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On new paranormed sequence spaces

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Abstract In this paper we introduce and study a new sequence space which is defined by lacunary almost convergent sequence . Further some inclusion relations, several topological properties have been considered which fill up a gap in the existing literature.

1 Introduction

Let s be the set of all sequences real or complex. By l_∞ and c, we denote the Banach spaces of bounded and convergent sequences $x = (x_k)$ normed by $\|x\| = \sup_k |x_k|$, respectively. Let D be the shift operator on s. That is,

$$Dx = \{x_n\}_{n=1}^{\infty}, \; D^2x = \{x_n\}_{n=2}^{\infty}, ...$$

and so on. It is evident that D is a bounded linear operator on ℓ_∞ onto itself and that $\left\|D^k\right\| = 1$ for every k.

It may be recalled that Banach limit L is a non-negative linear functional on ℓ_∞ such that L is invariant under the shift operator, that is, $L(Dx) = L(x) \; \forall x \in \ell_\infty$ and that $L(e) = 1$ where $e = (1, 1, 1, ...)$,(see, Banach [1]).

A sequence $x \in \ell_\infty$ is called almost convergent if all Banach Limits of x coincide. Various types of limits, including Banach limits, are considered in Das [1], Simons [10] and Sucheston [11]. Subsequently almost convergent sequences have been discussed in Duran[3], Kig[5], Nanda [8] and Schafere[9].

Let $\hat{c}$ denote the set of all almost convergent sequences. Lorentz [4] prove that

$$\hat{c} = \left\{x : \lim_m d_{mn}(x) \text{ exists uniformly in } n\right\}$$

where

$$d_{mn}(x) = \frac{x_n + x_{n+1} + ... + x_{n+m}}{m+1}$$

By a lacunary $\theta = (k_r)$; $r = 0, 1, 2, ...$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1}$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$.

Recently Das and Mishra[2] have introduced the space $\hat{c}_\theta$ of lacunary almost convergent sequences as follows:

$$\hat{c}_\theta = \left\{x : \lim_r t_{rn}(x - L) \text{ exists uniformly in } n\right\}$$

where

$$t_{rn}(x) = \frac{1}{h_r} \sum_{k \in I_r} x_{k+n}$$

Note that in the special case where $\theta = 2^r$, we have $\hat{c}_\theta = \hat{c}$.

The aim of this paper is to study a new sequence $\overline{N}_\theta$, which emerges naturally from the concept of almost convergence and lacunary sequence. We also consider the spaces $\overline{N}_\theta(p)$, which generalize $\overline{N}_\theta$ in the same way as $l(p)$ generalize l (see [10]). We discuss a related sequence space. Also some matrix transformation was characterized.

Let $\{p_r\}$ be bounded sequence of positive real numbers. We define

$$\overline{N}_\theta(p) = \left\{ x : \sum_r |t_{rn}|^{p_r} \quad \text{converges uniformly in n} \right\}$$

and

$$\overline{\overline{N}}_\theta(p) = \left\{ x : \sup_n \sum_r |t_{rn}|^{p_r} < \infty \right\}.$$

(Here and afterwards summation without limits runs from 0 to ∞). If $p_r = p$ for all r, we write $\overline{N}^p_\theta$ and $\overline{\overline{N}}^p_\theta$ in place of $\overline{N}_\theta(p)$ and $\overline{\overline{N}}_\theta(p)$ respectively. If $p = 1$, we write $\overline{N}_\theta$, $\overline{\overline{N}}_\theta$ for $\overline{N}_\theta(p)$ and $\overline{\overline{N}}_\theta(p)$ respectively.

It is now a natural question whether $\overline{N}_\theta(p) = \overline{\overline{N}}_\theta(p)$. We are only able to prove that $\overline{N}_\theta(p) \subset \overline{\overline{N}}_\theta(p)$. We have

Theorem 1.1. $\overline{N}_\theta(p) \subset \overline{\overline{N}}_\theta(p)$.

Proof. Suppose $x \in \overline{N}_\theta(p)$. Then there is a constant R such that

$$\sum_{r \geq R+1} |t_{rn}|^{p_r} \leq 1. \tag{1.1}$$

Hence it is enough to show that, for fixed r, $|t_{rn}|^{p_r}$ is bounded, or, equivalently, that t_{rn} is bounded. It follows from (1.1) that $|t_{rn}| \leq 1$ for $r \geq R + 1$ and all n. But if $r \geq 2$,

$$(h_r + 1)t_{rn} - h_r t_{r-1,n} = x_{k_r+n}. \tag{1.2}$$

Applying (1.2) with any fixed $r \geq R+1$ we deduce that x_{k_r+n} is bounded. Hence t_{rn} is bounded for all r. Thus the theorem is proved. □

Write $M = \max(1, \sup p_r)$. For $x \in \overline{N}_\theta(p)$ define

$$g_p(x) = \sup_n \Big(\sum_r |t_{rn}|^{p_r}\Big)^{\frac{1}{M}}; \tag{1.3}$$

this is exist because of Theorem 1.1. We write,

Theorem 1.2. *(i) $\overline{N}_\theta(p)$ is a complete linear topological space parnormed by g_p.*

(ii)$\overline{N}_\theta(p) \subset \overline{N}_\theta(q)$ for $p_r \leq q_r$.

