Palestine Journal of Mathematics

Vol. 12(1)(2023) , 285–299

NONUNIFORM AFFINE SYSTEMS IN $L^2(\mathbb{R})$

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Communicated by Harikrishnan Panackal

MSC 2010 Classifications: Primary 42C40, 42C15; Secondary 43A70, 65T60.

Keywords and phrases: Nonuniform affine frame; Sesquilinear operator; Fourier transform; Dual.

Abstract In this paper, we introduce the notion of nonuniform affine system, nonuniform quasi-affine system and nonuniform co-affine systems in $L^2(\mathbb{R})$. We provide the complete characterization of nonuniform affine frame and the corresponding nonuniform quasi-affine frames by means of an operator associated with the corresponding systems. Furthermore, we obtain the characterization of all such translation invariant operators. Finally, we show that there does not exist any frame associated to the nonuniform co-affine system in $L^2(\mathbb{R})$.

1 Introduction

Signals are in general non-stationary. A complete representation of non-stationary signals requires frequency analysis that is local in time, resulting in the time-frequency analysis of signals. In real life application all signals are not obtained from uniform shifts; so there is a natural question regarding analysis and decompositions of this types of signals by a stable mathematical tool. Gabardo and Nashed [18] and Gabardo and Yu [19] filled this gap by the concept of nonuniform multiresolution analysis and nonuniform wavelets based on the theory of spectral pairs for which the associated translation set $\Lambda = \{0, r/N\} + 2\mathbb{Z}$ is no longer a discrete subgroup of $\mathbb{R}$ but a spectrum associated with a certain one-dimensional spectral pair and the associated dilation is an even positive integer related to the given spectral pair.

The notion of frames was first introduced by Duffin and Shaeffer [15] in connection with some deep problems in non-harmonic Fourier series, and more particularly with the question of determining when a family of exponentials $\{e^{i\alpha_n t} : n \in \mathbb{Z}\}$ is complete for $L^2[a, b]$. Frames did not generate much interest outside non- harmonic Fourier series until the seminal work by Daubechies et al.[16]. After their pioneer work, the theory of frames began to be studied widely and deeply, particularly in the more specialized context of wavelet frames and Gabor frames. Frames provide a useful model to obtain signal decompositions in cases where redundancy, robustness, over-sampling, and irregular sampling play a role. Today, the theory of frames has become an interesting and fruitful field of mathematics with abundant applications in signal processing, image processing, harmonic analysis, Banach space theory, sampling theory, wireless sensor networks, optics, filter banks, quantum computing, and medicine. Sharma and Manchanda [28] presented a necessary and sufficient conditions for nonuniform wavelet frames in the frequency domain. For more about frames and their applications, we refer to [1, 2, 3, 4, 5, 6, 7, 8, 9, 13, 26, 27].

In order to obtain translation invariance of the discrete wavelet transform, the concept of quasi-affine frames in Euclidean spaces was introduced by Ron and Shen [25], where they proved that quasi-affine frames are invariant by translations with respect to elements of $\mathbb{Z}^n$. They also proved that if X is the affine system generated by a finite set $\Psi \subset L^2(\mathbb{R}^n)$ and associated with a dilation matrix A, and $\tilde{X}$ is the corresponding quasi-affine system, then X is an affine frame if and only if $\tilde{X}$ is a quasi-affine frame, provided the Fourier transforms of the functions in Ψ satisfy some mild decay conditions. Chui, Shi, and Stöckler [14] gave an alternative proof of this fact, and relaxing assumption of the decay conditions. Bownik [12] used this result to provide a new characterization of multiwavelets on $L^2(\mathbb{R}^n)$. Co-affine system is the another concept related to the above context which was initially defined in [20] for the case of $\mathbb{R}$ where the authors proved that the co-affine system can never be a frame for $L^2(\mathbb{R})$ and was subsequently extended to $L^2(\mathbb{R}^n)$ by Johnson [23]. Some of the other interesting articles dealing with these concepts can be found in [10, 11, 21, 22].

The main purpose of this paper is to introduce the notion of nonuniform affine system, nonuniform quasi-affine system and nonuniform co-affine systems in $L^2(\mathbb{R})$. We also define the frames associated with nonuniform affine and nonuniform quasi-affine systems. We show that the necessary and sufficient condition for a nonuniform affine system generated by a finite family of functions is a nonuniform affine frame is that the corresponding nonuniform quasi-affine system forms a nonuniform quasi-affine frame. This characterization is obatined by means of an operator associated with the corresponding systems known as Sesquilinear operator. Furthermore, we obtain the characterization of all such translation invariant operators. Finally, we show that there does not exist any frame associated to the nonuniform co-affine system in $L^2(\mathbb{R})$.

The remainder of the paper is organized as follows. In Section 2, we introduce the notion of nonuniform affine and nonuniform quasi-affine systems in Euclidean space and discuss frame properties of these systems via sesquilinear operators. we also show translation invariance of these operators. In Section 3, we define the nonuniform affine dual and nonuniform quasi-affine dual of a finite subset Ψ of $L^2(\mathbb{R})$ and show that a finite subset Φ of $L^2(\mathbb{R})$ with cardinality same as that of Ψ is an affine dual of Ψ if and only if it is a quasi-affine dual of Φ. In Section 4, nonuniform co-affine system in $L^2(\mathbb{R})$ is introduced and then we show that there does not exist a frame corresponding to nonuniform co-affine system.

2 Nonuniform Affine and Quasi-Affine Frames in $L^2(\mathbb{R})$

For an integer $N \geq 1$ and an odd integer r with $1 \leq r \leq 2N-1$ such that r and N are relatively prime, we define

$$\Lambda = \left\{0, \frac{r}{N}\right\} + 2\mathbb{Z} = \left\{\frac{rk}{N} + 2n : n \in \mathbb{Z}, k = 0, 1\right\}.$$

It is easy to verify that Λ is not necessarily a group nor a uniform discrete set, but is the union of $\mathbb{Z}$ and a translate of $\mathbb{Z}$. Moreover, the set Λ is the spectrum for the spectral set $\Upsilon = \left[0, \frac{1}{2}\right) \cup \left[\frac{N}{2}, \frac{N+1}{2}\right)$ and the pair (Λ, Υ) is called a *spectral pair* [17, 18, 19].

Definition 2.1. Fourier transform of a function $f \in L^2(\mathbb{R})$ is defined by

$$\mathcal{F}\{f(x)\}(\xi) = \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\xi}\, dx,$$

For $j \in \mathbb{Z}$ and $a, b \in \mathbb{R}$, let D_j, T_a and E_b and be the unitary operators acting on $f \in L^2(\mathbb{R})$ given by dilations, translations, and modulations, respectively:

$$\begin{aligned}
&T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad T_a f(x) = f(x-a) \quad \text{(Translation by } a\text{)},\\
&E_b : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad E_b f(x) = e^{2\pi i b x} f(x) \quad \text{(Modulation by } b\text{)},\\
&D_j : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad D_j f(x) = (2N)^{\frac{j}{2}} f((2N)^j x) \quad \text{(N-Dilation operator)}
\end{aligned}$$

Using the definition of the above operators, the following result can easily be proved.

