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Power GCDQ and LCMQ Matrices Defined on GCD-Closed Sets over Euclidean Domains

Yahia AWAD[1]*, Ragheb MGHAMES[2] and Haissam CHEHADE[3]

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Abstract In this paper, we present a full generalization for the power $GCDQ$ and $LCMQ$ matrices defined on q-ordered gcd-closed sets over Euclidean Domains. Structure theorems, determinants, reciprocals, inverses, and p-norms are also presented. In addition, Some examples are given in the Euclidean domain $\mathbb{Z}[i]$.

1 Introduction

Let $T = \{t_1, t_2, \ldots, t_m\}$ be a well ordered set of m distinct positive integers with $t_1 < t_2 < \cdots < t_m$. If r is any real number, then the power GCD matrix defined on T is $(T^r)_{m\times m} = (t_i, t_j)^r$, where (t_i, t_j) is the greatest common divisor of t_i and t_j, and the power LCM matrix on T is $[T^r]_{m\times m} = [t_i, t_j]^r$, where $[t_i, t_j]$ is the least common multiple of t_i and t_j. Set T is said to be factor-closed if $t \in T$ for any divisor t of $t_i \in T$, and it is gcd-closed if $(t_i, t_j) \in T$, for all t_i and t_j in T. In 1876, Smith [20] showed that if T is factor-closed then $\det(T) = \prod_{i=1}^{m} \phi(t_i)$ and $\det[T] = \prod_{i=1}^{m} \phi(t_i)\pi(t_i)$, where ϕ is Euler's totient function and π is a multiplicative function such that $\pi(p^k) = -p$. In 1988/92, Beslin and Ligh [3, 4, 5, 6, 19] factorized the GCD and LCM matrices if T is gcd-closed set, and computed their determinants. Later, in 1992, Borque and Ligh [7] conjectured that the LCM matrix on a gcd-closed set is invertible. In 1996, Chun [8] introduced the concept of power GCD and LCM matrices and gave general formulas for their structures, determinants and inverses over the domain of natural numbers. In 1998, Hong [15, 16] showed inductively that if T is gcd-closed and $m \leq 3$, then $\det(T)$ divides $\det[T]$. Otherwise, if $m \geq 4$ then there exist a gcd-closed set such that $\det(T)$ does not divide $\det[T]$. In 1990, Li [18] gave a generalization of Smith's determinant by obtaining the value of $\det(T)$ if T is defined on arbitrary ordered sets of distinct positive integers. In 2009, Hong et al. [17] generalized the power GCD matrices defined on factor-closed sets over unique factorization domains. In 1996, Haukkanen and Sillanpaa [14] studied the GCD and LCM matrices defined on lcm-closed and gcd-closed sets. In 1997, Haukkanen [13], in his famous paper "On Smith's Determinant" gave a counter example for the conjecture of Bourque-Ligh that the least common multiple matrix on any gcd-closed set is invertible. El-Kassar et al. [9, 10, 11, 12] extended many results concerning GCD and LCM matrices defined on factor-closed sets to arbitrary principal ideal domains. Recently, Awad et al. [2] gave a generalization for the power GCD and LCM matrices defined on gcd-closed sets over unique factorization domains, where all the obtained results in the previously published articles are considered as a special case of the presented generalization if the unique factorization domain R is taken to be the domain of natural integers $\mathbb{Z}$. Now, since there are no measures in unique factorization domains and the p-ordering process used in [2] is somehow complicated, so it is better to use new measures in order to make the ordering process more clear and sharper. Moreover, it is well-known that every Euclidean domain is a unique factorization domain but the converse is not true. In [1], Awad et al. generalized the Reciprocal power $GCDQ$ matrices and power $LCMQ$ matrices defined on arbitrary and factor-closed q-ordered sets are presented over any Euclidean domain S with measure $q(x)$ for every $x \in S$, where $q(x)$ is the norm of x defined on S. In our paper, we generalize all the obtained results in [2] to any Euclidean domain S by modifying the Jordan totient function over Euclidean domains with measure function $q(x)$. First, we generalize the power $GCDQ$ and $LCMQ$ matrices defined on q-ordered gcd-closed sets over any Euclidean domain S. Then, we present complete characterizations for their decompositions, determinants, reciprocals, and inverses. In addition, some examples in the Euclidean domain $\mathbb{Z}[i]$ are given.

In what follows, let S be an Euclidean domain of measure q with a complete prime residue system $P = \{p_{1,}p_2, \ldots, p_m\}$, and $T = \{t_1, t_2, \ldots, t_m\}$ be a set of non-zero non-associate elements in S with measure q. If $\{p_1, p_2, \ldots, p_i, \ldots\}$ is a well-ordered listing of all primes in P of S that divide all the elements of T, the q-ordering $<_q$ on S is defined as follows: $t_i <_q t_j$ if $q(t_i) < q(t_j)$, which is a well-defined linear ordering on S. Hence, if the set $T = \{t_1, t_2, \ldots, t_m\}$ such that $t_1 <_q t_2 <_q \cdots <_q t_m$, then we say that T is q-ordered. Through this paper, we denote by $E(t)$ to be complete set of distinct non associate divisors d of any $t \in S$, and by $J_{k,s}(t)$ on $S - \{0\}$ to be the Jordan totient multiplicative function defined as follows: If x is a non-zero element in S with the unique factorization, up to associates, $x = up_1^{\alpha_1} p_2^{\alpha_2} \ldots p_m^{\alpha_m}$, where α_i are positive integers and u is a unit in S, then $J_{k,s}(x) = \prod_{i=1}^{m} q[p_i]^{k(\alpha_i-1)}[q(p_i)^k - 1]$ and $q(x)^k = \sum_{d \in E(x)} J_{k,s}(d)$.

