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Descending Endomorphisms of Groups

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Abstract We define a descending endomorphism of a group as an endomorphism that induces a corresponding endomorphism in any homomorphic image of the group, such that the composition of the descending endomorphism with the homomorphism equals the composition of the homomorphism with the induced endomorphism. After proving that descending endomorphisms of a certain class of Abelian groups, including all finitely generated Abelian groups, are universal power endomorphisms, we characterise the descending endomorphisms of direct products of groups, and thus obtain a procedure to determine all the descending endomorphisms of a direct product using the descending endomorphisms of the direct factors. As a natural outcome of this theory, we also obtain a characterisation of the direct products whose normal subgroups are direct products of normal subgroups of the direct factors.

1 Introduction

One of the most fundamental and useful techniques in the study of groups, especially finite groups, is to "descend" to a quotient group and to "lift" any results obtained back to the original group. We encounter a problem when we attempt to apply this technique to study group endomorphisms – not all endomorphisms descend to quotients. That is, not every endomorphism of a group may induce a corresponding endomorphism in a quotient group. In the language of category theory, the map that carries groups to their endomorphism monoids is not a functor from the category of groups and epimorphisms to the category of monoids. Motivated by this observation, we propose a definition of *descending* endomorphisms in this paper, and study some of their basic properties. A natural first question is how to determine the descending endomorphisms of a direct product in terms of the descending endomorphisms of its direct factors. We obtain a complete answer to this question, and in doing so, also characterise the direct products all of whose normal subgroups are themselves direct products of normal subgroups of the respective factors. The same characterisation was previously obtained by Miller in [6], but the proof given there depends on a result of Suzuki [12] regarding the direct decomposition of the lattice of subgroups of a group. Our proof avoids this result and only uses the theory of descending endomorphisms developed in this paper.

Literature Survey

Before proceeding to the main content, we will briefly survey some similar concepts in the literature. Our definition of a descending endomorphism is in terms of a commutative diagram, namely (2.1), involving two given groups and a given epimorphism between them. There are three other variations of this idea, that can be obtained by a combination of reversing or retaining the directions of the arrows and making them epic or monic.

If the arrows are made monic, without reversing them, then we obtain the definition of endomorphisms with the property that they *extend* to endomorphisms of any group into which the original group is embedded. Schupp, in [10], has shown that inner automorphisms can be characterised as exactly those automorphisms that have this property. Pettet proved that this characterisation remains valid even when the universe is restricted to certain classes of finite groups [8]. Pettet also showed later in [9] that, interestingly, the dual of this condition characterises inner automorphisms as well – i.e., that inner automorphisms are exactly those that can always be *lifted* to groups of which the original group is a homomorphic image – which corresponds to reversing the epic arrows in (2.1).

If we reverse the arrows in (2.1) and make them monic as well, then the endomorphism thus defined must *restrict* to an endomorphism of every subgroup of the group. Such endomorphisms are called power endomorphisms, and are well-studied [4, 1, 7]. Again, it is interesting to note that in the case of Abelian groups, this notion coincides with that of descending endomorphisms.

In view of the characterisation of descending endomorphisms given in Lemma 2.4, another closely related definition is that of a normal automorphism – i.e., an automorphism that bijectively maps every normal subgroup to itself – first described by Lubotzky in [5]. A stronger version of this definition has recently been studied in [11] by Stanojkovski, who defines an intense automorphism as one that maps every subgroup bijectively to some conjugate of itself.

Organisation

The main content of this paper is organised into two sections. In Section 2, we define descending endomorphisms of groups, establish some preliminary results showing that they are "well-behaved", characterise them in terms of normal subgroups, and finally characterise the descending endomorphisms of a large class of Abelian groups. This characterisation is used crucially in obtaining the main results of the paper.

In Section 3, we discuss the descending endomorphisms of direct products of a finite number of groups, beginning with the observation that any descending endomorphism of the product is a pointwise product of descending endomorphisms of the direct factors. The failure of the converse to hold in general prompts the investigation leading to our main results (Theorems 3.6, 3.12, and 3.14), from which we obtain Miller's result (Theorem 3.15).

2 Definitions and Preliminary Results

In this section, we define descending endomorphisms of groups and study their basic properties.