Proposition 2.2. For $N \in \mathbb{N}$, $j \in \mathbb{Z}$ and $a \in \mathbb{R}$, we have the following properties hold:

$$\begin{aligned}
&(i) \quad D_j : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \textit{ is unitary map}.\\
&(ii) \quad D_j T_a f(x) = T_{(2N)^{-j}a} D_j f(x).\\
&(iii) \quad \mathcal{F}\left(D_j f(x)\right)(\xi) = D_{-j} \mathcal{F}\{f(x)\}(\xi).\\
&(iv) \quad \mathcal{F}\left(T_a f(x)\right)(\xi) = E_{-a} \mathcal{F}(f(x)(\xi).
\end{aligned}$$

For a given family of functions $\Psi = \{\psi_1, \psi_2, \ldots, \psi_{2N-1}\}$ in $L^2(\mathbb{R})$, define the *nonuniform affine system*

$$\mathcal{A}(\Psi) = \Big\{\psi_{\ell,j,\lambda}(x) : j \in \mathbb{Z}, \lambda \in \Lambda, 1 \le \ell \le 2N-1\Big\}, \tag{2.1}$$

where $\psi_{\ell,j,\lambda}(x) = (2N)^{j/2}\psi_\ell\left((2N)^j x - \lambda\right)$.The *nonuniform quasi-affine system* generated by $\Psi \subset L^2(\mathbb{R})$ is defined by

$$\mathcal{A}^q(\Psi) = \left\{\tilde{\psi}_{\ell,j,\lambda}(x) : j \in \mathbb{Z}, \lambda \in \Lambda, 1 \le \ell \le 2N-1\right\}, \tag{2.2}$$

where

$$\tilde{\psi}_{\ell,j,\lambda}(x) = \begin{cases} D_j T_\lambda \psi_\ell(x) = (2N)^{j/2}\psi_\ell\left((2N)^j x - \lambda\right), & j \ge 0, \lambda \in \Lambda, \\ \\ (2N)^{j/2} T_\lambda D_j \psi_\ell(x) = (2N)^j \psi_\ell\left((2N)^{-j}(x-\lambda)\right), & j < 0, \lambda \in \Lambda. \end{cases}$$

Definition 2.2. The nonuniform affine system $\mathcal{A}(\Psi)$ defined by (2.1) is called *nonuniform affine Bessel family* if there exists a constant $B > 0$ such that

$$\sum_{\eta \in \mathcal{A}(\Psi)} |\langle f, \eta\rangle|^2 \le B\|f\|_2^2 \quad \text{for all } f \in L^2(\mathbb{R}). \tag{2.3}$$

Definition 2.3. The nonuniform affine system $\mathcal{A}(\Psi)$ defined by (2.1) is called *nonuniform affine frame* if there exist constants A and B with $0 < A \le B < \infty$ such that

$$A\|f\|_2^2 \le \sum_{\eta \in \mathcal{A}(\Psi)} |\langle f, \eta\rangle|^2 \le B\|f\|_2^2, \tag{2.4}$$

holds for every $f \in L^2(\mathbb{R})$. The largest constant A and the smallest constant B satisfying (2.4) are called the *optimal lower* and *upper frame bounds*, respectively. A frame is a *nonuniform tight frame* if A and B are chosen so that $A = B$ and is a *nonuniform Parseval frame* if $A = B = 1$.

Similarly, the system $\mathcal{A}^q(\Psi)$ defined by (2.2) is called a *nonuniform quasi-affine Bessel family* if there exists a constant $\tilde{B} > 0$ such that (2.3) holds when B is replaced by $\tilde{B}$ and $\mathcal{A}(\Psi)$ is replaced by $\mathcal{A}^q(\Psi)$. It is called a *nonuniform quasi-affine frame* if there exists a constant $\tilde{A}$ and $\tilde{B} > 0$ such that(2.4) holds when A is replaced by $\tilde{A}$, B is replaced by $\tilde{B}$ and $\mathcal{A}(\Psi)$ is replaced by $\mathcal{A}^q(\Psi)$.

For two family of functions $\Psi = \{\psi_1, \psi_2, \ldots, \psi_{2N-1}\}$ and $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_{2N-1}\}$ in $L^2(\mathbb{R})$, we define a *sesquilinear operator* $\mathcal{S}_{\Psi,\Phi} : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to \mathbb{C}$ by

$$\mathcal{S}_{\Psi,\Phi}(f,g) = \sum_{\ell=1}^{2N-1} \sum_{j \in \mathbb{Z}} \sum_{\lambda \in \Lambda} \langle f, \psi_{\ell,j,\lambda}\rangle \langle \varphi_{\ell,j,\lambda}, g\rangle, \quad f, g \in L^2(\mathbb{R}). \tag{2.5}$$

It should be noted that if $\mathcal{A}(\Psi)$ and (Φ) are nonuniform affine Bessel families, then $\mathcal{S}_{\Psi,\Phi}$ defines a bounded operator. In the similar manner, we define the operator $\tilde{\mathcal{S}}_{\Psi,\Phi}$ by

$$\tilde{\mathcal{S}}_{\Psi,\Phi}(f,g) = \sum_{\ell=1}^{2N-1} \sum_{j \in \mathbb{Z}} \sum_{\lambda \in \Lambda} \langle f, \tilde{\psi}_{\ell,j,k}\rangle \langle \tilde{\varphi}_{\ell,j,k}, g\rangle, \quad f, g \in L^2(\mathbb{R}). \tag{2.6}$$

It is easy to see that $\mathcal{S}_{\Psi,\Phi}$ is dilation invariant, that is, $\mathcal{S}_{\Psi,\Phi}(D_J f, D_J g) = \mathcal{S}_{\Psi,\Phi}(f,g)$ for all $J \in \mathbb{Z}$, and $\tilde{\mathcal{S}}_{\Psi,\Phi}$ is invariant by translations with respect to λ, $\lambda \in \Lambda$. We write $\mathcal{S}_{\Psi,\Psi} = \mathcal{S}_\Psi$ and $\tilde{\mathcal{S}}_{\Psi,\Psi} = \tilde{\mathcal{S}}_\Psi$.

For $j \in \mathbb{Z}$, let $\mathcal{N}_j$ denotes a full collection of coset representatives of $\Lambda/(2N)^j\Lambda$, i.e.,

$$\mathcal{N}_j = \begin{cases} \{0,1,2,\dots,2(2N)^j-1\}, & j \geq 0 \\ \{0\}, & j<0. \end{cases}$$

For any $j \in \mathbb{Z}$ and $f,g \in L^2(\mathbb{R})$, we define

$$\mathcal{S}_j(f,g) = \sum_{\ell=1}^{2N-1} \sum_{\lambda\in\Lambda} \langle f, \psi_{\ell,j,\lambda}\rangle \langle \varphi_{\ell,j,k}, g\rangle$$

and

$$\tilde{\mathcal{S}}_j(f,g) = \sum_{\ell=1}^{2N-1} \sum_{\lambda\in\Lambda} \left\langle f, \tilde{\psi}_{\ell,j,\lambda}\right\rangle \left\langle \tilde{\varphi}_{\ell,j,\lambda}, g\right\rangle.$$

In order to prove main results in this paper, we first state and prove the following lemmas.