2 Preliminaries

A zero-divisor x is a non-zero element of a ring S such that there is a non-zero element y in S with $xy = 0$. An integral domain is a commutative ring with unity and with no zero-divisors. Two elements x and y of an integral domain S are said to be associates if $x = uy$, where u is a unit in S.

Definition 2.1. An integral domain S is said to be an Euclidean Domain (ED), if there is an arithmetic function $d : S - \{0\} \to N \cup \{0\}$, which satisfies that for every pair of non-zero elements a and b of S

(i) $d(a) \leq d(ab)$

(ii) $b = aq + r$, where either $r = 0$ or $d(r) < d(a)$.

For example, if $S = Z[i]$ is the ring of Gaussian integers and $q : S - \{0\} \to N \cup \{0\}$ defined by $q(a + bi) = a^2 + b^2$, then S is an ED with measure q.

Definition 2.2. Let $T = \{t_1, t_2, ..., t_m\}$ be a set of non-zero non-associate elements in an ED S with measure q, and let $\{p_1, p_2, ..., p_i, ..\}$ be an ordered listing of all primes in P of S that divide all the elements of T. In addition, assume that $\{p_1, p_2, ..., p_i, ..\}$ has the order inherited from the well ordering of P. The q-ordering on S is defined via the following scheme: $t_i <_q t_j$ if $q(t_i) < q(t_j)$.

We note that the relation $<_q$ is a well-defined linear ordering defined on S. Hence, if the set $T = \{t_1, t_2, ..., t_m\}$ such that $t_1 <_q t_2 <_q ... <_q t_m$, then we say that T is q-ordered.

Definition 2.3. Let S be an ED of measure q, and let $P = \{p_1, p_2, ..., p_m\}$ be a complete prime residue system of S. Define the Jordan totient multiplicative function $J_{k,s}(t)$ on $S - \{0\}$ as follows: If x is a non-zero element in S with the unique factorization, up to associates, $x = up_1^{\alpha_1} p_2^{\alpha_2} ... p_m^{\alpha_m}$, where p_i are distinct and non associate elements in P, α_i are positive integers, and u is a unit in S, then

$$J_{k,s}(x) = \prod_{i=1}^{m} q[p_i]^{k(\alpha_i-1)}[q(p_i)^k - 1]$$

with $J_{k,s}(u) = 1$.

Theorem 2.4. *If S is an ED with measure q and prime residue system P, and $E(x)$ is a complete set of distinct non associate divisors d of $x \in S$, then*

$$q\left(x\right)^{k} = \sum_{d\in E(x)} J_{k,s}(d).$$

Proof. Since $J_{k,s}\left(x\right)$ is multiplicative, then $f(x) = \sum\limits_{d\in E(x)} J_{k,s}(d)$ is also multiplicative, and

$$f(p_i^{\alpha_i}) = \sum_{d\in E(p_i^{\alpha_i})} J_{k,s}(d) = 1 + q\left(p_i\right)^{k(1-1)}\left[q(p_i)^k - 1\right] + ... + q\left(p_i\right)^{k(\alpha_i-1)}\left[q(p_i)^k - 1\right]$$
$$= 1 + q\left(p_i\right)^{k} - 1 + ... + q\left(p_i\right)^{k\alpha_i} - q\left(p_i\right)^{k(\alpha_i-1)} = q\left(p_i\right)^{k\alpha_i} = q\left(p_i^{\alpha_i}\right)^{k}$$

□

3 Power GCDQ Matrices Defined on gcd-Closed Sets Over Euclidean Domains

Definition 3.1. Let S be an ED with prime residue system P and measure q. If $T = \{t_1, t_2, ..., t_m\}$ is a q-ordered set of non-zero non-associate elements in S. Then, the r^{th} power $GCDQ$ matrix defined on T is the $m \times m$ matrix$(T^r)_q$ whose ij^{th} entries are $(t_{ij})_r = q\left((t_i, t_j)\right)^r$, where r is any real number.

Example 3.2. $T = \{1, 1+i, 2\}$ is q-ordered set in $\mathbb{Z}[i]$ with the measure function $q\left(a+bi\right) = a^2 + b^2$. The 2^{nd} power $GCDQ$ matrix is

$$(T^2)_q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 16 \end{bmatrix}.$$

3.1 Factorizations of Power GCDQ Matrices Defined on gcd-Closed Sets Over Euclidean Domains

In this section, we study the factorizations of r^{th} power $GCDQ$ matrices defined on gcd-closed sets over EDs.