Definition 2.1. An endomorphism δ of a group G is a *descending endomorphism* if, for every epimorphism $\varphi\colon G \twoheadrightarrow Q$ of G onto a group Q, there exists an endomorphism $\overline{\delta}$ of Q such that the diagram in (2.1) commutes.

$$\begin{array}{ccc} G & \xrightarrow{\delta} & G \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi} \\ Q & \xrightarrow[\overline{\delta}]{} & Q \end{array} \tag{2.1}$$

Then $\overline{\delta}$ is unique and is said to be the *descended endomorphism* or *descent* of δ to Q along φ, or the endomorphism *induced by* δ.

In other words, given a group homomorphism, any descending endomorphism of the domain induces an endomorphism of the homomorphic image. The true usefulness of this property comes from the fact that this induced endomorphism also inherits the descending property.

Theorem 2.2. *Let G be a group, and $\varphi\colon G \twoheadrightarrow Q$ an epimorphism. If δ is a descending endomorphism of G, then its descent $\overline{\delta}$ to Q is a descending endomorphism of Q.*

Proof. Let $\psi\colon Q \twoheadrightarrow K$ be an epimorphism from Q onto any group K, and let $\overline{\overline{\delta}}$ be the descent of δ to K along the epimorphism $\psi \circ \varphi$. Then in the following diagram, the upper square and outer rectangle commute.

$$\begin{array}{ccc} G & \xrightarrow{\delta} & G \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ Q & \xrightarrow[\overline{\delta}]{} & Q \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi} \\ K & \xrightarrow[\overline{\overline{\delta}}]{} & K \end{array}$$

This implies that, as φ is surjective, the lower square also commutes, showing that $\overline{\overline{\delta}}$ is the descent of $\overline{\delta}$ to K along ψ. Thus, $\overline{\delta}$ is a descending endomorphism of Q. □

The set of all descending endomorphisms of G, which we shall denote as $\operatorname{Des} G$, clearly forms a submonoid of the endomorphism monoid $\operatorname{End} G$. When G is an Abelian group, $\operatorname{End} G$ forms a ring, and then $\operatorname{Des} G$ is a subring of $\operatorname{End} G$. In particular, the identity automorphism and the trivial endomorphism (which maps every element to the identity element) are always descending endomorphisms of any group.

The group $\operatorname{Aut} G$ of all automorphisms of G is a subgroup of $\operatorname{End} G$. It is not hard to see that $\operatorname{Des} G$ is invariant under conjugation by elements of $\operatorname{Aut} G$. That is, if $\alpha \in \operatorname{Aut} G$ and $\delta \in \operatorname{Des} G$, then $\alpha \circ \delta \circ \alpha^{-1} \in \operatorname{Des} G$. This follows from the more general result given below.

Lemma 2.3. *If G and H are isomorphic groups with an isomorphism $f\colon G \to H$, and δ is a descending endomorphism of G, then $f \circ \delta \circ f^{-1}$ is a descending endomorphism of H.*

Proof. Note that $f \circ \delta \circ f^{-1}$ is an endomorphism of H, and denote it by η. As f is surjective and $\eta \circ f = f \circ \delta$, η is the descent of δ along f, and by Theorem 2.2, is descending. □

We shall call a subgroup H of a group G *invariant* under an endomorphism ε of G if $\varepsilon(H) \leq H$. Now, if N is a normal subgroup of G (hereafter written as $N \trianglelefteq G$), and δ is a descending endomorphism of G, then observe that δ descends to the quotient G/N (along the canonical projection from G to G/N) if and only if it leaves N invariant. In fact, by the First Isomorphism Theorem, if $\varphi\colon G \twoheadrightarrow Q$ is an epimorphism, then $Q \cong G/\ker\varphi$. This gives us a characterisation of descending endomorphisms in terms of normal subgroups.

Lemma 2.4. *An endomorphism δ of a group G is a descending endomorphism of G if and only if for every normal subgroup N of G, $\delta(N) \leq N$. Then the descent of δ to any quotient G/N along the canonical projection is defined by $\overline{\delta}(xN) = \delta(x)N$ for all $xN \in G/N$.* □

Lemma 2.4 shows that in particular, all inner automorphisms and power endomorphisms are descending endomorphisms. A *power endomorphism* is one that leaves all subgroups invariant [1]. Equivalently, it is one that maps every element to an integer power of itself. A power endomorphism is *universal* if it maps every element to the same power – i.e., if there exists an integer k such that the endomorphism maps every element x of the group to x^k. As all subgroups of any Abelian group A are normal, $\operatorname{Des} A$ must be the set of all power endomorphisms of A. In fact, a stronger statement holds in the case of a large class of Abelian groups.