Lemma 2.1. Let $\Psi = \{\psi_1,\psi_2,\dots,\psi_{2N-1}\}$ and $\Phi = \{\varphi_1,\varphi_2,\dots,\varphi_{2N-1}\}$ be two family of functions in $L^2(\mathbb{R})$. For a fixed $L \in \mathbb{N}$, $j \geq -L$ and $f,g \in L^2(\mathbb{R})$, we have

$$\tilde{\mathcal{S}}_j(f,g) = (2N)^{-L} \sum_{\sigma\in\mathcal{N}_L} \mathcal{S}_j(T_\sigma f, T_\sigma g).$$

Proof. For $j \geq 0$, $\tilde{\mathcal{S}}_j(f,g) = \mathcal{S}_j(f,g) = \mathcal{S}_j(T_\sigma f, T_\sigma g)$ for any $\sigma \in \Lambda$. Now, for any integer j such that $-J \leq j < 0$, $\mathcal{S}_j$ is invariant with respect to translation by $(2N)^{-j}\sigma$, $\sigma \in \Lambda$. That is,

$$\mathcal{S}_j(T_{(2N)^j\sigma} f, T_{(2N)^j\sigma} g) = \mathcal{S}_j(f,g), \quad \sigma \in \Lambda.$$

Note that, for any $m \leq L$,

$$\sum_{\sigma=0}^{(2N)^L-1} a_\sigma = \sum_{\gamma=0}^{(2N)^m-1} \sum_{\beta=0}^{(2N)^{L-m}-1} a_{\beta(2N)^m+\gamma}.$$

Hence,

$$\sum_{\sigma\in\mathcal{N}_L} a_\sigma = \sum_{\sigma=(2N)^L}^{2(2N)^L-1} a_\sigma = \sum_{\gamma=0}^{(2N)^m-1} \sum_{\beta=0}^{(2N)^{L-m}-1} a_{\beta(2N)^m+\gamma+(2N)^L}.$$

Therefore, we have

$$\begin{aligned} (2N)^{-L} \sum_{\sigma\in\mathcal{N}_J} \mathcal{S}_j(T_\sigma f, T_\sigma g) &= (2N)^{-L} \sum_{\gamma=0}^{(2N)^{-j}-1} \sum_{\beta=0}^{(2N)^{L+j}-1} \mathcal{S}_j(T_{\beta(2N)^{-j}+\gamma+(2N)^L} f, T_{\sigma(2N)^{-j}+\gamma+(2N)^L} g) \\ &= (2N)^{-L} \sum_{\gamma=(2N)^{-j}}^{2(2N)^{-j}-1} \sum_{\sigma=(2N)^{L+j}-1}^{2(2N)^{L+j}-2} \mathcal{S}_j(T_{\beta(2N)^{-j}+\gamma} f, T_{\beta(2N)^{-j}+\gamma} g). \end{aligned}$$

Since

$$\gamma \in \left\{(2N)^{-j}, (2N)^{-j}+1, \dots, 2(2N)^{-j}-1\right\}$$

and

$$\beta \in \left\{(2N)^{J+j}-1, (2N)^{J+j}, 2(2N)^{J+j}-2\right\},$$

we have $\beta = (2N)^{L+j}+r$ and $\gamma = (2N)^{-j}+s$,
where

$$r \in \{-1,0,1,\dots,(2N)^{L+j}-2\},\ s \in \{0,1,\dots,(2N)^{-j}-1\}.$$

Hence,

$$\beta(2N)^{-j}+\gamma = (2N)^L + r(2N)^{-j} + (2N)^{-j} + s = \left((2N)^{L+j}+r+1\right)(2N)^{-j}+s,$$

so that

$$\beta(2N)^{-j} + \gamma = \left((2N)^{L+j} + r + 1\right)(2N)^{-j} + \left(\gamma - (2N)^{-j}\right).$$

Therefore, we obtain

$$\begin{aligned}
(2N)^{-L}\sum_{\sigma\in\mathcal{N}_J}\mathcal{S}_j(T_\sigma f, T_\sigma g) &= (2N)^{-L}\sum_{\gamma=(2N)^{-j}}^{2(2N)^{-j}-1}\ \sum_{\beta=(2N)^{L+j}-1}^{2(2N)^{L+j}-2}\mathcal{S}_j(T_{(\gamma-(2N)^{-j})}f, T_{(\gamma-(2N)^{-j})}g)\\
&= (2N)^{j}\sum_{\gamma=0}^{(2N)^{-j}-1}\mathcal{S}_j(T_\gamma f, T_\gamma g)\\
&= (2N)^{j}\sum_{\gamma=0}^{(2N)^{-j}-1}\mathcal{S}_j(T_{-\gamma} f, T_{-\gamma} g)\\
&= (2N)^{j}\sum_{\gamma=0}^{(2N)^{-j}-1}\sum_{\ell=1}^{2N-1}\sum_{\lambda\in\Lambda}\langle T_{-\gamma}f, \psi_{\ell,j,\lambda}\rangle\langle\varphi_{\ell,j,\lambda}, T_{-\gamma}g\rangle\\
&= \sum_{\gamma=0}^{(2N)^{-j}-1}\sum_{\ell=1}^{2N-1}\sum_{\lambda\in\Lambda}\langle f, \tilde{\psi}_{\ell,j,(2N)^{-j}\lambda+\gamma}\rangle\langle\tilde{\varphi}_{\ell,j,(2N)^{-j}\lambda+\gamma}, g\rangle\\
&= \sum_{\ell=1}^{2N-1}\sum_{\lambda\in\Lambda}\langle f, \tilde{\psi}_{\ell,j,\lambda}\rangle\langle\tilde{\varphi}_{\ell,j,\lambda}, g\rangle\\
&= \tilde{\mathcal{S}}_j(f,g).
\end{aligned}$$

This completes the proof of the lemma.

Now we proceed to state and prove the second lemma in which we prove two important properties of the operators $\mathcal{S}_j$ and $\tilde{\mathcal{S}}_j$ when $\Phi = \Psi$.

Lemma 2.2. Let $\Psi = \{\psi_1, \psi_2, \dots, \psi_{2N-1}\} \subset L^2(\mathbb{R})$. Put $\Phi = \Psi$ in the definitions of $\mathcal{S}_j$ and $\tilde{\mathcal{S}}_j$. If $f \in L^2(\mathbb{R})$ has compact support, then

(a) $\lim\limits_{J\to\infty}\sum\limits_{j<0}\tilde{\mathcal{S}}_j(D_J f, D_J f) = 0.$

(b) $\lim\limits_{J\to\infty}(2N)^{-J}\sum\limits_{j<-J}\sum\limits_{\sigma\in\mathcal{N}_J}\mathcal{S}_j(T_\sigma f, T_\sigma f) = 0.$

Proof. We have,

$$\sum_{j<0}\tilde{\mathcal{S}}_j(D_J f, D_J f) = \sum_{j<0}\sum_{\ell=1}^{2N-1}\sum_{\lambda\in\Lambda}|\langle D_J f, (2N)^{j/2}T_\lambda D_j\psi_\ell\rangle|^2.$$

Let $\Delta = \operatorname{supp} f$. Since

$$\begin{aligned}
\langle D_J f, T_\lambda D_j\psi_\ell\rangle &= \langle f, D_{-J}T_\lambda D_j\psi_\ell\rangle\\
&= \langle f, T_{(2N)^{-J}\lambda}D_{-J}D_j\psi_\ell\rangle\\
&= \langle T_{-(2N)^{-J}\lambda}f, D_{j-J}\psi_\ell\rangle.
\end{aligned}$$

we have,

$$\begin{aligned}
|\langle D_J f, T_\lambda D_j \psi_\ell\rangle|^2 &= \left|\int_{\mathbb{R}} f\left(x+(2N)^{-J}\lambda\right)\overline{D_{j-J}\psi_\ell(x)}dx\right|^2 \\
&\le \|f\|_2^2 \int_{\Delta-(2N)^{-J}\lambda} |D_{j-J}\psi_\ell(x)|^2 dx \\
&= \|f\|_2^2 \int_{\Delta-(2N)^{-J}\lambda} |(2N)^{(j-J)/2}\psi_\ell\left((2N)^{-(j-J)}x\right)|^2 dx \\
&= \|f\|_2^2 \int_{(2N)^{-(j-J)}(\Delta-(2N)^{-J}\lambda)} |\psi_\ell(x)|^2 dx \\
&= \|f\|_2^2 \int_{(2N)^{-(j-J)}\Delta-(2N)^{-j}\lambda} |\psi_\ell(x)|^2 dx.
\end{aligned}$$

Thus,

$$\sum_{j<0}\tilde{\mathcal{S}}_j(D_J f, D_J f) \le \|f\|_2^2 \sum_{j<0}(2N)^j \sum_{\lambda\in\Lambda}\int_{(2N)^{-(j-J)}\Delta-(2N)^{-j}\lambda}\sum_{\ell=1}^{2N-1}|\psi_\ell(x)|^2 dx.$$