Theorem 3.3. *Let S be an ED with prime residue system P and measure q. If $T = \{t_1, t_2, ..., t_m\}$ is any q-ordered set of non-zero non-associate elements in S. Then, the r^{th} power $GCDQ$ matrix is decomposed as $\left(T^r\right)_q = EG_rE^T$.*

Proof. Let $D = \{y_1, y_2, ..., y_n\}$ be the minimal gcd-closed set containing T in S, and $E(x)$ be a complete set of distinct non-associate divisors d of x in S. Define the $n \times n$ diagonal matrix G_r as:

$$G_r = diag\left(\sum_{\substack{d\in E(y_1), d\notin E(y_u)\\ q(y_u)<q(y_1)}} J_{r,s}(d), \sum_{\substack{d\in E(y_2), d\notin E(y_u)\\ q(y_u)<q(y_2)}} J_{r,s}(d), ..., \sum_{\substack{d\in E(y_n), d\notin E(y_u)\\ q(y_u)<q(y_n)}} J_{r,s}(d)\right)$$

and $E_{m\times n} = (e_{ij})$ such that $e_{ij} = 1$ if $y_j \in E(t_i)$ and 0 otherwise. Then,

$$\begin{aligned}
(EG_rE^T)_{ij} &= \sum_{k=1}^{n} e_{ik} \sum_{\substack{d\in E(y_k), d\notin E(y_u)\\ q(y_u)<q(y_k)}} J_{r,s}(d) e_{jk} \\
&= \sum_{\substack{y_k\in E(t_i)\\ y_k\in E(t_j)}} \sum_{\substack{d\in E(y_k), d\notin E(y_u)\\ q(y_u)<q(y_k)}} J_{r,s}(d) \\
&= \sum_{y_k\in E((t_i,t_j))} \sum_{\substack{d\in E(y_k), d\notin E(y_u)\\ q(y_u)<q(y_k)}} J_{r,s}(d) \\
&= [q(t_i, t_j)_p]^r.
\end{aligned}$$

□

Example 3.4. Let $T = \{1, 1 + i, 2, 5\}$ be a q-ordered gcd-closed set in $\mathbb{Z}[i]$. Then,

$$EG_2E^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 624 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 4 & 4 & 1 \\ 1 & 4 & 16 & 1 \\ 1 & 1 & 1 & 625 \end{bmatrix} = (T^2)_q.$$

Theorem 3.5. *Let S be an ED with prime residue system P and measure q. If $T = \{t_1, t_2, ..., t_m\}$ is any q-ordered set of non-zero non-associate elements in S. Then, the r^{th} power $GCDQ$ matrix is decomposed into a product of $m\times n$ matrix G_r and $m\times n$ incidence matrix B_r for some positive integer $n \geq m$.*

Proof. Let $D = \{y_1, y_2, ..., y_n\}$ be the minimal gcd-closed set containing T in S, and $E(x)$ be a complete set of distinct non-associate divisors d of x in S. Define the $n \times n$ diagonal matrix G_r as in the above theorem, and let B_r be defined as $b_{ij} = 1$ if $g_{ji} \neq 0$ and 0 otherwise. Then,

$$\begin{aligned}
(G_rB_r)_{ij} &= \sum_{k=1}^{n} (g_{ik}b_{kj}) = \sum_{\substack{y_k\in E(t_i)\\ y_k\in E(t_j)}} \sum_{\substack{d\in E(y_k), d\notin E(y_u)\\ q(y_u)<q(y_k)}} J_{r,s}(d) \\
&= \sum_{y_k\in E((t_i,t_j)_p)} \sum_{\substack{d\in E(y_k), d\notin E(y_u)\\ q(y_u)<q(y_k)}} J_{r,s}(d) \ = [q(t_i, t_j)_p]^r.
\end{aligned}$$

□

Example 3.6. Let T be defined as above, then

$$G_2B_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 3 & 12 & 0 \\ 1 & 0 & 0 & 624 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 4 & 4 & 1 \\ 1 & 4 & 16 & 1 \\ 1 & 1 & 1 & 625 \end{bmatrix}.$$

Theorem 3.7. *Let S be an ED with prime residue system P and measure q. If $T = \{t_1, t_2, ..., t_m\}$ is any q-ordered set of non-zero non-associate elements in S. Then, the r^{th} power $GCDQ$ matrix is decomposed as a product of $m \times n$ matrix G_r and its corresponding transpose G_r^T.*

Proof. Let $\overline{F}$ be an extension of the field of fractions F of S in which $J_{r,s}(t)$ has a root for every $t \in T$, set $D = \{y_1, y_2, ..., y_n\}$ to be the smallest gcd-closed set containing T in S, and $E(x)$ to be a complete set of distinct non-associate divisors d of x in S. Define the $m \times n$ matrix G_r such that $g_{ij} = \sqrt{\sum_{d \in E(y_j), d \notin E(y_u), q(y_u) < q(y_j)} J_{r,s}(d)}$ if $y_j \in E(t_i)$ and 0 otherwise. Then,

$$(G_r G_r^T)_{ij} = \sum_{k=1}^{n} (g_{ik} g_{jk}) = \sum_{\substack{y_k \in E(t_i) \\ y_k \in E(t_j)}} \sqrt{\sum_{\substack{d \in E(y_k), d \notin E(y_u) \\ q(y_u) < q(y_k)}} J_{r,s}(d)} \sqrt{\sum_{\substack{d \in E(y_k), d \notin E(y_u) \\ q(y_u) < q(y_k)}} J_{r,s}(d)}$$

$$= \sum_{y_k \in E((t_i, t_j)_p)} \left(\sum_{\substack{d \in E(y_k), d \notin E(y_u) \\ q(y_u) < q(y_k)}} J_{r,s}(d) \right) = [q(t_i, t_j)_p]^r.$$