Theorem 2.5. *If A is an Abelian group satisfying any one of the following properties, then the descending endomorphisms of A are exactly its universal power endomorphisms.*

(i) A is a direct product of cyclic groups.

(ii) A has finite exponent.

(iii) A has an element of infinite order.

Proof. In each of the following cases, let δ be any descending endomorphism of A.

(i) Let $A = \prod_{i \in I} \langle g_i \rangle$, considered as an internal direct product, and define $x = \prod_{i \in I} g_i$. As A is Abelian, there exists $n_i \in \mathbb{Z}$ such that $\delta(g_i) = g_i^{n_i}$, for each $i \in I$, and there exists $n \in \mathbb{Z}$ such that $\delta(x) = x^n$. Then $\delta(x) = \prod_{i \in I} \delta(g_i)$ implies that $g_i^{n_i} = g_i^n$, for each $i \in I$. As all elements of A are products of g_i, $i \in I$, this implies that δ is a universal endomorphism of A that maps every element to its n^{th} power.

(ii) Let A have finite exponent m. Since an Abelian group of finite exponent can be expressed as the direct sum of cyclic groups [3, Theorem 6], let $A = \bigoplus_{i \in I} \langle g_i \rangle$, considered as an internal direct sum. Let $j, k \in I$, and define $x = \prod_{i \in I} x_i$, where $x_i = g_i$ for $i = j, k$, and $x_i = 1$ for $i \in I \setminus \{j, k\}$. As A is Abelian and has exponent m, for each $i \in I$, $\delta(g_i) = g_i^{n_i}$, for some $n_i \in \{0, \ldots, m-1\}$, and $\delta(x) = x^n$, for some $n \in \{0, \ldots, m-1\}$. Then $\delta(x) = \prod_{i \in I} \delta(x_i)$ implies that $n_j = n = n_k$. Since this is true if we fix j and vary k, we have $n_i = n_j$ for all $i \in I$, and hence δ is a universal power endomorphism.

(iii) Let A be an Abelian group and x an element of infinite order in A. Then $\delta(x) = x^n$, for some $n \in \mathbb{Z}$. Now, for any $y \in A$, consider the restriction of δ to the subgroup $\langle x, y \rangle$. From (i), this restriction must be a universal power endomorphism of $\langle x, y \rangle$, which implies that $\delta(y) = y^n$. As y is arbitrary, δ is a universal power endomorphism. □

We note that Theorem 2.5 implies Theorem 3.4.1 of [1], which states that any power endomorphism of an Abelian group is universal on every finitely generated subgroup. As mentioned in the paragraph preceding Theorem 2.5, the statement does not hold for all Abelian groups. Example 2.6 describes such a case.

Example 2.6. Let p be any prime and $\mathbb{Z}_{p^\infty}$ the Prüfer p-group, written additively. Define a map δ on $\mathbb{Z}_{p^\infty}$ that maps each element x to the element $x + px + p^2x + \cdots$. Note that this is well defined, since each x has finite order p^k for some k, making $p^k x$ and later terms zero. Then δ is an endomorphism of $\mathbb{Z}_{p^\infty}$, and since it is a power map (written as a multiplication because of additive notation), it is descending. However, it is not a universal power endomorphism.

Before concluding this section, we give an example of a descending endomorphism that is not an inner automorphism or a power endomorphism.

Example 2.7. Consider the symmetric group S_3 generated by the permutations $r = (123)$ and $s = (12)$. Let δ be the endomorphism of S_3 that maps r to 1 and fixes s. Then δ is neither an inner automorphism nor a power endomorphism. The only proper, non-trivial normal subgroup of S_3 is $\langle r \rangle$, and $\delta(\langle r \rangle) = 1 \leq \langle r \rangle$. Hence, $\delta \in \operatorname{Des} S_3$.

3 Descending Endomorphisms of Direct Products of Groups

Direct products are, arguably, the simplest way of constructing larger groups from given groups, and conversely, the simplest way of decomposing a given group into smaller groups. Given such a decomposition, we wish to obtain information about the group by studying its direct factors. The existing literature contains a number of results that serve this purpose. For instance, Goursat's lemma [2] completely determines all subgroups of a direct product of two groups in terms of subgroups of its direct factors, and Miller gives a procedure to construct all normal subgroups of a direct product of groups in terms of normal subgroups of its direct factors [6].