Note that $(2N)^{-(j-J)}\Delta-(2N)^{-j}\lambda = (2N)^{-j}\left((2N)^J\Delta-\lambda\right)$. Since Δ is compact and $|(2N)^J\Delta| = (2N)^{-J}|\Delta|$, we can choose J_0 large enough so that $(2N)^J\Delta \subseteq [0,N]$ if $J \ge J_0$. Since $\{[0,N]+\lambda : \lambda\in\Lambda\}$ is a disjoint collection, it follows that $\left\{(2N)^{-j}\left((2N)^J\Delta-\lambda\right) : \lambda\in\Lambda\right\}$ is also a disjoint collection. Hence,

$$\begin{aligned}
\sum_{j<0}\tilde{\mathcal{S}}_j(D_J f, D_J f) &\le \|f\|_2^2 \sum_{j<0}(2N)^j \int_{\bigcup\limits_{\lambda\in\Lambda}(2N)^{-j}((2N)^J\Delta-\lambda)} \sum_{\ell=1}^{2N-1}|\psi^l(x)|^2 dx \\
&= \|f\|_2^2 \int_{\mathbb{R}} G_J(x)\sum_{\ell=1}^{2N-1}|\psi^l(x)|^2 dx,
\end{aligned}$$

where

$$G_J = \sum_{j<0}(2N)^j \chi_{\bigcup\limits_{\lambda\in\Lambda}(2N)^{-j}((2N)^J\Delta-\lambda)}.$$

Observe that $|G_J(x)| \le \sum\limits_{j<0}(2N)^j = \frac{1}{2N-1}$. Since $\sum\limits_{\ell=1}^{2N-1}|\psi^l|^2 \in L^1(0,N)$, if we can show that $G_J \to 0$ a.e. as $J\to\infty$, then by Lebesgue dominated convergence theorem, the last integral above will converge to 0 as $J\to\infty$.
Let $E = \{x\in\mathbb{R} : x = -(2N)^{-j}\lambda$ for some $j<0$ and $\lambda\in\Lambda\}$. If $x\notin E$, then $(2N)^j x+\lambda \ne 0$ for any $j<0$ and $\lambda\in\Lambda$ so that $|(2N)^j x+\lambda| = (2N)^\sigma$ for some $\sigma\in\mathbb{Z}$. Thus, $(2N)^j x+\lambda \notin (2N)^J\Delta$ if $J>-\sigma$. That is, $x\notin(2N)^{-j}((2N)^J\Delta-\lambda)$ if $J>-\sigma$. Since E is a set of measure zero, it follows that $G_J\to 0$ a.e. as $J\to\infty$. This proves part (a) of the lemma.

To prove part (b), we have

$$
\begin{aligned}
&(2N)^{-J}\sum_{j<-J}\sum_{\sigma\in\mathcal{N}_J}\mathcal{S}_j(T_\sigma f, T_\sigma f)\\
&= (2N)^{-J}\sum_{\sigma=(2N)^J}^{2(2N)^J-1}\sum_{j<-J}\sum_{\ell=1}^{2N-1}\sum_{\lambda\in\Lambda}|\langle T_\sigma f, \psi_{\ell,j,\lambda}\rangle|^2\\
&= (2N)^{-J}\sum_{\sigma=(2N)^J}^{2(2N)^J-1}\sum_{j<-J}\sum_{\ell=1}^{2N-1}\sum_{\lambda\in\Lambda}|\langle T_{\sigma-(2)^j\lambda} f, D_j\psi_{\ell}\rangle|^2\\
&\le (2N)^{-J}\|f\|_2^2\sum_{\sigma=(2N)^J}^{2(2N)^J-1}\sum_{j<-J}\sum_{\ell=1}^{2N-1}\sum_{\lambda\in\Lambda}\int_{\Delta+\sigma-(2N)^j\lambda}|D_j\psi_\ell(x)|^2dx\\
&= (2N)^{-J}\|f\|_2^2\sum_{\sigma=(2N)^J}^{2(2N)^J-1}\sum_{j<-J}\sum_{\ell=1}^{2N-1}\sum_{\lambda\in\Lambda}\int_{(2N)^{-j}(\Delta+\sigma-(2N)^j\lambda)}|\psi_\ell(x)|^2dx.\\
&\le (2N)^{-J}\|f\|_2^2\sum_{\sigma=(2N)^J}^{2(2N)^N-1}\sum_{j<-J}\sum_{\lambda\in\Lambda}\int_{(2N)^{-j}(\Delta+\sigma-(2N)^j\lambda)}\sum_{\ell=1}^{2N-1}|\psi_\ell(x)|^2dx\\
&= \|f\|_2^2\int_{\mathbb{R}}\mathcal{H}_J(x)\sum_{\ell=1}^{2N-1}|\psi_\ell(x)|^2dx,
\end{aligned}
$$

where

$$\mathcal{H}_J = (2N)^{-J}\sum_{\sigma=(2N)^J}^{2(2N)^J-1}\sum_{j<-J}\sum_{\lambda\in\Lambda}\chi_{(2N)^{-j}(\Delta+\sigma)-\lambda}.$$

As in part (a), to complete the proof, we need to show that $\mathcal{H}_J \to 0$ a.e. as $J\to\infty$.
Note that $(2N)^J\Delta\subseteq[0,N]$ if $J\ge J_0$. For such an J, consider the set $(2N)^J(\Delta+\sigma)$, where $\sigma\in\mathcal{N}_J$. If $x\in(2N)^J(\Delta+\sigma)$, then $x=y+(2N)^J\sigma$ for some $y\in(2N)^J\Delta\subseteq[0,N]$. Since $|y|\le 1$ and $|(2N)^J\sigma| = (2N)^{-J}\cdot(2N)^{J+1}=2N$. Thus, $(2N)^{-j}(\Delta+\sigma)\subseteq(2N)^{-j-J-1}[0,N)$ for any $j<J$.
For $J\ge J_0$, fix $j<-J$ and $\lambda\in\Lambda$. Then

$$(2N)^{-j}(\Delta+\lambda)-\lambda\subseteq(2N)^{-j}\left((2N)^{-J_0}[0,N)+\sigma\right)-\lambda.$$

Since the set Δ is compact, each $\Delta+\lambda_0$ can intersect with only finitely many sets of the form $\Delta+\lambda$, $\lambda\in\Lambda$. So there exists an integer $\omega\in\Lambda$ such that each $x\in\mathbb{R}$ can belong to at most ω such sets. Thus, in particular, any $x\in\mathbb{R}$ can belong to at most ω sets in the collection $\{(2N)^{-j}(\Delta+\sigma)-\lambda : \sigma\in\mathcal{N}_J\}$. Each of these sets is contained in $(2N)^{-j-J-1}[0,N)-\lambda$, hence so is their union. Thus,

$$\sum_{\sigma=(2N)^J}^{2(2N)^J-1}\chi_{(2N)^{-j}(\Delta+\sigma)-\lambda}\le\omega\chi_{(2N)^{-j-J-1}[0,N)-\lambda}.$$

Now,

$$
\begin{aligned}
\sum_{j<-J}\chi_{(2N)^{-j-J-1}[0,N)-\lambda} &= \sum_{i>0}\chi_{(2N)^{i-1}[0,N)-\lambda}\\
&= \chi_{\bigcup_{i>0}(2N)^{i-1}[0,N)-\lambda} = \chi_{[0,N)-\lambda}\le 1.
\end{aligned}
$$

Combining all the above estimates, we get

$$\mathcal{H}_J(x)\le(2N)^{-J}\cdot\omega.$$

Therefore, it follows that $\mathcal{H}_J(x) \to 0$ as $J \to \infty$ uniformly in x. This completes the proof of the lemma.

