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Multipliers of Hankel type transformable generalized functions

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Abstract In this paper the Zemanian space of Hankel type transformable functions $H_{\alpha,\beta}$ is shown to be nuclear, Schwartz, montel and reflexive. The Space O is completely characterized as the set of multipliers of $H_{\alpha,\beta}$ and $H'_{\alpha,\beta}$.

Finally certain topologies are considered on O and continuity properties of the multiplication operation with respect to those topologies are studied.

1 Introduction

Following Zemanian [8], we introduce the space $H_{\alpha,\beta}$ which consists of all those infinitely differentiable functions $\varphi = \varphi(x)$ defined on $I = (0, \infty)$ such that

$$\rho_{m,k}^{\alpha,\beta}(\varphi) = \sup_{x \in I} |x^m (x^{-1}D)^k x^{2\beta-1} \varphi(x)| < \infty, m, k \in \mathbb{N}, (\alpha - \beta) \geq -\frac{1}{2}.$$

Endowed with the topology generated by the family of seminorms $\{\rho_{m,k}^{\alpha,\beta}\}_{(m,k)\in\mathbb{N}\times\mathbb{N}}$, $H_{\alpha,\beta}$ is a Frechet Space.

This topology of $H_{\alpha,\beta}$ can also be defined by means of seminorms

$$\tau_{m,k}^{\alpha,\beta}(\varphi) = \sup_{x \in I} |(1 + x^2)^m (x^{-1}D)^k x^{2\beta-1} \varphi(x)|, m, k \in \mathbb{N}, \varphi \in H_{\alpha,\beta}, (\alpha - \beta) \geq -\frac{1}{2}.$$

By following the technique used in Zemanian [8], one can show that the vector space O of all those $\theta \in C^\infty(I)$ such that for every $k \in \mathbb{N}$ there exist $n_k \in \mathbb{N}$, $A_k > 0$ satisfying

$$|(x^{-1}D)^k \theta(x)| \leq A_k (1 + x^2)^{n_k}, x \in I$$

is a space of multipliers for $H_{\alpha,\beta}$. In characterizing O as the space of multipliers for $H'_{\alpha,\beta}$, we use the reflexivity of $H_{\alpha,\beta}$, which derives from the fact, previously established that $H_{\alpha,\beta}$ is nuclear.

One can easily note that most of the properties established here for $H_{\alpha,\beta}$, $H'_{\alpha,\beta}$ and O are similar to the corresponding ones for the schwartz space $\mathcal{S}$, its dual $\mathcal{S}'$ (the space of tempered distributions) and their space of multipliers O_M.

In this paper author is motivated by the work done by Betancor and Marrero [2].

2 Multipliers of $H_{\alpha,\beta}$.

A function $\theta = \theta(x)$ defined on I is said to be a multiplier for $H_{\alpha,\beta}$ if the map $\varphi \to \theta\varphi$ is continuous from $H_{\alpha,\beta}$ into $H_{\alpha,\beta}$. The main object of this section is to characterize the space of multipliers of $H_{\alpha,\beta}$.

Lemma 2.1. *For every $r, s \in \mathbb{R}$ the holds*

$$\frac{1 + r^2}{1 + s^2} \leq 2(1 + |r - s|^2).$$

Lemma 2.2. *Let* $a \in D(I)$ *be such that* $0 \le a \le 1$*, supp* $a = [1/2, 3/2]$ *and* $a(1) = 1$*. Also, let* $\{x_j\}_{j\in\mathbb{N}}$ *be a sequence of real numbers satisfying* $x_0 > 1$ *and* $x_{j+1} > x_j + 1$*.*
Define

$$\varphi(x) = x^{2\alpha} \sum_{j=0}^{\infty} \frac{a(x - x_j + 1)}{(1 + x_j^2)^j}, x \in I. \tag{2.1}$$