In a similar vein, we now proceed to describe all descending endomorphisms of the direct product of two arbitrary groups. As an immediate application of the theory so developed, we obtain an alternative proof of Miller's characterisation of direct products whose normal subgroups are themselves direct products of the respective direct factors [6].

We first define some notation that will be employed throughout this section. Let $G = G_1 \times G_2$ be the direct product of two groups G_1 and G_2. In each statement that follows, consider the index i as taking the values 1 and 2. Denote by π_i, the canonical projection from G onto G_i. For any $\delta \in \operatorname{Des} G$, let $\delta|_{G_i}$ be its restriction to G_i. As any normal subgroup of a direct factor is normal in G, it follows from Lemma 2.4 that $\delta|_{G_i}$ is always a descending endomorphism of G_i. Thus, every $\delta \in \operatorname{Des} G$ can be written as a pointwise product of descending endomorphisms $\delta_1 \in \operatorname{Des} G_1$ and $\delta_2 \in \operatorname{Des} G_2$, where $\delta_i = \delta|_{G_i}$.

Definition 3.1. If $\delta_i \in \operatorname{Des} G_i$, $i = 1, 2$, then the *pointwise product* of δ_1 and δ_2, denoted by $\delta_1 \times \delta_2$, is the endomorphism of $G_1 \times G_2$ defined by $(\delta_1 \times \delta_2)(x_1, x_2) = (\delta_1(x_1), \delta_2(x_2))$, for each $(x_1, x_2) \in G_1 \times G_2$.

Thus, in general, $\operatorname{Des} G_1 \times G_2$ is a submonoid of $\operatorname{Des} G_1 \times \operatorname{Des} G_2$, the Cartesian product of $\operatorname{Des} G_1$ and $\operatorname{Des} G_2$.

Let N be a fixed normal subgroup of G, and let $N_i = \pi_i(N)$ be its projection into G_i. Then $[G_1, N_1] \times [G_2, N_2] \leq N \leq N_1 \times N_2$, and all three of these subgroups are normal in G. There is a bijection between the subgroups of G containing $[G_1, N_1] \times [G_2, N_2]$, and the subgroups of the quotient $G/([G_1, N_1] \times [G_2, N_2])$.

If H is any subgroup of G that contains $[G_1, N_1] \times [G_2, N_2]$, then with respect to N, we denote by $\overline{H}$ the unique subgroup of $G/([G_1, N_1] \times [G_2, N_2])$ corresponding to H under the bijection defined by the correspondence theorem.

Explicitly, $\overline{H} = H/([G_1, N_1] \times [G_2, N_2])$, and in particular, $\overline{G} = G/([G_1, N_1] \times [G_2, N_2])$. Moreover, $H \trianglelefteq G$ if and only if $\overline{H} \trianglelefteq \overline{G}$. As $\overline{N_1 \times N_2}$ is central in $\overline{G}$, every subgroup H of G lying between $[G_1, N_1] \times [G_2, N_2]$ and $N_1 \times N_2$ will be normal in G.

Now, $\overline{G} \cong \overline{G}_1 \times \overline{G}_2$ via a canonical isomorphism (where $\overline{G}_i = G_i/[G_i, N_i]$). By abuse of notation, we may identify these two groups. Let δ_i be a fixed descending endomorphism of G_i. Then the pointwise product $\delta_1 \times \delta_2$ is an endomorphism of G, but is not necessarily descending. However, δ_i descends to $\overline{\delta_i} \in \operatorname{Des} \overline{G}_i$ along the canonical epimorphism from G_i to $\overline{G}_i$, and $\overline{\delta_1} \times \overline{\delta_2} \in \operatorname{End} \overline{G}_1 \times \overline{G}_2$. Again, by abuse of notation, we identify this pointwise product with a corresponding endomorphism of $\overline{G}$. Thus, $\overline{\delta_1} \times \overline{\delta_2}$ maps each element $(x, y)[G_1, N_1] \times [G_2, N_2] \in \overline{G}$ to $(\delta_1(x), \delta_2(y))[G_1, N_1] \times [G_2, N_2]$.