Theorem 2.1. Let $\Psi = \{\psi_1, \psi_2, \dots, \psi_{2N-1}\} \subset L^2(\mathbb{R})$. Then

(a) $\mathcal{A}(\Psi)$ is a Bessel family if and only if $\mathcal{A}^q(\Psi)$ is a Bessel family. Moreover, their exact upper bounds are equal.

(b) $\mathcal{A}(\Psi)$ is an nonuniform affine frame if and only if $\mathcal{A}^q(\Psi)$ is a nonuniform quasi-affine frame. Moreover, their exact lower and upper bounds are equal.

Proof. Taking $\Phi = \Psi$ in the definitions of $\mathcal{S}_j$ and $\tilde{\mathcal{S}}_j$. Suppose that $\mathcal{A}(\Psi)$ is a Bessel family with upper bound $B \geq 0$. Then, by Lemma 2.1, for all $f \in L^2(\mathbb{R})$, we have

$$\begin{aligned}
\tilde{\mathcal{S}}_\Psi(f,f) &= \sum_{j=-\infty}^{\infty} \tilde{\mathcal{S}}_j(f,f) = \lim_{L\to\infty} \sum_{j\geq -L} \tilde{\mathcal{S}}_j(f,f) \\
&= \lim_{L\to\infty} (2N)^{-L} \sum_{\sigma\in\mathcal{N}_L} \sum_{j\geq -L} \mathcal{S}_j(T_\sigma f, T_\sigma f) \\
&\leq \lim_{L\to\infty} (2N)^{-L} \sum_{\sigma\in\mathcal{N}_L} \mathcal{S}_\Psi(T_\sigma f, T_\sigma f) \\
&\leq \lim_{L\to\infty} (2N)^{-L} \sum_{\sigma\in\mathcal{N}_L} B\|T_\sigma f\|_2^2 = B\|f\|_2^2.
\end{aligned}$$

Thus, the nonuniform quasi-affine system $\mathcal{A}^q(\Psi)$ is also a Bessel family with upper bound B.

Conversely, let us assume that $\mathcal{A}^q X(\Psi)$ is a Bessel family with upper bound $C \geq 0$. Further, assume that there exists $f \in L^2(\mathbb{R})$ with $\|f\|_2 = 1$ and $\mathcal{S}_\Psi(f,f) > C$. We have

$$\begin{aligned}
\sum_{j=-J}^{\infty} \mathcal{S}_j(f,f) &= \sum_{j=-J}^{\infty} \sum_{\ell=1}^{2N-1} \sum_{\lambda\in\Lambda} |\langle f, \psi_{\ell,j,\lambda}\rangle|^2 \\
&= \sum_{j=0}^{\infty} \sum_{\ell=1}^{2N-1} \sum_{\lambda\in\Lambda} |\langle f, \psi_{\ell,j-J,k}\rangle|^2 \\
&= \sum_{j=0}^{\infty} \sum_{\ell=1}^{2N-1} \sum_{\lambda\in\Lambda} |\langle D_J f, \psi_{\ell,j,\lambda}\rangle|^2 \\
&= \sum_{j=0}^{\infty} \mathcal{S}_j(D_J f, D_J f).
\end{aligned}$$

Since $\mathcal{S}_\Psi(f,f) = \lim\limits_{J\to\infty} \sum\limits_{j=-J}^{\infty} \mathcal{S}_j(f,f) > C$, there exists $J \in \mathbb{N}$ such that

$$\sum_{j=-J}^{\infty} \mathcal{S}_j(f,f) = \sum_{j=0}^{\infty} \mathcal{S}_j(D_J f, D_J f) > C.$$

Now,

$$\tilde{\mathcal{S}}_\Psi(D_J f, D_J f) \geq \sum_{j=0}^{\infty} \tilde{\mathcal{S}}_j(D_J f, D_J f) = \sum_{j=0}^{\infty} \mathcal{S}_j(D_J f, D_J f) > C.$$

If $g = D_J f$, then we have $\|g\|_2 = \|D_J f\|_2 = \|f\|_2 = 1$ but $\tilde{\mathcal{S}}_\Psi(g,g) > C$. This is a contradiction to the fact that $\mathcal{A}^q(\Psi)$ is a nonuniform Bessel family with upper bound C. This proves part (a) of the theorem.

We will now prove part (b). We have already dealt with the upper bounds in part (a) of the theorem. Therefore, we need only to consider the lower bounds $\tilde{A}$ and A. Suppose that $\mathcal{A}(\Psi)$ is an nonuniform affine frame with lower frame bound A. Then, for all $f \in L^2(\mathbb{R})$ with compact support, we have

$$\begin{aligned}
\tilde{\mathcal{S}}_\Psi(f,f) &= \lim_{L\to\infty}(2N)^{-L}\sum_{\sigma\in\mathcal{N}_L}\sum_{j\geq -L}\mathcal{S}_j(T_\sigma f, T_\sigma f)\\
&= \lim_{L\to\infty}(2N)^{-L}\sum_{\sigma\in\mathcal{N}_L}\sum_{j\in\mathbb{Z}}\mathcal{S}_j(T_\sigma f, T_\sigma f) \quad \text{(by Lemma ??(b))}\\
&= \lim_{L\to\infty}(2N)^{-L}\sum_{\sigma\in\mathcal{N}_L}\mathcal{S}_\Psi(T_\sigma f, T_\sigma f)\\
&\geq \lim_{L\to\infty}(2N)^{-L}\sum_{\sigma\in\mathcal{N}_L}A\|T_\sigma f\|_2^2 = A\|f\|_2^2.
\end{aligned}$$

The set of all such f is dense in $L^2(\mathbb{R})$. So this holds for all $f \in L^2(\mathbb{R})$. Hence, $\tilde{A} \geq A$.
To show that $\tilde{A} \leq A$, we assume that it is not true and get a contradiction. Thus, there exists $\epsilon > 0$, $f \in L^2(\mathbb{R})$ with $\|f\| = 1$ such that $\mathcal{S}_\Psi(f,f) \leq \tilde{A} - \epsilon$.

Without loss of generality, we can assume that f has compact support. Since $\mathcal{S}_\Psi$ is dilation invariant, we also get

$$\mathcal{S}_\Psi(D_J f, D_J f) \leq \tilde{A} - \epsilon.$$

By Lemma 2.2 (a), there exists $J \in \mathbb{N}$ such that

$$\sum_{j<0}\tilde{\mathcal{S}}_j(D_J f, D_J f) < \epsilon/2.$$

Hence,

$$\begin{aligned}
\tilde{\mathcal{S}}_\Psi(D_J f, D_J f) &< \sum_{j\geq 0}\tilde{\mathcal{S}}_j(D_J f, D_J f) + \epsilon/2\\
&= \sum_{j\geq 0}\mathcal{S}_j(D_J f, D_J f) + \epsilon/2\\
&\leq \mathcal{S}_\Psi(D_J f, D_J f) + \epsilon/2 \leq \tilde{A} - \epsilon/2.
\end{aligned}$$

This contradicts the definition of the lower bound $\tilde{A}$ of $\tilde{\mathcal{A}}(\Psi)$ and completes the proof of the theorem.

We have seen that $\mathcal{S}_{\Psi,\Phi}$ is dilation invariant and $\tilde{\mathcal{S}}_{\Psi,\Phi}$ is invariant by translations with respect to λ, $\lambda \in \Lambda$. In the next theorem, we obtain the characterization of translation invariance of $\mathcal{S}_{\Psi,\Phi}$.