Then $\varphi \in H_{\alpha,\beta}$*.*

Proof. As the functions $a(x - x_j + 1)$ have pairwise disjoint supports, we note that the sum on the right-hand side of (2.1) is finite. Indeed, if $m, k \in \mathbb{N}$ and $x_j - 1/2 \le x \le x_j + 1/2$, we may write

$$(1 + x^2)^m (x^{-1}D)^k x^{2\beta-1}\varphi(x) = \left(\frac{1 + x^2}{1 + x_j^2}\right)^m \frac{(x^{-1}D)^k x^{2\alpha+2\beta-1} a(x - x_j + 1)}{(1 + x_j^2)^{j-m}}.$$

Now by Lemma 2.1, we conclude that $\tau^{\alpha,\beta}_{m,k}(\varphi) < \infty$, thus it shows that $\varphi \in H_{\alpha,\beta}$ as required. Thus proof is completed. □

Theorem 2.3. *(Characterization of multipliers of* $H_{\alpha,\beta}$*):*
The following statements are equivalent to each other.

i. *The function* $\theta = \theta(x) \in C^\infty(I)$*, and for every* $k \in \mathbb{N}$ *there exists* $n_k \in \mathbb{N}$ *such that* $(1 + x^2)^{-\eta_k}(x^{-1}D)^k\theta(x)$ *is bounded on I.*

ii. *The product* $\theta\varphi$ *lies in* $H_{\alpha,\beta}$ *whenever* $\varphi \in H_{\alpha,\beta}$*, and the map* $\varphi \to \theta\varphi$ *is a continuous endomorphism of* $H_{\alpha,\beta}$*.*

iii. *The function* θ *is infinitely differentiable on I, for every* $k \in \mathbb{N}$ *and every* $\varphi \in H_{\alpha,\beta}$ *the function* $\varphi(x)(x^{-1}D)^k\theta(x) \in H_{\alpha,\beta}$*, and the map* $\varphi(x) \to \varphi(x)(x^{-1}D)^k\theta(x)$ *is a continuous endomorphism of* $H_{\alpha,\beta}$*.*

Proof. By following technique used in Zemanian [8, p. 134], one can easily prove that (i) implies (ii).
To show that (ii) implies (iii) : Let us consider the function $\varphi \in H_{\alpha,\beta}$ defined by

$$\varphi(x) = x^{2\alpha}e^{-x^2} \tag{2.2}$$

By (ii) above,

$$\psi(x) = x^{2\alpha}\theta(x)e^{-x^2} \tag{2.3}$$

lies in $H_{\alpha,\beta}$, and hence

$$\theta(x) = x^{2\alpha}\psi(x)e^{-x^2} \tag{2.4}$$

is infinitely differentiable on I.
Now, it is sufficient to show that $(x^{-1}D)^k\theta(x)$ is a multiplier of $H_{\alpha,\beta}$ whenever θ is . But this can be easily established by induction on k, taking into account the formula

$$\varphi(x)(x^{-1}D)\theta(x) = x^{2\alpha}(x^{-1}D)x^{2\beta-1}\theta(x)\varphi(x) - \theta(x)x^{2\alpha}(x^{-1}D)x^{2\beta-1}\varphi(x),$$
$$\alpha + \beta = \frac{1}{2}, (\alpha - \beta) \ge -\frac{1}{2}.$$

along with the fact that if $\varphi \in H_{\alpha,\beta}$ then so is $x^{2\alpha}(x^{-1}D)^k\theta(x)\varphi(x)$.
Finally, let $\theta(x)$ be satisfy (iii). As (2.2) belongs to $H_{\alpha,\beta}$, so does (2.3). Thus $\theta(x)$ can be represented by (2.4), and in particular the limit $\lim_{x\to 0^+}\theta(x)$ exists. Now according to (iii), each $(x^{-1}D)^k\theta(x)$ is a multiplier of $H_{\alpha,\beta}$, and we conclude that $\lim_{x\to 0^+}(x^{-1}D)^k\theta(x)$ exists for all $k \in \mathbb{N}$.