Proposition 3.2. *If $\delta_i \in \operatorname{Des} G_i$, $i = 1, 2$, then $\delta_1 \times \delta_2$ descends to any quotient of $G = G_1 \times G_2$ of the form $G/(M_1 \times M_2)$ where $M_i \trianglelefteq G_i$.*

Proof. $G/(M_1 \times M_2) \cong (G_1/M_1) \times (G_2/M_2)$, on which the pointwise product of the descents of δ_1 and δ_2 acts as an endomorphism, and this induces a corresponding endomorphism of $G/(M_1 \times M_2)$. □

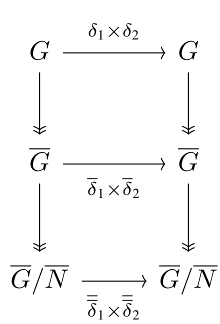
Remark 3.3. In Proposition 3.2, the pointwise product $\delta_1 \times \delta_2$ may not be a descending endomorphism of G. However, it is guaranteed to descend to quotients of the particular form mentioned.

Example 3.4. Let $G = \{1, h, k, hk\}$ be the Klein 4-group. G can be written as the direct product $G = H \times K$ where $H = \{1, h\}$ and $K = \{1, k\}$. Let δ_1 be the identity automorphism of H, and δ_2 the trivial endomorphism of K. Then their pointwise product $\delta_1 \times \delta_2$ is an endomorphism of G, and $(\delta_1 \times \delta_2)(hk) = \delta_1(h)\delta_2(k) = h$. Both δ_1 and δ_2 are descending endomorphisms of the two respective direct factors, but $\delta_1 \times \delta_2$ is not a descending endomorphism of G, since $N = \{1, hk\} \trianglelefteq G$ but $(\delta_1 \times \delta_2)(N) = \{1, h\} \not\leq N$. However, $\delta_1 \times \delta_2$ does descend to all the quotients of the form $G/(M_1 \times M_2)$. For example, it descends to the identity automorphism of $G/(1 \times K)$ and to the trivial endomorphism of $G/(H \times 1)$.

As a first step in simplifying the problem of determining whether a given pointwise product of descending endomorphisms is a descending endomorphism of the direct product, we derive the following result dealing with invariance of a given normal subgroup under such a pointwise product.

Lemma 3.5. *Let $G = G_1 \times G_2$. For $\delta_1 \in \operatorname{Des} G_1$ and $\delta_2 \in \operatorname{Des} G_2$, the endomorphism $\delta_1 \times \delta_2$ leaves $N \trianglelefteq G$ invariant if and only if $\overline{\delta}_1 \times \overline{\delta}_2 \in \operatorname{End} \overline{G}$ leaves $\overline{N} \trianglelefteq \overline{N_1 \times N_2}$ invariant.*

Proof. Consider the following diagram, in which each epimorphism is a canonical projection.



From Proposition 3.2, we know that $\delta_1 \times \delta_2$ induces the endomorphism $\overline{\delta}_1 \times \overline{\delta}_2$ in $\overline{G}$. It is obvious (see proof of Theorem 2.2) that the outer rectangle commutes if and only if the lower square commutes – i.e., if and only if $\overline{\delta}_1 \times \overline{\delta}_2$ leaves $\overline{N}$ invariant. However, since $G/N \cong \overline{G}/\overline{N}$ canonically, the outer rectangle commutes if and only if $\delta_1 \times \delta_2$ leaves N invariant. □

Now to determine whether $\delta_1 \times \delta_2$ is descending – i.e., leaves all normal subgroups of $G = G_1 \times G_2$ invariant, we need to apply Lemma 3.5 to all normal subgroups of G. The first main result of this paper shows that this can be further simplified.