Theorem 2.2. Let $\Psi = \{\psi_1, \psi_2, \ldots, \psi_{2N-1}\}$ and $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_{2N-1}\}$ generate two nonuniform affine Bessel families. Then $\mathcal{S}_{\Psi,\Phi}$ is translation invariant if and only if $\mathcal{S}_{\Psi,\Phi} = \tilde{\mathcal{S}}_{\Psi,\Phi}$.

Proof. Suppose that $\mathcal{S}_{\Psi,\Phi}$ is translation invariant. Then, as in the proof of Theorem **??**, for all $f, g \in L^2(\mathbb{R})$ with compact support, we have

$$\begin{aligned}
\tilde{\mathcal{S}}_{\Psi,\Phi}(f,g) &= \lim_{L\to\infty}(2N)^{-L}\sum_{\sigma\in\mathcal{N}_L}\mathcal{S}_{\Psi,\Phi}(T_\sigma f, T_\sigma g)\\
&= \lim_{L\to\infty}(2N)^{-L}\sum_{\sigma\in\mathcal{N}_L}\mathcal{S}_{\Psi,\Phi}(f,g) = \mathcal{S}_{\Psi,\Phi}(f,g),
\end{aligned}$$

where we have used the translation invariance of $\mathcal{S}_{\Psi,\Phi}$. By density and the boundedness of the operators $\mathcal{S}_{\Psi,\Phi}$ and $\tilde{\mathcal{S}}_{\Psi,\Phi}$, the equality holds for all $f, g \in L^2(\mathbb{R})$.

Conversely, assume that $\mathcal{S}_{\Psi,\Phi} = \tilde{\mathcal{S}}_{\Psi,\Phi}$. Then for $m \in \Lambda$,

$$\begin{aligned}
\mathcal{S}_{\Psi,\Phi}(T_m f, T_m g) &= \tilde{\mathcal{S}}_{\Psi,\Phi}(T_m f, T_m g)\\
&= \sum_{\ell=1}^{2N-1}\sum_{j\geq 0}\sum_{\lambda\in\Lambda}\langle T_m f, D_j T_\lambda\psi_\ell\rangle\langle D_j T_\lambda\varphi_\ell, T_m g\rangle\\
&\quad + \sum_{\ell=1}^{2N-1}\sum_{j<0}\sum_{\lambda\in\Lambda}\langle T_m f, (2N)^{j/2}T_\lambda D_j\psi_\ell\rangle\langle(2N)^{j/2}T_\lambda D_j\varphi_\ell, T_m g\rangle.
\end{aligned}$$

Since $m \in \Lambda$, there exists a unique $m_0 \in Lambda$ such that $-m = m_0$. Therefore, in the first sum, we have

$$\begin{aligned} \langle T_m f, D_j \tau_\lambda \psi_\ell \rangle &= \langle f, T_{m_0} D_j T_\lambda \psi_\ell \rangle \\ &= \langle f, D_j T_{(2N)^j m_0 + \lambda} \psi_\ell \rangle. \end{aligned}$$

Similarly, we can write $\langle D_j T_\lambda \varphi_\ell, T_m g \rangle = \langle D_j T_{(2N)^j m_0 + \lambda} \varphi_\ell, g \rangle$.
For a fixed $j \geq 0$, we have $\{\lambda + (2N)^j m_0) : \lambda \in \Lambda\} = \{\lambda : \lambda \in \Lambda\}$. Therefore, for a fixed $j \geq 0$, we have

$$\begin{aligned} \sum_{\lambda \in \Lambda} \langle T_m f, D_j T_\lambda \psi_\ell \rangle \langle D_j T_\lambda \varphi_\ell, T_m g \rangle &= \sum_{\lambda \in \Lambda} \langle f, D_j T_\lambda \psi_\ell \rangle \langle D_j T_\lambda \varphi_\ell, g \rangle \\ &= \sum_{\lambda \in \Lambda} \langle f, \psi_{\ell,j,\lambda} \rangle \langle \varphi_{\ell,j,\lambda}, g \rangle. \end{aligned}$$

In the second sum, we have

$$\langle T_m f, (2N)^{j/2} T_\lambda D_j \psi_\ell \rangle = \langle f, (2N)^{j/2} T_{m_0 + \lambda} D_j \psi_\ell \rangle.$$

By a similar argument as above, we get for each $j < 0$,

$$\sum_{\lambda \in \Lambda} \langle T_m f, (2N)^{j/2} T_\lambda D_j \psi_\ell \rangle \langle (2N)^{j/2} T_\lambda D_j \varphi_\ell, T_m g \rangle = \sum_{\lambda \in \Lambda} \langle f, \tilde{\psi}_{\ell,j,\lambda} \rangle \langle \tilde{\varphi}_{\ell,j,\lambda}, g \rangle.$$

Hence, we have

$$\mathcal{S}_{\Psi,\Phi}(T_m f, T_m g) = \tilde{\mathcal{S}}_{\Psi,\Phi}(f, g) = \mathcal{S}_{\Psi,\Phi}(f, g).$$

This proves that $\mathcal{S}_{\Psi,\Phi}$ is invariant by translations with respect to m, where $m \in \Lambda$. Since $\mathcal{S}_{\Psi,\Phi}$ is invariant with respect to dilations, it follows that it is invariant with respect to all x of the form $x = (2N)^j m$, $m \in \Lambda$, $j \in \mathbb{Z}$. But such elements are dense in $\mathbb{R}$. This can be seen as follows. Since $\{[0, N] + m : m \in \Lambda\}$ is a partition of $\mathbb{R}$, $\{(2N)^j [0, N] + (2N)^j m : m \in \Lambda\}$ is also a partition of $\mathbb{R}$ for any $j \in \mathbb{Z}$. Hence, if $x \in \mathbb{R}$, then for each $j \in \mathbb{Z}$, there exists a unique $m \in \Lambda$ and $y \in [0, N]$ such that $x = (2N)^j y + (2N)^j m$ so that $|x - (2N)^j m| = |(2N)^j y|$. Since $|y| \leq 1$, we can choose j sufficiently large to make $|x - (2N)^j m|$ as small as we want. Since $\mathcal{S}_{\Psi,\Phi}$ is a bounded operator and translation is a continuous operation, it follows that $\mathcal{S}_{\Psi,\Phi}$ is invariant with respect to translation by all elements of $\mathbb{R}$. This completes the proof of the theorem.

3 Duals Nonuniform Affine and Quasi-Affime Systems in $L^2(\mathbb{R})$

We start this section with the definition of nonuniform affine duals.

Definition 3.1. Let $\Psi = \{\psi_1, \psi_2, \ldots, \psi_{2N-1}\}$ and $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_{2N-1}\}$ be two subsets of $L^2(\mathbb{R})$ such that $\mathcal{A}(\Psi)$ and $\mathcal{A}(\Phi)$ are Bessel families. Then Φ is called a *nonuniform affine dual* of Ψ if $\mathcal{S}_{\Psi,\Phi}(f, g) = \langle f, g \rangle$ for all $f, g \in L^2(\mathbb{R})$, that is,

$$\sum_{\ell=1}^{2N-1} \sum_{j \in \mathbb{Z}} \sum_{\lambda \in \Lambda} \langle f, \psi_{\ell,j,k} \rangle \langle \varphi_{\ell,j,k}, g \rangle \quad \text{for all } f, g \in L^2(\mathbb{R}). \tag{3.1}$$

We say that Φ is a *nonuniform quasi-affine dual* of Ψ if $\tilde{\mathcal{S}}_{\Psi,\Phi}(f, g) = \langle f, g \rangle$ for all $f, g \in L^2(\mathbb{R})$, that is,

$$\sum_{\ell=1}^{2N-1} \sum_{j \in \mathbb{Z}} \sum_{\lambda \in \Lambda} \langle f, \tilde{\psi}_{\ell,j,k} \rangle \langle \tilde{\varphi}_{\ell,j,k}, g \rangle \quad \text{for all } f, g \in L^2(\mathbb{R}). \tag{3.2}$$

Since $\mathcal{S}_{\Psi,\Phi}$ and $\tilde{\mathcal{S}}_{\Psi,\Phi}$ are sesquilinear operators, it follows from the polarization identity that (3.1) or (3.2) holds if and only if it holds for all $f = g$ in $L^2(K)$.