Arguing by contradiction, let us assume that (i) is false. Then there exist $k \in \mathbb{N}$ and a sequence $\{x_j\}_{j\in\mathbb{N}}$ of real numbers, which by what has been just proved, may be chosen so that $x_0 > 1$ and $x_{j+1} > x_j + 1$, such that:

$$|(x^{-1}D)^k\theta(x)_{|_{x=x_j}}| > (1 + x_j^2)^j.$$

The function $\varphi \in H_{\alpha,\beta}$ constructed by means of $\{x_j\}_{j\in\mathbb{N}}$ as in Lemma 2.2 plainly satisfies

$$|x_j^{2\beta-1}\varphi(x_j)(x^{-1}D)^k\theta(x)_{|_{x=x_j}}| > a(1) = 1(j \in \mathbb{N}), (\alpha-\beta) \geq -\frac{1}{2}$$

contradicting (iii). Thus proof is completed. □

3 Topology and properties of the space of multipliers.

Following Zemanian [8], we denote by O the linear space of all those $\theta \in C^\infty(I)$ such that for every $k \in \mathbb{N}$ there exist $n_k \in \mathbb{N}, A_k > 0$ satisfying

$$|(x^{-1}D)^k\theta(x)| \leq A_k(1+x^2)^{n_k}, x \in I.$$

The equivalence between the conditions (i) and (ii) in Theorem 2.3 above characterizes O as the space of multipliers of $H_{\alpha,\beta}$, with independence of the value of the real parameter $(\alpha-\beta)$. However, once $(\alpha-\beta)$ has been fixed, the condition (iii) suggests to introduce on O the family of seminorms

$$\Gamma_{\alpha,\beta} = \{\eta^{\alpha,\beta}_{\varphi,k} : \varphi \in H_{\alpha,\beta}, k \in \mathbb{N}\},$$

where

$$\eta^{\alpha,\beta}_{\varphi,k}(\theta) = \sup_{x\in I}|x^{2\beta-1}\varphi(x)(x^{-1}D)^k\theta(x)|$$

As the map $\varphi(x) \to x^{\nu-\alpha+\beta}\varphi(x) = \psi(x)$ establishes an isomorphism between $H_{\alpha,\beta}$ and H_ν for any $\alpha-\beta, \nu \in \mathbb{R}$, the equality $\eta^{\alpha,\beta}_{\varphi,k}(\theta) = \eta^{\alpha,\beta}_{\psi,k}(\theta)$ holds whenever $k \in \mathbb{N}$ and $\theta \in O$. Therefore, all families $\Gamma_{\alpha,\beta}(\alpha-\beta \in \mathbb{R})$ define one and the same topology on O. In the sequel, unless otherwise stated, it will always be assumed that O is endowed with this topology, and $(\alpha-\beta)$ will be any real number.

Some Remarks:

i. If $\theta \in C^\infty(I)$ is such that $\eta^{\alpha,\beta}_{\varphi,k}(\theta) < \infty$ for every $\varphi \in H_{\alpha,\beta}$ and $k \in \mathbb{N}$, then $\theta \in O$. Indeed, fix $\varphi \in H_{\alpha,\beta}, m, k \in \mathbb{N}$ and for $0 \leq p \leq k$ define $\varphi_p \in H_{\alpha,\beta}$ by

$$\varphi_p(x) = (1+x^2)^m x^{2\alpha}(x^{-1}D)^{k-p}x^{2\beta-1}\varphi(x), x \in I.$$

As

$$(1+x^2)^m(x^{-1}D)x^{2\beta-1}(\theta\varphi)(x) = \sum_{p=0}^{k}\binom{k}{p}x^{2\beta-1}\varphi_p(x)(x^{-1}D)^p\theta(x), x \in I,$$

necessarily

$$\tau^{\alpha,\beta}_{m,k}(\theta\varphi) \leq \sum_{p=0}^{k}\binom{k}{p}\eta^{\alpha,\beta}_{\varphi_p,p}(\theta) \tag{3.1}$$

In general

$$\tau^{\alpha,\beta}_{m,k}(\varphi(x)(\frac{1}{x}D)^k\theta(x)) \leq \sum_{p=0}^{k}\binom{k}{p}\eta^{\alpha,\beta}_{\varphi_p,p+n}(\theta), n \in \mathbb{N}.$$

Our assertion now follows as in the proof that (iii) implies (i) in Theorem 2.3.