Theorem 3.6. *For any $\delta_1 \in \operatorname{Des} G_1$, $\delta_2 \in \operatorname{Des} G_2$, the pointwise product $\delta_1 \times \delta_2$ is a descending endomorphism of $G_1 \times G_2$ if and only if for all $N_i \trianglelefteq G_i$, $(\overline{\delta}_1 \times \overline{\delta}_2)|_{\overline{N}_1 \times \overline{N}_2}$ is a descending endomorphism of $\overline{N}_1 \times \overline{N}_2$.*

Proof. Lemma 3.5 implies that it is sufficient to check whether, for each $N \trianglelefteq G$, the descent $\overline{\delta}_1 \times \overline{\delta}_2$ to $\overline{G}_1 \times \overline{G}_2 \cong G/([G_1, N_1] \times [G_2, N_2])$ leaves $N/([G_1, N_1] \times [G_2, N_2])$ invariant. But each pair $N_1 \trianglelefteq G_1$ and $N_2 \trianglelefteq G_2$ corresponds, in general, to several $N \trianglelefteq G$ (namely all N with $\pi_i(N) = N_i$, $i = 1, 2$). For each such pair N_1 and N_2, the corresponding $\overline{\delta}_1 \times \overline{\delta}_2$ leaves all $\overline{N} \trianglelefteq \overline{N}_1 \times \overline{N}_2$ invariant exactly when the restriction of $\overline{\delta}_1 \times \overline{\delta}_2$ to $\overline{N}_1 \times \overline{N}_2$ is a descending endomorphism of $\overline{N}_1 \times \overline{N}_2$. □

Since $\overline{N}_1$ and $\overline{N}_2$ are Abelian, so is their direct product, and therefore $(\overline{\delta}_1 \times \overline{\delta}_2)|_{\overline{N}_1 \times \overline{N}_2}$ being a power endomorphism of $\overline{N}_1 \times \overline{N}_2$ is a necessary and sufficient condition for it to be descending. When $\overline{N}_1$ and $\overline{N}_2$ are torsion, we get another equivalent condition in terms of the orders of their elements.

Theorem 3.7. *Let $A = A_1 \times A_2$ be the direct product of torsion Abelian groups A_1 and A_2, and let $\delta_i \in \operatorname{Des} A_i$, $i = 1, 2$. Then*

(i) for each $a_i \in A_i$ there exists $m_i \in \mathbb{Z}$ such that , $\delta_i(a_i) = a_i^{m_i}$, $i = 1, 2$, and

(ii) the pointwise product $\delta_1 \times \delta_2$ is a descending endomorphism of A if and only if for all $a_1 \in A_1$, $a_2 \in A_2$, $m_1 \equiv m_2 \mod \gcd(|a_1|, |a_2|)$.

Proof. The claim in (i) follows from the fact that A_1 and A_2 are Abelian. Now, for all $(a_1, a_2) \in A_1 \times A_2$, $(\delta_1 \times \delta_2)(a_1, a_2) = (a_1^{m_1}, a_2^{m_2})$. But, since $A_1 \times A_2$ is also Abelian, its descending endomorphisms are exactly its power endomorphisms, and therefore, $\delta_1 \times \delta_2$ is descending if and only if there exists an integer m such that

$$m \equiv m_1 \mod |a_1|$$
$$m \equiv m_2 \mod |a_2|.$$

By a well-known generalisation of the Chinese remainder theorem, such an integer m exists if and only if $m_1 \equiv m_2 \mod \gcd(|a_1|, |a_2|)$. □

The generalisations of Theorems 3.6 and 3.7 to direct products of any finite number of groups follow by straightforward application of mathematical induction.

Theorem 3.8. *Let $G = \prod_{i=1}^{k} G_i$ be the direct product of k groups $G_1, \ldots, G_k$. Let $\delta_i \in \operatorname{Des} G_i$, $i = 1, \ldots, k$, and let $\delta = \prod_{i=1}^{k} \delta_i$ denote their pointwise product. Then δ is a descending endomorphism of G if and only if for all $N_i \trianglelefteq G_i$, $i = 1, \ldots, k$, the restriction $\overline{\delta}|_{\overline{N}}$ is a descending endomorphism of $\overline{N}$, where $\overline{N} = \prod_{i=1}^{k} (N_i/[G_i, N_i])$ and $\overline{\delta} = \prod_{i=1}^{k} \overline{\delta}_i$.* □

Theorem 3.9. *Let $A = \prod_{i=1}^{k} A_i$ be the direct product of k torsion Abelian groups $A_1, \ldots, A_k$. If $\delta_i \in \operatorname{Des} A_i$, $i = 1, \ldots, k$, then*

(i) for each $a_i \in A_i$ there exists $m_i \in \mathbb{Z}$ such that $\delta_i(a_i) = a_i^{m_i}$, $i = 1, \ldots, k$, and

(ii) the pointwise product $\delta_1 \times \cdots \times \delta_k$ is a descending endomorphism of A if and only if for every pair $i, j \in \{1, \ldots, k\}$, and for all $a_i \in A_i$ and $a_j \in A_j$, $m_i \equiv m_j \mod \gcd(|a_i|, |a_j|)$. □

Remark 3.10. In Theorem 3.9, whenever A_i has finite exponent n_i, we may replace $|a_i|$ in (ii) by n_i, thus making the condition independent of individual elements $a_i \in A_i$.