Theorem 3.1. Let $\Psi = \{\psi_1, \psi_2, \ldots, \psi_{2N-1}\} \subset L^2(\mathbb{R})$ generate a nonuniform affine Bessel family. Then $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_{2N-1}\}$ is an affine dual of Ψ if and only if it is a nonuniform quasi-affine dual of Ψ.

Proof. We first assume that Φ is a nonuniform affine dual of Ψ. So $\mathcal{S}_{\Psi,\Phi}(f,g) = \langle f, g\rangle$ for all $f, g \in L^2(\mathbb{R})$. Since $\langle T_y f, T_y g\rangle = \langle f, g\rangle$ for all $y \in \mathbb{R}$ and for all $f, g \in L^2(\mathbb{R})$, it follows that $\mathcal{S}_{\Psi,\Phi}$ is translation invariant. Hence, by Theorem 2.2, we have

$$\tilde{\mathcal{S}}_{\Psi,\Phi}(f,g) = \mathcal{S}_{\Psi,\Phi}(f,g) = \langle f, g\rangle \quad \text{for all } f, g \in L^2(\mathbb{R}).$$

Therefore, Φ is a nonuniform quasi-affine dual of Ψ.

Conversely, assume that Φ is a nonuniform quasi-affine dual of Ψ. Let $f \in L^2(\mathbb{R})$ be a function with compact support. By Lemma 2.2 (a), we have

$$\sum_{j<0} \tilde{\mathcal{S}}_j(D_J f, D_J f) \to 0 \text{ as } J \to \infty.$$

i.e.,

$$\sum_{j<0} \sum_{\ell=1}^{2N-1} \sum_{\lambda\in\Lambda} \langle D_J f, \tilde{\psi}_{\ell,j,\lambda}\rangle \langle \tilde{\varphi}_{\ell,j,\lambda}, D_J f\rangle \to 0 \text{ as } J \to \infty. \tag{3.3}$$

Now, since Φ is a nonuniform quasi-affine dual of Ψ, we have

$$\begin{aligned}
\|f\|^2 &= \|D_J f\|^2 \\
&= \langle D_J f, D_J f\rangle \\
&= \sum_{\ell=1}^{2N-1} \sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda} \langle D_J f, \tilde{\psi}_{\ell,j,\lambda}\rangle \langle \tilde{\varphi}_{\ell,j,\lambda}, D_J f\rangle \\
&= \sum_{\ell=1}^{2N-1} \sum_{j\geq 0} \sum_{\lambda\in\Lambda} \langle D_J f, \tilde{\psi}_{\ell,j,\lambda}\rangle \langle \tilde{\varphi}_{\ell,j,\lambda}, D_J f\rangle + \sum_{\ell=1}^{2N-1} \sum_{j<0} \sum_{\lambda\in\Lambda} \langle D_J f, \tilde{\psi}_{\ell,j,\lambda}\rangle \langle \tilde{\varphi}_{\ell,j,\lambda}, D_J f\rangle.
\end{aligned}$$

The second term goes to 0 as J goes to ∞, by (3.3). Hence,

$$\sum_{\ell=1}^{2N-1} \sum_{j\geq 0} \sum_{\lambda\in\Lambda} \langle D_J f, \tilde{\psi}_{\ell,j,\lambda}\rangle \langle \tilde{\varphi}_{\ell,j,\lambda}, D_J f\rangle \to \|f\|_2^2 \text{ as } J \to \infty. \tag{3.4}$$

But,

$$\begin{aligned}
\sum_{\ell=1}^{2N-1} \sum_{j\geq 0} \sum_{\lambda\in\Lambda} \langle D_J f, \tilde{\psi}_{\ell,j,\lambda}\rangle \langle \tilde{\varphi}_{\ell,j,\lambda}, D_J f\rangle &= \sum_{\ell=1}^{2N-1} \sum_{j\geq 0} \sum_{\lambda\in\Lambda} \langle D_J f, \psi_{\ell,j,\lambda}\rangle \langle \varphi_{\ell,j,\lambda}, D_J f\rangle \\
&= \sum_{\ell=1}^{2N-1} \sum_{j\geq 0} \sum_{\lambda\in\Lambda} \langle f, \psi_{\ell,j-J,\lambda}\rangle \langle \varphi_{\ell,j-J,\lambda}, f\rangle \\
&= \sum_{\ell=1}^{2N-1} \sum_{j\geq -J} \sum_{\lambda\in\Lambda} \langle f, \psi_{\ell,j,\lambda}\rangle \langle \varphi_{\ell,j,\lambda}, f\rangle.
\end{aligned}$$

Hence, by (3.4), we have

$$\sum_{\ell=1}^{2N-1} \sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda} \langle f, \psi_{\ell,j,\lambda}\rangle \langle \varphi_{\ell,j,\lambda}, f\rangle = \|f\|_2^2.$$

This shows that (3.1) holds for all $f = g$ with compact support. Since such functions are dense in $L^2(\mathbb{R})$, (3.1) holds for all $f = g$ in $L^2(\mathbb{R})$. This completes the proof of the theorem.

4 4. Nonuniform Co-Affine Systems in $L^2(\mathbb{R})$

We know that the nonuniform quasi-affine system $\mathcal{A}^q(\Psi)$ was obtained by reversing the dilation and translation operations for negative scales $j < 0$ and then by renormalizing from the nonuniform affine-system $\mathcal{A}(\Psi)$. Therefore, a natural question arises: what happens if we reverse these operations for each scale $j \in \mathbb{Z}$. In this section we answer this question. We first make the following definition.

Definition 4.1. Let $\Psi = \{\psi_1, \psi_2, \ldots, \psi_{2N-1}\} \subset L^2(\mathbb{R})$ and let the sequence of scalars be $\alpha = \{\alpha_{\ell,j} : 1 \leq \ell \leq 2N-1, j \in \mathbb{Z}\}$. The weighted *nonuniform co-affine system* $\mathcal{A}^*(\Psi, \alpha)$ generated by Ψ and α is defined as

$$\mathcal{A}^*(\Psi, \alpha) = \left\{\psi^*_{\ell,j,\lambda} = \alpha_{\ell,j} T_\lambda D_j \psi_\ell : 1 \leq \ell \leq 2N-1, j \in \mathbb{Z}, \lambda \in \Lambda\right\}. \tag{4.1}$$

Let $\mathcal{A}^*(\Psi, \alpha)$ be a weighted nonuniform co-affine system generated by Ψ and α. For $f \in L^2(\mathbb{R})$, define

$$\Gamma_f(x) = \sum_{\ell=1}^{2N-1} \sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda} |\langle T_x f, \mathcal{A}^*_{\ell,j,\lambda}\rangle|^2.$$

It is clear that $\Gamma_f(x+\lambda) = \Gamma_f(x)$ for all $\lambda \in \Lambda$ i.e., Γ_f is integral-periodic.