ii. The topology of O may be also generated by means of the family of seminorms, $\{\eta^{\alpha,\beta}_{m,k;\varphi} : (m,k) \in \mathbb{N}\times\mathbb{N}, \varphi \in H_{\alpha,\beta}\}$,
where

$$\eta^{\alpha,\beta}_{m,k;\varphi}(\theta) = \tau^{\alpha,\beta}_{m,k}(\theta\varphi), m, k \in \mathbb{N}, \varphi \in H_{\alpha,\beta}.$$

Let $k \in \mathbb{N}$ and for every $\varphi \in H_{\alpha,\beta}$ and every $p \in \mathbb{N}$ with $0 \leq p \leq k$, define $\varphi_p \in H_{\alpha,\beta}$ by

$$\varphi_p(x) = x^{2\alpha}(x^{-1}D)^p x^{2\beta-1}\varphi(x), x \in I.$$

If $\varphi \in H_{\alpha,\beta}$ and $\theta \in O$, the equality

$$x^{2\beta-1}\varphi(x)(x^{-1}D)^k\theta(x) = \sum_{p=0}^{k}(-1)^p \binom{k}{p} (x^{-1}D)^{k-p}x^{2\beta-1}(\theta\varphi_p)(x), x \in I$$

then shows that

$$\eta^{\alpha,\beta}_{\varphi,k}(\theta) \leq \sum_{p=0}^{k} \binom{k}{p} \eta^{\alpha,\beta}_{0,k-p;\varphi_p}(\theta).$$

By using (3.1) and this estimate, our assertion is proved.

Theorem 3.1. *The identity map $O \to \varepsilon(I)$ is continuous.*

Proof. It is sufficient to observe that

$$D^k\theta(x) = \frac{1}{x^{2\beta-1}\varphi(x)} \sum_{p=0}^{k} C_p x^{a(p)} x^{2\beta-1}\varphi(x)(x^{-1}D)^{b(p)}\theta(x), x \in I$$

for every $k \in \mathbb{N}$ and every $\theta \in O$, where $\varphi(x) = x^{2\alpha}e^{-x^2}(x \in I) \in H_{\alpha,\beta}, C_p > 0(0 \leq p \leq k)$ are suitable constants, and $a(p) \leq k, b(p) \leq k(0 \leq p \leq k)$ denote non-negative integers, with $C_k = 1$ and $a(k) = b(k) = k$.This completes the proof □

Theorem 3.2. *The linear topological space O is locally convex, Housdorff, non-metrizable and complete.*

Proof. It is enough to check the completeness property . Let $\{\theta_l\}_{l\in J}$ be a Cauchy net in O. As O injects continuously into $\varepsilon(I), \{\theta_l\}_{l\in J}$ is also a Cauchy net in $\varepsilon(I)$ being complete,$\{\theta_l\}_{l\in J}$ converges to some $\theta \in \varepsilon(I)$ in $\varepsilon(I)$. We must show that $\theta \in O$ and that $\{\theta_l\}_{l\in J}$ converges to θ in the topology of O. Fix $\varphi \in H_{\alpha,\beta}, k \in \mathbb{N}, \epsilon > 0$. By assumption, there exists $l_0 = l_0(\varphi, k, \epsilon) \in J$ such that

$$\eta^{\alpha,\beta}_{\varphi,k}(\theta_l - \theta_{l'}) < \epsilon, l, l' \geq l_0. \tag{3.2}$$

Let us consider $x \in I, \delta > 0$. As $\{\theta_l\}_{l\in J}$ converges to θ in $\varepsilon(I)$, there holds

$$|x^{2\beta-1}\varphi(x)(x^{-1}D)^k(\theta - \theta_{l'}(x)| < \delta \tag{3.3}$$

for some $l' = l'(\varphi, x, \delta) \geq l_0$. The combination of (3.2) and (3.3) yields

$$|x^{2\beta-1}\varphi(x)(x^{-1}D)^k(\theta - \theta_l(x)| < \epsilon + \delta, l \geq l_0,$$

and from the arbitrariness of x and δ, we infer that

$$\eta^{\alpha,\beta}_{\varphi,k}(\theta - \theta_l) < \epsilon, l \geq l_0.$$

By the inequality

$$\eta^{\alpha,\beta}_{\varphi,k}(\theta) \leq \eta^{\alpha,\beta}_{\varphi,k}(\theta - \theta_l) + \eta^{\alpha,\beta}_{\varphi,k}(\theta_l), l \geq l_0.$$

We finally prove that $\theta \in O$ and $\{\theta_l\}_{l\in J}$ converges to θ in O. Thus proof is completed. □

Now in the next theorem, we state several continuity properties of certain operators on O.