□

Example 3.8. Let T be defined as above, then

$$G_2 G_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \sqrt{3} & 0 & 0 \\ 1 & \sqrt{3} & \sqrt{12} & 0 \\ 1 & 0 & 0 & \sqrt{624} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \sqrt{3} & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{12} & 0 \\ 0 & 0 & 0 & \sqrt{624} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 4 & 4 & 1 \\ 1 & 4 & 16 & 1 \\ 1 & 1 & 1 & 625 \end{bmatrix}$$

3.2 Determinants of Power GCDQ Matrices Defined on gcd-Closed Sets Over Euclidean Domains

Theorem 3.9. *Let S be an ED with prime residue system P and measure q. If $T = \{t_1, t_2, ..., t_m\}$ is any q-ordered gcd-closed set of non-zero non-associate elements in S. Then, the determinant of the r^{th} power $GCDQ$ defined on T in S is:*

$$\det(T^r)_q = \prod_{i=1}^{m} \left(\sum_{d \in E(t_i), d \notin E(t_u), qt_u < t_i < q(t_i)} J_{r,s}(d) \right).$$

Proof. Since T is gcd-closed set in S, then $T \approx D$ and $(T^r)_q = E G_r E^T$. Hence,

$$\det(T^r)_q = \det(E G_r E^T) = \det(E) \det(G_r) \det(E^T) = 1 \times \det(G_r) \times 1$$

$$= \det(G_r) = \prod_{i=1}^{m} \left(\sum_{d \in E(t_i), d \notin E(t_u), q(t_u) < t_i < q(t_i)} J_{r,s}(d) \right).$$

□

Example 3.10. Let T be defined as above, then

$$det[(T^2)]_q = \sum_{\substack{d \in E(1) \\ d \notin E(t_u) \\ q(t_u) < q(1)}} J_{r,s}(d) \sum_{\substack{d \in E(1+i) \\ d \notin E(t_u) \\ q(t_u) < q(1+i)}} J_{r,s}(d) \sum_{\substack{d \in E(2) \\ d \notin E(t_u) \\ q(t_u) < q(2)}} J_{r,s}(d) \sum_{\substack{d \in E(5) \\ d \notin E(t_u) \\ q(t_u) < q(5)}} J_{r,s}(d)$$

$$= 1 \times 3 \times 12 \times 624 = 22464.$$

Remark 3.11. Note that we may prove the above theorem by using the factorization $(T^r)_q = G_r B_r$ or $(T^r)_q = G_r G_r^T$.

Corollary 3.12. *(Beslin-ligh result) If $T = \{t_1, t_2, ..., t_m\}$ is a gcd-closed set of positive integers, then* $\det(T) = \prod_{i=1}^{m} \sum_{d | t_i,\ d \nmid t_u,\ t_u < t_i} \phi(d)$.

Proof. By Theorem 65, $\det(T^r)_q = \prod_{i=1}^{m}\left(\sum_{d\in E(t_i), d\notin E(t_u), q(t_u)<t_i<q(t_i)} J_{r,s}(d)\right)$. Let $r = 1$ and $S = \mathbb{Z}$ with $q = |.|$, then $J_{1,\mathbb{Z}} = \phi$, where ϕ is Euler's totient function. Therefore, $\det(T^1)_{|.|} = \prod_{i=1}^{m}\left(\sum_{d\in E(t_i), d\notin E(t_u), qt_u<t_i<q(t_i)} J_{1,\mathbb{Z}}(d)\right) = \prod_{i=1}^{m}\left(\sum_{d|t_i,\ d\nmid t_u,\ t_u<t_i} \phi(d)\right)$. □

3.3 p-Norms of the Power GCDQ Matrices Defined on gcd-Closed Sets Over Euclidean Domains

In the following, we investigate the l_p norms of any r^{th} power $GCDQ$ matrix defined on gcd-closed sets over EDs.

Definition 3.13. Let S be an mn-dimensional Euclidean space domain with measure q, and let $A \in S^{m\times n}$. If $p \in \mathbb{N}$, define the p-Norm to be the function that maps A to a real number $\|A\|_p$ such that

$$\|A\|_p = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}(q(a_{ij}))^p\right)^{\frac{1}{p}}.$$

Theorem 3.14. *Let S be an ED with a prime residue system P and measure q, and let $T = \{t_1, t_2, \ldots, t_m\}$ be a q-ordered gcd-closed set of non-zero non-associate elements in S. Then, for any real number r and integer p, the l_p norm of the r^{th} power $GCDQ$ matrix is equal to:*

$$\|(T^r)_q\|_p = \left(\sum_{i=1}^{m}\sum_{j=1}^{m}\left(\sum_{y_k\in E((t_i,t_j))}\ \sum_{\substack{d\in E(y_k), d\notin E(y_u)\\ q(y_u)<q(y_k)}} J_{pr,s}(d)\right)\right)^{\frac{1}{p}}.$$