In order to prove the main result of this section, we first state and prove the following lemma

Lemma 4.1. If $\mathcal{A}^*(\Psi, \alpha)$ is a Bessel system for $L^2(\mathbb{R})$, then for each $f \in L^2(\mathbb{R})$, we have

$$\int_0^N \Gamma_f(x)dx = \int_{\mathbb{R}} \sum_{\ell=1}^{2N-1} \sum_{j\in\mathbb{Z}} |\alpha_{\ell,j}|^2 (2N)^{-j} \left|\hat{\psi}_\ell\left((2N)^{-j}\xi\right)\right|^2 \left|\hat{f}(\xi)\right|^2 d\xi.$$

Proof. By invoking Plancherel theorem, we have

$$\begin{aligned}
\int_0^N \Gamma_f(x)dx &= \int_0^N \sum_{\ell=1}^{2N-1} \sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda} |\langle T_x f, \alpha_{\ell,j} T_\lambda D_j \psi_\ell\rangle|^2 dx \\
&= \sum_{\ell=1}^{2N-1} \sum_{j\in\mathbb{Z}} |\alpha_{\ell,j}|^2 \int_0^N \sum_{\lambda\in\Lambda} |\langle f, T_{-x+\lambda} D_j \psi_\ell\rangle|^2 dx \\
&= \sum_{\ell=1}^{2N-1} \sum_{j\in\mathbb{Z}} |\alpha_{l,j}|^2 \int_{\mathbb{R}} |\langle f, T_{-x} D_j \psi_\ell\rangle|^2 dx \\
&= \sum_{\ell=1}^{2N-1} \sum_{j\in\mathbb{Z}} |\alpha_{\ell,j}|^2 \int_{\mathbb{R}} \left|\int_{\mathbb{R}} \hat{f}(\xi)(2N)^{-j/2} \overline{\hat{\psi}_\ell\left((2N)^j\xi\right)} \chi_x(\xi) d\xi\right|^2 dx \\
&= \sum_{\ell=1}^{2N-1} \sum_{j\in\mathbb{Z}} |\alpha_{\ell,j}|^2 (2N)^{-j} \int_{\mathbb{R}} \left|\left(\hat{f}\, \overline{\hat{\psi}_\ell\left((2N)^j\cdot\right)}\right)^\vee (x)\right|^2 dx \\
&= \sum_{\ell=1}^{2N-1} \sum_{j\in\mathbb{Z}} |\alpha_{l,j}|^2 (2N)^{-j} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 |\hat{\psi}_\ell\left((2N)^j\xi\right)|^2 d\xi.
\end{aligned}$$

This completes the proof of the lemma.

Now we proceed show that there does not exist any frame corresponding to nonuniform co-affine frame in $L^2(\mathbb{R})$.

Theorem 4.1. Let $\Psi = \{\psi_1, \psi_2, \ldots, \psi_{2N-1}\}$ be a family of functions in $L^2(\mathbb{R})$ and $\alpha = \{\alpha_{\ell,j} : 1 \leq \ell \leq 2N-1, j \in \mathbb{Z}\}$ be a sequence of scalars. Then $\mathcal{A}^*(\Psi, \alpha)$ defined by (4.1) cannot be a frame for $L^2(\mathbb{R})$.

Proof. Suppose that $\mathcal{A}^*(\Psi,\alpha)$ defined by (4.1) is a frame with bounds A_Ψ and B_Ψ. That is,

$$A_\Psi\|f\|_2^2 \le \sum_{\ell=1}^{2N-1}\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}|\langle f,\psi^*_{\ell,j,\lambda}\rangle|^2 \le B_\Psi\|f\|_2^2 \quad \text{for all } f\in L^2(\mathbb{R}).$$

On taking $f=\psi^*_{\ell_0,j_0,\lambda_0}$ for a fixed $j_0\in\mathbb{Z}$, $\lambda_0\in\Lambda$ and $1\le\ell_0\le 2N-1$, we have

$$\|\psi^*_{\ell_0,j_0,\lambda_0}\|^4 \le \sum_{\ell=1}^{2N-1}\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}|\langle\psi^*_{\ell_0,j_0,\lambda_0},\psi^*_{\ell,j,\lambda}\rangle|^2 \le B_\Psi\|\psi^*_{,\ell_0,j_0,\lambda_0}\|_2^2.$$

which implies $\|\psi^*_{\ell_0,j_0,\lambda_0}\|_2^2\le B_\Psi$.

Since $\|\psi^*_{\ell_0,j_0,\lambda_0}\|_2=|\alpha_{\ell_0,j_0}|\|\psi_{\ell_0}\|_2$, it follows that

$$|\alpha_{\ell_0,j_0}|^2 \le B_\Psi\|\psi_{\ell_0}\|_2^{-2} \quad \text{for all } j_0\in\mathbb{Z} \text{ and } 1\le\ell_0\le 2N-1. \tag{4.2}$$

By the definition of Γ_f, we have

$$A_\Psi\|f\|_2^2 \le \Gamma_f(x) \le B_\Psi\|f\|_2^2 \quad \text{for all } f\in L^2(\mathbb{R}).$$

Integrating over $[0,N]$ and using Lemma 4.1, we obtain

$$A_\Psi\|f\|_2^2 \le \int_{\mathbb{R}}\sum_{\ell=1}^{2N-1}\sum_{j\in\mathbb{Z}}|\alpha_{\ell,j}|^2(2N)^{-j}|\hat\psi_\ell\left((2N)^j\xi\right)|^2|\hat f(\xi)|^2d\xi \le B_\Psi\|f\|_2^2 \quad \text{for all } f\in L^2(\mathbb{R}).$$

which implies,

$$A_\Psi \le \sum_{\ell=1}^{2N-1}\sum_{j\in\mathbb{Z}}|\alpha_{\ell,j}|^2(2N)^{-j}|\hat\psi_\ell\left((2N)^j\xi\right)|^2 \le B_\Psi \quad \text{for a.e. } \xi\in\mathbb{R}.$$

Now, integrating over $[1,2N]$ after making the substitution $\xi\to(2N)^n\xi, n\in\mathbb{Z}$, we have

$$\begin{aligned} A_\Psi(2N-1) &\le \int_0^{2N}\sum_{\ell=1}^{2N-1}\sum_{j\in\mathbb{Z}}|\alpha_{\ell,j}|^2(2N)^{-j}|\hat\psi_\ell((2N)^{j+n}\xi)|^2d\xi \\ &= \int_0^{2N}\sum_{\ell=1}^{2N-1}\sum_{j\in\mathbb{Z}}|\alpha_{\ell,j-n}|^2(2N)^{-j+n}|\hat\psi_\ell((2N)^{j}\xi)|^2d\xi \\ &= (2N)^n\sum_{\ell=1}^{2N-1}\sum_{j\in\mathbb{Z}}\int_0^{(2N)^{j+1}}|\alpha_{\ell,j-n}|^2|\hat\psi_\ell(\xi)|^2d\xi. \end{aligned}$$

Using (4.2), we obtain

$$\begin{aligned} A_\Psi(2N-1) &\le (2N)^n\sum_{\ell=1}^{2N-1}\sum_{j\in\mathbb{Z}}B_\Psi\|\psi_\ell\|_2^{-2}\int_0^{(2N)^j}|\hat\psi_\ell(\xi)|^2d\xi \\ &= (2N)^n\sum_{\ell=1}^{2N-1}B_\Psi\|\psi_\ell\|_2^{-2}\sum_{j\in\mathbb{Z}}\int_0^{(2N)^j}|\hat\psi_\ell(\xi)|^2d\xi \\ &= (2N)^n\sum_{\ell=1}^{2N-1}B_\Psi \\ &= (2N)^n(2N-1)B_\Psi. \end{aligned}$$

Thus,

$$A_\Psi \le (2N)^nB_\Psi \quad \text{for each } n\in\mathbb{Z}.$$

By letting $n\to-\infty$, we see that $A_\Psi=0$. Hence, $\mathcal{A}^*(\Psi,\alpha)$ cannot be a frame for $L^2(\mathbb{R})$. This completes the proof of the theorem.

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Received: December 24th, 2021
Accepted: November 4th, 2022