Theorem 3.3. *The following statements hold:*

i. The bilinear map $O \times O \to O$, defined by $(\theta, \varphi) \to \theta\varphi$ is separately continuous.

ii. If $R(x) = P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials and Q does not vanish in $[0, \infty)$, then the map $\theta(x) \to R(x^2)\theta(x)$ is continuous from O to O.

iii. For every $k \in \mathbb{N}$, the map $\theta(x) \to (x^{-1}D)^k\theta(x)$ is continuous from O to O.

Proof. Let $k \in \mathbb{N}$ and $\theta \in O$, and for $0 \leq p \leq k$, let $n_p \in \mathbb{N}, A_p > 0$ be such that

$$|(x^{-1}D)^p \theta(x)| \leq A_p(1+x^2)^{n_p}, x \in I.$$

Set

$$\varphi_p(x) = (1+x^2)^{n_p}\varphi(x), x \in I, \varphi \in H_{\alpha,\beta}.$$

One can easily deduce that $\varphi_p \in H_{\alpha,\beta}$. Now the formula

$$x^{2\beta-1}\varphi(x)(x^{-1}D)^k(\theta\varphi)(x) = \sum_{p=0}^{k} \begin{pmatrix} k \\ p \end{pmatrix} x^{2\beta-1}\phi_p(x)\frac{(x^{-1}D)^p\theta(x)}{(1+x^2)^{n_p}}(x^{-1}D)^{k-p}\varphi(x),$$

valid for all $x \in I$, leads to the inequality

$$\eta^{\alpha,\beta}_{\varphi,k}(\theta\varphi) \leq \sum_{p=0}^{k} \begin{pmatrix} k \\ p \end{pmatrix} A_p \eta^{\alpha,\beta}_{\varphi_p,k-p}(\varphi).$$

Thus (i) is proved.
By using Lemma 5.3.1 of Zemanian [7] and (i) above, we can immediately prove(ii). As

$$\eta^{\alpha,\beta}_{\varphi,k}((x^{-1}D)^k\theta(x)) = \eta^{\alpha,\beta}_{\varphi,k+p}(\theta),$$

proof of (iii) is clear. Thus proof of the theorem is completed. □

Theorem 3.4. *The bilinear map $O \times H_{\alpha,\beta} \to H_{\alpha,\beta}$ defined by $(\theta, \varphi) \to \theta\varphi$ is separately continuous.*

Proof. Proof is clear from Theorem 2.3 and part (i) stated in some remarks preceding Theorem 3.1. □

Theorem 3.5. *The map $\varphi(x) \to x^{2\beta-1}\varphi(x)$ is continuous from $H_{\alpha,\beta}$ into O.*

Proof. Proof is immediate from the following:

$$\eta^{\alpha,\beta}_{\varphi,k}(x^{2\beta-1}\psi(x)) \leq \sup_{x\in \mathrm{I}} |x^{2\beta-1}\varphi(x)|\lambda^{\alpha,\beta}_{0,k}(\psi), \psi, \varphi \in H_{\alpha,\beta}, k \in \mathbb{N}.$$

□

Lemma 3.6. *The test space $\mathcal{D}(I)$ is not dense in $x^{2\beta-1}H_{\alpha,\beta}$ with respect to the topology of O.*

Proof. Let $\psi \in H_{\alpha,\beta}$ and assume that $\{x^{2\beta-1}a_l(x)\}_{l\in J}$ is a net in $\mathcal{D}(I)$ converging to $x^{2\beta-1}\psi(x)$ in the topology of O. For $k \in \mathbb{N}, \epsilon > 0$, there exists $l_0 = l_0(k,\epsilon) \in J$, with

$$|e^{-x^2}(x^{-1}D)^k x^{2\beta-1}(a_{l_0} - \psi)(x)| < \epsilon/e, x \in I.$$

Now, for $x \in (0,1)$, we may write

$$|(x^{-1}D)^k x^{2\beta-1}(a_{l_0} - \psi)(x)| < e|e^{-x^2}(x^{-1}D)^k x^{2\beta-1}(a_{l_0} - \psi)(x)| < \epsilon.$$

