, Department of Mathematics, School of Mathematical Sciences, Kavayitri Bahinabai Chaudhari North Maharashtra University, Jalgaon - 425 001, (M.S.), INDIA.
E-mail: kfpawar@nmu.ac.in

Received: October 2nd, 2021
Accepted: November 23rd, 2021