In the beginning of this section, we made the observation that $\operatorname{Des} G_1 \times G_2 \subseteq \operatorname{Des} G_1 \times \operatorname{Des} G_2$, and Theorems 3.6 and 3.7 tell us exactly which elements of the latter are present in the former. Now we characterise all the direct products for which this inclusion becomes an equality.

Definition 3.11 ([6]). A group G is *super-perfect* if $[G, N] = N$ for all $N \trianglelefteq G$.

We note that this definition of the term super-perfect is as given in [6], and is unrelated to the more familiar notion of superperfection appearing in group homology.

Theorem 3.12. $\operatorname{Des} G_1 \times G_2 = \operatorname{Des} G_1 \times \operatorname{Des} G_2$ *if and only if one of the following holds.*

(i) G_1 *or* G_2 *is super-perfect.*

(ii) For all $N_1 \trianglelefteq G_1$ *and* $N_2 \trianglelefteq G_2$, $\overline{N}_1$ *and* $\overline{N}_2$ *are torsion groups, and the order of any element of* $\overline{N}_1$ *is relatively prime to that of any element of* $\overline{N}_2$.

Proof. Let $\delta_i \in \operatorname{Des} G_i$, $i = 1, 2$, and let $N \trianglelefteq G_1 \times G_2$. If the condition in (i) holds, then either $\overline{N}_1 = 1$ and $\overline{\delta}_1$ is the identity map, or similarly, $\overline{\delta}_2$ is the identity map. In either case, $\overline{\delta}_1 \times \overline{\delta}_2 \in \operatorname{Des} \overline{N}_1 \times \operatorname{Des} \overline{N}_2$, and therefore by Theorem 3.6, $\delta_1 \times \delta_2 \in \operatorname{Des} G_1 \times G_2$. If, on the other hand, the condition in (ii) holds, then Theorem 3.7 implies that $\overline{\delta}_1 \times \overline{\delta}_2 \in \operatorname{Des} \overline{N}_1 \times \operatorname{Des} \overline{N}_2$.

Conversely, assume that $\operatorname{Des} G_1 \times G_2 = \operatorname{Des} G_1 \times \operatorname{Des} G_2$, and suppose that neither G_1 nor G_2 is super-perfect.

First, observe that $\overline{N}_1$ must be a torsion group. For otherwise, by Theorem 2.5, every descending endomorphism of $\overline{N}_1 \times \overline{N}_2$ must be a universal power endomorphism. But on taking $\delta_1 \in \operatorname{Des} G_1$ to be the trivial endomorphism and $\delta_2 \in \operatorname{Des} G_2$ to be the identity endomorphism, we find that $\overline{\delta}_1 \times \overline{\delta}_2 \in \operatorname{End} \overline{N}_1 \times \overline{N}_2$ is not a universal power endomorphism, which implies (by Theorem 3.6) that $\delta_1 \times \delta_2 \notin \operatorname{Des} G_1 \times G_2$, contradicting our assumption. Similarly, $\overline{N}_2$ must also be torsion.

Now, since $\overline{N}_1$ and $\overline{N}_2$ are torsion Abelian groups, let $a_i \in \overline{N_i}$ and let $n_i = |a_i|$, $i = 1, 2$. We must show that n_1 and n_2 are relatively prime. Again, take δ_1 and δ_2 to be, respectively, the trivial and identity endomorphisms of G_1 and G_2. Then from Theorem 2.5, $\overline{\delta}_i(a_i) = a_i^{m_i}$, for some $m_i \in \mathbb{Z}$. Clearly,

$$m_1 \equiv 0 \mod n_1 \tag{3.1}$$

$$m_2 \equiv 1 \mod n_2. \tag{3.2}$$

By Theorem 3.6, $\delta_1 \times \delta_2 \in \operatorname{Des} G_1 \times G_2$ implies that $\overline{\delta}_1 \times \overline{\delta}_2 \in \operatorname{Des} \overline{N}_1 \times \overline{N}_2$, which implies (by Theorem 3.7) that