Proof. It can be proved by "standart" arguments that g_p is a paranorm on $\overline{N}_\theta(p)$ and also, with the paranorm topology, the space $\overline{N}_\theta(p)$ is complete. As one step in the proof we shall only show that for fixed x, $\lambda x \to 0$ as $\lambda \to 0$. If $x \in \overline{N}_\theta(p)$, then given $\epsilon > 0$ there is a R such that, for all r

$$\sum_{r \geq R} |t_{rn}|^{p_r} < \varepsilon. \tag{1.4}$$

So if $0 < \lambda \leq 1$, then

$$\sum_{r \geq R} |t_{rn}(\lambda x)|^{p_r} \leq \sum_{r \geq R} |t_{rn}(x)|^{p_r} \leq \varepsilon,$$

and since, for fixed R,

$$\sum_{r=0}^{R-1} |t_{rn}(\lambda x)|^{p_r} \to 0$$

as $\lambda \to 0$, this completes proof. If $p_r = p$ for all r, then g_p is a norm for $p \geq 1$ and p-norm for $0 \leq p \leq 1$. To prove (ii), let $x \in \overline{N}_\theta(p)$. Then there is an integer R such that (1.1) holds. Hence for $r \geq R$, $|t_{rn}| \leq 1$. So that

$$|t_{rn}|^{q_r} \leq |t_{rn}|^{p_r}$$

and this completes the proof. □

Theorem 1.3. (i) *Let* $inf\ p_r > 0$. *Then* $\overline{\overline{N}}_\theta(p)$ *is a complete linear topological space paranormed by* g_p.

(ii) $\overline{\overline{N}}_\theta(p) \subset \overline{\overline{N}}_\theta(q)$ *for* $p_r \leq q_r$.

Proof. (i) It can be proved by " standard " arguments. It may, however, be noted that there is an essential difference between the proof of Theorem 1.3(i) and that of Theorem 1. 2(i). If we are given that$x \in \overline{\overline{N}}_\theta(p)$ we can not assert (1.4). We now use the assumption that $inf p_r > 0$.
Let $\rho = inf p_r > 0$. Then for $|\lambda| \leq 1$, $|\lambda|^{p_r} \leq |\lambda|^\rho$, so that $g_p(\lambda x) \leq |\lambda|^\rho\ g_p(x)$. The result new clearly follows.
(ii) The proof differs from that of Theorem 1. 2(ii), since we can not assert (1.1). If $x \in \overline{\overline{N}}_\theta(p)$, then $\sum_r |t_{rn}|^{p_r}$ is bounded. So t_{rn} is bounded for all r, n; say $|t_{rn}| \leq S$. We may suppose that $S \geq 1$. Then

$$\sum_r |t_{rn}|^{q_r} \leq \sum_r S^{q_r - p_r} |t_{rn}|^{p_r} \leq R^M \sum_r |t_{rn}|^{p_r}.$$

Hence, the result follows.

□

2 Topological Results

In this section we discuss some topological results for $\overline{N}_\theta(p)$ and $\overline{\overline{N}}_\theta(p)$ we now prove

Theorem 2.1. *If a set* $B \subset \overline{\overline{N}}_\theta(p)$ *is compact, then given* $\varepsilon > 0$ *there is some* $j_0 = j_0(\varepsilon)$ *such that for all* n

$$\Big(\sum_{r=j+1}^{\infty} |t_{rn}(x)|^{p_r}\Big)^{\frac{1}{M}} < \varepsilon,\ \forall x \in B, j \geq j_0.$$

Proof. Let $\varepsilon > 0$ be given and for every $x \in \overline{\overline{N}}_\theta(p)$, let

$$U(x, \tfrac{\epsilon}{2}) = \{y \in \overline{\overline{l}}(p) : g_p(y - x) < \tfrac{\epsilon}{2}\}.$$

Then $\{U(x, \frac{\epsilon}{2})\}_{x \in \overline{N}_\theta(p)}$ is a open cover for B. Since B is compact, there is $x^1, x^2, \cdots, x^N \in B$ such that

$$B \subset \bigcup_{i=1}^{N} U(x^i, \frac{\epsilon}{2}).$$

For each i, there is j_i such that

$$\Big(\sum_{r=j+1}^{\infty} |t_{rn}(x^i)|^{p_r}\Big)^{\frac{1}{M}} < \frac{\epsilon}{2}, for j \geq j_i.$$

Let $j_0 = \max_{1 \leq i \leq N} j_i$. Then

$$(\sum_{r=j+1}^{\infty}|t_{rn}(x^i)|^{p_r})^{\frac{1}{M}} < \frac{\epsilon}{2}, for j \geq j_0.$$

Since $x \in B$ there is $i_0(1 \leq i_0 \leq N)$ such that,

$$(\sum_{r=j+1}^{\infty}|t_{rn}(x-x^{i0})|^{p_r})^{\frac{1}{M}} < \frac{\epsilon}{2}.$$

Let $j \geq j_0$. Then

$$(\sum_{r=j+1}^{\infty}|t_{rn}(x)|^{p_r})^{\frac{1}{M}} \leq (\sum_{r=0}^{\infty}|t_{rn}(x-x^{i0})|^{p_r})^{\frac{1}{M}} + (\sum_{r=j+1}^{\infty}|t_{rn}(x^{i0})|^{p_r})^{\frac{1}{M}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This completes the proof.