Proof. Let $\overline{F}$ be any extension field of the field of fractions F of S in which $\sum_{\substack{d\in E(t_i)\\ d\notin E(t_u)\\ t_u<_q t_i}} J_{pr,s}(d)$ has p^{th} root. Then,

$$\|(T^r)_q\|_p = \left(\sum_{i=1}^{m}\left(\sum_{j=1}^{m}[q(t_i,t_j)^r]^p\right)\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{m}\left(\sum_{j=1}^{m}[q(t_i,t_j)^{pr}]\right)\right)^{\frac{1}{p}}$$

$$= \left(\sum_{i=1}^{m}\sum_{j=1}^{m}\left(\sum_{y_k\in E((t_i,t_j))}\left(\sum_{\substack{d\in E(y_k), d\notin E(y_u)\\ q(y_u)<q(y_k)}} J_{pr,s}(d)\right)\right)\right)^{\frac{1}{p}}.$$

□

Example 3.15. Since $T = \{1, 1+i, 2+i\}$ is gcd-closed over $S = \mathbb{Z}[i]$, then

$$\|(T^2)_q\|_3 = \left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{d\in E(t_i,t_j)}\left(\sum_{\substack{d\in E(y_k), d\notin E(y_u)\\ q(y_u)<q(y_k)}} J_{6,s}(d)\right)\right)^{\frac{1}{3}}$$

$$= (J_{6,s}(1) + J_{6,s}(1) + J_{6,s}(1) + J_{6,s}(1) + (J_{6,s}(1) + J_{6,s}(1+i))$$
$$+ J_{6,s}(1) + J_{6,s}(1) + J_{6,s}(1) + (J_{6,s}(1) + J_{6,s}(2+i)))^{\frac{1}{3}}$$
$$= (1+1+1+1+1+63+1+1+1+1+15624)^{\frac{1}{3}}$$
$$= \sqrt[3]{15696}.$$

4 Reciprocals of Power GCDQ Matrices Defined on gcd-Closed Sets over Euclidean Domains

Definition 4.1. If $T = \{t_1, t_2, ..., t_m\}$ be a q-ordered set of non-zero non-associate elements in an ED S with measure q, then the Reciprocal r^{th} power $GCDQ$ matrix defined on T over S is the matrix $1/(T^r)_q$ whose ij^{th} entry is $\frac{1}{[q(t_i, t_j)]^r}$.

Theorem 4.2. *Let S be an ED with prime residue system P and measure q. If $T = \{t_1, t_2, ..., t_m\}$ is any q-ordered any set (not necessary gcd-closed) of non-zero non-associate elements in S. Then, the r^{th} power Reciprocal $GCDQ$ matrix $(1/(T^r)_q)$ is decomposed as $1/(T)_q = E(1/G_r)E^T$.*

Proof. Let $D = \{y_1, y_2, ..., y_n\}$ be the minimal gcd-closed set containing T in S, and let $E(x)$ be a complete set of distinct non associate divisors d of x in S. Define the $n \times n$ diagonal matrix $(1/G_r)$ as follows:

$$(1/G_r) = diag\left(\sum_{\substack{d\in E(y_1), d\notin E(y_u)\\ q(y_u)<q(y_1)}} \frac{1}{J_{r,s}(d)}, \sum_{\substack{d\in E(y_2), d\notin E(y_u)\\ q(y_u)<q(y_2)}} \frac{1}{J_{r,s}(d)}, ..., \sum_{\substack{d\in E(y_n), d\notin E(y_u)\\ q(y_u)<q(y_n)}} \frac{1}{J_{r,s}(d)}\right)$$

and E is $m \times n$ such that $e_{ij} = 1$ if $y_j \in E(t_i)$ and 0 otherwise. Then,

$$\begin{aligned}(E(1/G_r)E^T)_{ij} &= \sum_{k=1}^{n} e_{ik} \sum_{d\in E(y_k), d\notin E(y_u), q(y_u)<q(y_k)} \frac{1}{J_{r,s}(d)} \; e_{jk} \\ &= \sum_{y_k\in E(t_i), y_k\in E(t_j)} \; \sum_{d\in E(y_k), d\notin E(y_u), q(y_u)<q(y_k)} \frac{1}{J_{r,s}(d)} \\ &= \sum_{y_k\in E((t_i,t_j))} \; \sum_{d\in E(y_k), d\notin E(y_u), q(y_u)<q(y_k)} \frac{1}{J_{r,s}(d)} \\ &= 1/\left[q(t_i,t_j)\right]^r.\end{aligned}$$

□

Example 4.3. Let T be defined as above, then

$$1/(T^2)_q = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \frac{1}{4} & \frac{1}{4} & 1 \\ 1 & \frac{1}{4} & \frac{1}{16} & 1 \\ 1 & 1 & 1 & \frac{1}{625} \end{bmatrix}$$

and $E\ (1/G_2)_q\ E^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{3}{4} & 0 & 0 \\ 0 & 0 & -\frac{3}{16} & 0 \\ 0 & 0 & 0 & -\frac{624}{625} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$

Theorem 4.4. *Let S be an ED with prime residue system P and measure q. If $T = \{t_1, t_2, ..., t_m\}$ is any q-ordered any set (not necessary gcd-closed) of non zero non-associate elements in S. Then, the r^{th} power Reciprocal $GCDQ$ matrix $(1/(T^r)_q)$ is decomposed into a product of $m \times n$ matrix $(1/G_r)$ and $m \times n$ incidence matrix $(1/B_r)$ for some positive integer $n \geq m$.*