$$m_1 \equiv m_2 \mod \gcd(n_1, n_2). \tag{3.3}$$

From (3.1), (3.2), and (3.3), $\gcd(n_1, n_2) = 1$. □

As descending endomorphisms are those that leave all normal subgroups invariant, it is natural to seek an alternative characterisation of direct products $G_1 \times G_2$ such that $\operatorname{Des} G_1 \times G_2 = \operatorname{Des} G_1 \times \operatorname{Des} G_2$ in terms of normal subgroups, without involving quotient groups. There indeed exists a simple and elegant characterisation of this kind, in terms of decomposable normal subgroups, defined below.

Definition 3.13 ([6]). A normal subgroup N of $G_1 \times G_2$ is G_1-G_2 *decomposable* if $N = N_1 \times N_2$, where $N_i \trianglelefteq G_i$, $i = 1, 2$.

Observe that N_i in this definition is necessarily the projection of N onto the direct factor G_i.

Theorem 3.14. $\operatorname{Des} G_1 \times G_2 = \operatorname{Des} G_1 \times \operatorname{Des} G_2$ *if and only if every normal subgroup of* $G_1 \times G_2$ *is* G_1-G_2 *decomposable.*

Proof. Let $N \trianglelefteq G_1 \times G_2$, and let $N_i = \pi_i(N)$, be the projection of N onto G_i, $i = 1, 2$. Let id_i be the identity automorphism and ϵ_i the trivial endomorphism of G_i, $i = 1, 2$.

If $\operatorname{Des} G_1 \times G_2 = \operatorname{Des} G_1 \times \operatorname{Des} G_2$, then $\operatorname{id}_1 \times \epsilon_2 \in \operatorname{Des} G_1 \times \operatorname{Des} G_2$, and therefore, $(\operatorname{id}_1 \times \epsilon_2)(N) = N_1 \times 1 \leq N$. Similarly, $(\epsilon_1 \times \operatorname{id}_2)(N) = 1 \times N_2 \leq N$. Hence, $N_1 \times N_2 \leq N$, which implies that $N = N_1 \times N_2$, showing that N is G_1-G_2 decomposable.

On the other hand, if $N = N_1 \times N_2$, then for all $\delta_i \in \operatorname{Des} G_i$, $i = 1, 2$, $\delta_1 \times \delta_2$ leaves N invariant, since $(\delta_1 \times \delta_2)(N) = \delta_1(N_1) \times \delta_2(N_2) \leq N_1 \times N_2 = N$. □

Now, Miller's result follows immediately from Theorems 3.12 and 3.14.

Corollary 3.15 ([6, Theorem 1]). *Every normal subgroup of $G_1 \times G_2$ is G_1-G_2 decomposable if and only if one of the following holds:*

(i) G_1 *or* G_2 *is super-perfect.*

(ii) For all $N_1 \trianglelefteq G_1$ *and* $N_2 \trianglelefteq G_2$, $\overline{N}_1$ *and* $\overline{N}_2$ *are torsion groups, and the order of any element of* $\overline{N}_1$ *is relatively prime to that of any element of* $\overline{N}_2$. □

The following generalisation of the preceding results can be proved in an analogous manner.

Theorem 3.16. *Let* $G = \prod_{i=1}^{k} G_i$ *be the direct product of* k *groups* $G_1, \dots, G_k$. *Then the following are equivalent:*

(i) $\operatorname{Des} G = \prod_{i=1}^{k} \operatorname{Des} G_i$.

(ii) Either there exists at most one $i \in \{1, \dots, k\}$ *such that* G_i *is not super-perfect, or for all* $i = 1, \dots, k$ *and all* $N_i \trianglelefteq G_i$, $N_i/[G_i, N_i]$ *is torsion and the order of each of its elements is relatively prime to that of every element of* $N_j/[G_j, N_j]$ *for all* $N_j \trianglelefteq G_j$ *whenever* $i \neq j = 1, \dots, k$.

(iii) Every normal subgroup $N \trianglelefteq G$ *is a direct product* $N = \prod_{i=1}^{k} N_i$ *of normal subgroups* $N_i \trianglelefteq G_i$, $i = 1, \dots, k$. □

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