We now study local boundedness and β- convexity for $\overline{N}_\theta(p)$. For $0 \leq \beta \leq 1$, a nonvoid subset U of a linear space is said to be absolutely β- convex if $x, y \in U$ and $|\lambda|^\beta + |\mu|^\beta \leq 1$ together imply that $\lambda x + \mu y \in U$. A linear topological space X is said to be β-convex if every neighbourhood of $0 \in X$ contains an absolutely β- convex neigbourhood of $0 \in X$. A subset B of X is said to be bounded if for each neigbourhood U of $0 \in X$ there is a bounded neigbourhood of $0 \in X$.

□

Theorem 2.2. *Let* $0 < p_r \leq 1$. *Then (i)* $\overline{N}_\theta(p)$ *is locally bounded if* $inf p_r > 0$, *(ii)* $\overline{N}_\theta(p)$ *is* β-*convex for all* β *where* $0 < \beta < \liminf p_r$.

Proof. Let $\inf p_r = \theta > 0$. If $x \in \overline{N}_\theta(p)$ there is a constant $R > 0$ such that for all n

$$\sum_r |t_{rn}(x)|^{p_r} \leq R.$$

For this R and given $\delta > 0$, choose an integer $N > 1$ such that

$$N^\theta \geq \tfrac{R}{\delta}.$$

Since $\frac{1}{N} < 1$ and $p_r \geq \theta$ we have

$$\tfrac{1}{N^{p_r}} \leq \tfrac{1}{N^\theta}.$$

Therefore we have

$$\sum_r |t_{rn}(\frac{x}{N})|^{p_r} \leq \frac{R}{N^\theta} \leq \delta.$$

Thus

$$\{x : g_p(x) \leq R \subset Nx : g_p(x) \leq \delta\}.$$

This completes the proof of (i). To prove (ii), let $\beta \in (0, \liminf p_r)$. Then there is n_0 such that $\beta \leq p_r$ for $r > r_0$. Define

$$\bar{g}(x) = \sup[\sum_{r=1}^{r_0}|t_{rn}(x)|^\beta + \sum_{r=r_0+1}^{\infty}|t_{rn}(x)|^{p_r}].$$

Since $p_r \leq 1$ and $\beta \leq p_r$ for $r > r_0$, $\bar{g}$ is sub- additive and for $r > r_0$

$$|\lambda|^{p_r} \leq |\lambda|^{\beta}\,, \quad (0 < |\lambda| \leq 1)$$

and so

$$\bar{g}(\lambda x) \leq |\lambda|^{\beta}\bar{g}(x)\,, \quad (0 < |\lambda| \leq 1.)$$

Therefore for $0 < \delta < 1$, $\cup = \{x : \bar{g}(x) \leq \delta\}$ is an absolutely β- convex set and this completes the proof.

□

3 Matrix Transformation

Let A=(a_{nk}) be an infinite matrix of complex numbers. Let X and Y be any two subsets of space of all sequences of complex numbers. We write Ax = $(A_n(x))$ if $A_n(x)$ = $\sum_k a_{nk}x_k$ converges for each n.
If $x \in X$ implies that $Ax \in Y$, then we say that A defines a matrix transformation from X into Y and we denote it by $A : X \to Y$. By (X, Y) we mean the class of matrices A such that $A : X \to Y$. If in X and Y there is some notion of limit or sum, the we write (X,Y,P) to denote the subset of (X,Y) which preserves the limit or sum.

We now characterize some matrix transformations connecting $\overline{\overline{N}}_{\theta}^{p}$.
We write,

$$t_{rn}(Ax) = \frac{1}{h_r}\sum_{i \in I_r} A_{n+i}(x) = \sum_k a(n,k,r)x_k$$

where

$$a(n,k,m) = \frac{1}{h_r}\sum_{i \in I_r} a_{n+i,k}.$$

We have

Theorem 3.1. *Let* $1 \leq p < \infty$. *Then* $A \in (l_1, \overline{\overline{N}}_{\theta}^{p})$ *if and only if*

$$\sup_{n,k}\sum_r |a(n,k,r)|^p < \infty \tag{3.1}$$

$A \in (l_1, \overline{\overline{N}}_{\theta}^{1}, P)$ *if and only if (3.1) with p=1 holds and*

$$\sum_r a(n,k,r) = 1 \quad \textit{(for all n,k)} \tag{3.2}$$

$(l_1, \overline{\overline{N}}_{\theta}^{1}, P)$ *is closed and convex in* $(l_1, \overline{\overline{N}}_{\theta}^{1})$.

The proof is omitted.

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