Proof. Let $D = \{y_1, y_2, ..., y_n\}$ be the smallest gcd-closed set containing T in S, and $E(x)$ be a complete set of distinct non associate divisors d of x in S. Define the $m \times n$ matrix $(1/G_r)$ such that $g_{ij} = \sum_{d\in E(y_j), d\notin E(y_u), q(y_u)<q(y_j)} \frac{1}{J_{r,s}(d)}$ if $y_j \in E(t_i)$, and 0 otherwise. Let $(1/B_r)$ be defined

as $b_{ij} = 1$ if $g_{ji} \neq 0$ and 0 otherwise.

$$((1/G_r)\,(1/B_r))_{ij} = \sum_{k=1}^{n}(a_{ik}b_{kj}) = \sum_{y_k \in E(t_i), y_k \in E(t_j)} \; \sum_{d \in E(y_k), d \notin E(y_u), q(y_u)<q(y_k)} \frac{1}{J_{r,s}(d)}$$

$$= \sum_{y_k \in E((t_i,t_j)_p)} \; \sum_{d \in E(y_k), d \notin E(y_u), q(y_u)<q(y_k)} \frac{1}{J_{r,s}(d)} = \frac{1}{[q(t_i,t_j)]^r}.$$

□

Example 4.5. Let T be defined as above, then

$$(1/G_2)\,(1/B_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -\frac{3}{4} & 0 & 0 \\ 1 & -\frac{3}{4} & -\frac{3}{16} & 0 \\ 1 & 0 & 0 & -\frac{624}{625} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \frac{1}{4} & \frac{1}{4} & 1 \\ 1 & \frac{1}{4} & \frac{1}{16} & 1 \\ 1 & 1 & 1 & \frac{1}{625} \end{bmatrix}.$$

Theorem 4.6. *Let S be an ED with prime residue system P and measure q. If $T = \{t_1, t_2, ..., t_m\}$ is any q-ordered any set (not necessary gcd-closed) of non zero non-associate elements in S. Then, the r^{th} power Reciprocal GCDQ matrix $(1/\,(T^r)_q)$ is decomposed into a product of $m \times n$ matrix $(1/G_r)$ and its corresponding transpose $(1/G_r)^T$.*

Proof. Let $\overline{F}$ be an extension of the field of fractions F of S in which g_{ij}, defined above, has a root for every $t \in T$ and $D = \{y_1, y_2, ..., y_n\}$ be the smallest gcd-closed set containing T in S and $E(x)$ be a complete set of distinct non associate divisors d of x in S. Then,

$$((1/G_r)\,(1/G_r)^T)_{ij} = \sum_{k=1}^{n}(g_{ik}g_{jk})$$

$$= \sum_{y_k \in E(t_i), y_k \in E(t_j)} \left(\sqrt{\sum_{d \in E(y_k), d \notin E(y_u), q(y_u)<q(y_k)} \frac{1}{J_{r,s}(d)}} \right)^2$$

$$= \sum_{y_k \in E((t_i,t_j)_p)} \; \sum_{d \in E(y_k), d \notin E(y_u), q(y_u)<q(y_k)} \frac{1}{J_{r,s}(d)} = \frac{1}{[q(t_i,t_j)]^r}.$$

□

Example 4.7. Let T be defined as above, then

$$(1/G_2)\,(1/G_2)^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \sqrt{-\frac{3}{4}} & 0 & 0 \\ 1 & \sqrt{-\frac{3}{4}} & \sqrt{-\frac{3}{16}} & 0 \\ 1 & 0 & 0 & \sqrt{-\frac{624}{625}} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \sqrt{-\frac{3}{4}} & \sqrt{-\frac{3}{4}} & 0 \\ 0 & 0 & \sqrt{-\frac{3}{16}} & 0 \\ 0 & 0 & 0 & \sqrt{-\frac{624}{625}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \frac{1}{4} & \frac{1}{4} & 1 \\ 1 & \frac{1}{4} & \frac{1}{16} & 1 \\ 1 & 1 & 1 & \frac{1}{625} \end{bmatrix}.$$

4.1 Determinants of Reciprocal Power GCDQ Matrices on gcd-Closed Sets over Euclidean Domains

Theorem 4.8. *Let S be an ED with prime residue system P and measure q. If $T = \{t_1, t_2, ..., t_m\}$ is any q-ordered gcd-closed set of non-zero non-associate elements in S. Then, the determinant* *of the r^{th} power Reciprocal $GCDQ$ matrix $1/\left(T^r\right)_q$ defined on T in S is equal to the product of*

$$\det(1/\left(T^r\right)_q) = \prod_{i=1}^{m}\left(\sum_{d\in E(t_i),d\notin E(t_u),q(t_u)<q(t_i)}\frac{1}{J_{r,s}(d)}\right).$$

Proof. Let T be a q-ordered set in S. Since T is gcd-closed set, then $T \approx D$. Hence, by Theorem 4.2 we have $(1/T^r)_q = E\,(1/G_r)\,E^T$, where E is a lower triangular matrix with diagonal entries $e_{ij} = 1$. Therefore,

$$\det(1/T^r)_q = \det(E\,(1/G_r)\,E^T) = \det(E)\det(1/G_r)\det(E^T) = \det(1/G_r)$$
$$= \prod_{i=1}^{m}\left(\sum_{d\in E(t_i),d\notin E(t_u),q(t_u)<q(t_i)}\frac{1}{J_{r,s}(d)}\right).$$

□

Example 4.9. Let T be defined as above, then

$$\det[(T^{-2})]_q = -\frac{351}{2500},$$

and

$$\det(1/T^2)_q = \prod_{i=1}^{4}\left(\sum_{d\in E(t_i),d\notin E(t_u),q(t_u)<q(t_i)}\frac{1}{J_{2,s}(d)}\right)$$
$$= 1\times\left(-\frac{3}{4}\right)\left(-\frac{3}{16}\right)\left(-\frac{624}{625}\right) = -\frac{351}{2500}.$$

Note that we may prove the above theorem by using the factorizations $1/\left(T^r\right)_q = (1/G_r)\,(1/G_r)^T$ and $1/\left(T^r\right)_q = (1/G_r)\,(1/B_r)$.

4.2 p-Norms of the Reciprocal Power GCDQ Matrices Defined on gcd-Closed Sets Over Euclidean Domains

Theorem 4.10. *Let S be an ED with a prime residue system P and measure q, and let $T = \{t_1, t_2, \ldots, t_m\}$ be a q-ordered gcd-closed set of non-zero non-associate elements in S. Then, for any real number r and integer p, the l_p norm of the r^{th} power Reciprocal $GCDQ$ matrix is equal to:*

$$\left\|(1/T^r)_q\right\|_p = \left(\sum_{i=1}^{m}\sum_{j=1}^{m}\left(\sum_{y_k\in E((t_i,t_j))}\ \sum_{\substack{d\in E(y_k),d\notin E(y_u)\\ q(y_u)<q(y_k)}}\frac{1}{J_{pr,s}(d)}\right)\right)^{\frac{1}{p}}.$$

Proof. Similar to the proof of Theorem 3.14. □

5 Inverses of Power GCDQ Matrices on gcd-Closed Sets over Euclidean Domains

Theorem 5.1. *Let S be an ED with prime residue system P and measure q. If $T = \{t_1, t_2, ..., t_m\}$ is any q-ordered gcd-closed set of non-zero non-associate elements in S. Then, the inverse of the r^{th} power $GCDQ$ matrix is $\left(T^r\right)_q^{-1} = t_{ij}$ such that*

$$t_{ij} = \sum_{t_i\in E(t_k),t_j\in E(t_k)}\left(\frac{\mu_p\left(\frac{t_k}{t_i}\right)\mu_p\left(\frac{t_k}{t_j}\right)}{\sum\limits_{d\in E(t_k),d\notin E(t_u),q(t_u)<q(t_k)}J_{r,s}(d)}\right).$$

Proof. Let $T = \{t_1, t_2, ..., t_m\}$ be a q-ordered gcd-closed set of non zero non associate elements in S, then the r^{th} power $GCDQ$ matrix $(T^r)_q = EG_rE^T$. Thus,

$$((T^r)_q^{-1})_{ij} = (EG_rE^T)_{ij}^{-1} = \left((E^T)^{-1}(G_r)^{-1}(E)^{-1}\right)_{ij} = \left((E^{-1})^T(G_r)^{-1}(E)^{-1}\right)_{ij}$$
$$= \left(U^T(G_r)^{-1}U\right)_{ij} = \sum_{k=1}^{m} u_{ki}\frac{1}{g_{kk}}u_{kj}$$
$$= \sum_{t_i \in E(t_k), t_j \in E(t_k)} \mu_p\left(\frac{t_k}{t_i}\right) \frac{1}{\sum\limits_{d \in E(t_k), d \notin E(t_u), q(t_u) < q(t_k)} J_{r,s}(d)} \mu_p\left(\frac{t_k}{t_j}\right).$$

□

Example 5.2. let $T = \{1, 1+i, 2\}$ be a q-ordered gcd-closed set $\mathbb{Z}[i]$. Then,

$$(T^2)_q^{-1} = \begin{bmatrix} \frac{4}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{5}{12} & -\frac{1}{12} \\ 0 & -\frac{1}{12} & \frac{1}{12} \end{bmatrix}.$$

6 Power LCMQ Matrices on gcd-Closed Sets over Euclidean Domains

6.1 Structurs of Power LCMQ Matrices on gcd-Closed Sets over Euclidean Domains

Theorem 6.1. *Let S be an ED with prime residue system P and measure q. If $T = \{t_1, t_2, ..., t_m\}$ is any q-ordered gcd-closed set of non-zero non-associate elements in S. Then, the r^{th} power $LCMQ$ matrix $[T^r]_q$ can be decomposed, up to associates, as $[T^r]_q = D_rE\,(1/G)\,E^TD_r$.*

Proof. Let $D = \{y_1, y_2, ..., y_n\}$ be the smallest gcd-closed set containing T in S, and $E(x)$ be a complete set of distinct non associate divisors d of x in S. Define the $n \times n$ diagonal matrix $(1/G_r)$ as follows:

$$(1/G_r) = diag\Big(\sum_{\substack{d \in E(t_1) \\ d \notin E(t_u) \\ q(t_u) < q(t_1)}} \frac{1}{J_{r,s}(y_1)}, \sum_{\substack{d \in E(t_2) \\ d \notin E(t_u) \\ q(t_u) < q(t_2)}} \frac{1}{J_{r,s}(y_2)}, ..., \sum_{\substack{d \in E(t_n) \\ d \notin E(t_u) \\ q(t_u) < q(t_n)}} \frac{1}{J_{r,s}(y_n)}\Big).$$

and $E_{m\times n} = (e_{ij})$ such that $e_{ij} = 1$ if $y_j \in E(t_i)$ and 0 otherwise. Let D_r be $m \times m$ diagonal matrix whose diagonal entries are of the form $d_{ii} = q\,(t_i)^r$. Then,

$$(D_rE\,(1/G_r)\,E^TD_r)_{ij} = (D_r(1/T^r)_qD_r) = q\,(t_i)^r\ (1/T^r)_{ij}\ q\,(t_j)^r$$
$$= \frac{q\,(t_i)^r\,q\,(t_j)^r}{[q(t_i, t_j)]^r} = q\left[\frac{t_i^rt_j^r}{(t_i, t_j)^r}\right] = [q[t_i, t_j]_p]^r.$$

□

Example 6.2. let $T = \{1, 1+i, 2\}$ be a q-ordered gcd-closed set $\mathbb{Z}[i]$. Then,

$$D_2\left(1/T^2\right)_qD_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 625 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \frac{1}{4} & \frac{1}{4} & 1 \\ 1 & \frac{1}{4} & \frac{1}{16} & 1 \\ 1 & 1 & 1 & \frac{1}{625} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 625 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 4 & 16 & 625 \\ 4 & 4 & 16 & 2500 \\ 16 & 16 & 16 & 10000 \\ 625 & 2500 & 10000 & 625 \end{bmatrix} = [T^2]_q.$$

6.2 Determinants of Power LCMQ Matrices on gcd-Closed Sets over Euclidean Domains

Theorem 6.3. *Let S be an ED with prime residue system P and measure q. If $T = \{t_1, t_2, ..., t_m\}$ is any q-ordered gcd-closed set of non-zero non-associate elements in S. Then, the determinant of the r^{th} power $LCMQ$ matrix $[T^r]_q$ is equal to*

$$\det [T^r]_q = \prod_{i=1}^{m} \left(\sum_{d \in E(y_i), d \notin E(y_u), q(y_u) < q(y_i)} \frac{1}{J_{r,s}(t_i)} \right) q(t_i)^{2r}.$$

Proof. Let T be a q-ordered set in S. Since T is gcd- closed set, then $T \approx D$. But, $[T^r]_q = D_r E \left(1/G_r\right) E^T D_r$ and E is lower triangular matrix with diagonal entry $e_{ii} = 1$. Therefore,

$$\begin{aligned}\det [T^r]_q &= \det(D_r E \left(1/G_r\right) E^T D_r) \\ &= \prod_{i=1}^{m} q(t_i)^r \times \det \left(1/G_r\right) \times \prod_{i=1}^{m} q(t_i)^r \\ &= \prod_{i=1}^{m} \left(\sum_{d \in E(y_i), d \notin E(y_u), q(y_u) < q(y_i)} \frac{1}{J_{r,s}(t_i)} \right) q(t_i)^{2r}.\end{aligned}$$

□

Example 6.4. let $T = \{1, 1+i, 2, 5\}$ be a q-ordered gcd-closed set in $\mathbb{Z}[i]$.

$$\begin{aligned}\det[T^2]_q &= \prod_{i=1}^{4} \left(\sum_{d \in E(y_i), d \notin E(y_u), q(y_u) < q(y_i)} \frac{1}{J_{2,s}(t_i)} \right) q(t_i)^{2\times 2} \\ &= (1 \times 1) \times (-\frac{3}{4} \times 16) \times (-\frac{3}{16} \times 256) \times (-\frac{624}{625} \times 390625) \\ &= -224640000.\end{aligned}$$

6.3 p-Norms of the Power LCMQ Matrices Defined on gcd-Closed Sets Over Euclidean Domains

Theorem 6.5. *Let S be an ED with a prime residue system P and measure q, and let $T = \{t_1, t_2, \ldots, t_m\}$ be a q-ordered gcd-closed set of non-zero non-associate elements in S. Then, for any real number r and integer p, the l_p norm of the r^{th} power Reciprocal $GCDQ$ matrix is equal to:*

$$\|[T^r]_q\|_p = \left(\sum_{i=1}^{m} \sum_{j=1}^{m} \left(\sum_{y_k \in E((t_i,t_j))} \sum_{\substack{d \in E(y_k), d \notin E(y_u) \\ q(y_u) < q(y_k)}} \left(\frac{q\left(t_i\right) q\left(t_j\right)}{q(t_i, t_j)} \right)^{pr} \right) \right)^{\frac{1}{p}}.$$

Proof. Similar to the proof of Theorem 3.14. □

Declarations

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Author information

Yahia AWAD[1*], (Corresponding author)
Department of Mathematics and Physics, Lebanese International University, Bekaa, Lebanon.
E-mail: `yehya.awad@liu.edu.lb; ORCID: https://orcid.org/0000-0001-9878-2482.`

Ragheb MGHAMES[2],
Department of Mathematics and Physics, Lebanese International University, Bekaa, Lebanon.
E-mail: `ragheb.mghames@liu.edu.lb`

Haissam CHEHADE[3],
Department of Mathematics and Physics, International University of Beirut, Beirut, Lebanon.
E-mail: `haissam.chehade@b-iu.edu.lb`

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