Therefore, to every $k \in \mathbb{N}$ and every $n = 1, 2, 3, \ldots$ there corresponds $l_n \in J, x_n \in (0, 1/n)$ such that

$$|(x^{-1}D)^k x^{2\beta-1}\psi(x)_{|x=x_n}| \leq |(x^{-1}D)^k x^{2\beta-1}(\alpha_{l_n} - \psi)(x)_{|x=x_n}| + |(x^{-1}D)^k x^{2\beta-1}a_{l_n}(x)_{|x=x_n}| < 1/n,$$

hence,

$$\lim_{n\to\infty}(x^{-1}D)^k x^{2\beta-1}\psi(x)_{|x=x_n} = 0.$$

However, the particularizations $\psi(x) = x^{2\alpha}e^{-x^2}$ and $k = 0$ lead to

$$\lim_{x\to 0^+}(x^{-1}D)^k x^{2\beta-1}\psi(x) = 1,$$

which is contradiction to the assumption. Thus proof is completed. □

Theorem 3.7. *Let $(\alpha - \beta) \geq -1/2$. Given $\theta \in O$, the function $x^{2\alpha}\theta(x)$ defines an element in $H'_{\alpha,\beta}$ by the formula*

$$\langle x^{2\alpha}\theta(x), \varphi(x)\rangle = \int_0^\infty x^{2\alpha}\theta(x)\varphi(x)dx, \varphi \in H_{\alpha,\beta}, \tag{3.4}$$

and the map $\theta(x) \to x^{2\alpha}\theta(x)$ is continuous from O into $H'_{\alpha,\beta}$.

Proof. Let $\theta \in O, \varphi \in H_{\alpha,\beta}$ and choose $r \in \mathbb{N}, A_r > 0$ satisfying

$$|\theta(x)| \leq A_r(1 + x^2)^r, x \in I.$$

Further, let $s \in \mathbb{N}, s > (3\alpha + \beta)$ be such that

$$C_s^{\alpha,\beta} = \int_0^\infty \frac{x^{4\alpha}}{(1 + x^2)^s}dx < \infty.$$

By multiplying and dividing the integrand in (3.4) By $x^{2\beta-1}(1 + x^2)^s$ we find that

$$|\langle x^{2\alpha}\theta(x), \varphi(x)\rangle| \leq A_r C_s^{\alpha,\beta}\tau_{r+s,0}^{\alpha,\beta}(\varphi),$$

and that

$$|\langle x^{2\alpha}\theta(x), \varphi(x)\rangle| \leq C_s^{\alpha,\beta}\eta_{\psi,0}^{\alpha,\beta}(\theta),$$

where $\psi(x) = (1 + x^2)^s\varphi(x) \in H_{\alpha,\beta}$.
Thus proof is completed. □

4 Multipliers of $H'_{\alpha,\beta}$.

In this section our aim is to characterize O as the space of multipliers of $H'_{\alpha,\beta}((\alpha-\beta) \in \mathbb{R})$. The reflexivity of $H_{\alpha,\beta}$ will be needed for that purpose. First we shall prove the following Lemma.

Lemma 4.1. *Let $m, k \in \mathbb{N}$, and let $\varphi \in H_{\alpha,\beta}$. Then*

$$\sum_{k=0}^{m} \sup_{x\in I} |(1 + x^2)^m(x^{-1}D)^k x^{2\beta-1}\psi(x)| \leq (m + 1)\sum_{k=0}^{m+1}\int_0^\infty |(1 + t^2)^{m+1}(t^{-1}D)^k t^{2\beta-1}\varphi(t)|dt.$$

Proof. We have

$$\begin{aligned}(1 + x^2)^m(x^{-1}D)^k x^{2\beta-1}\varphi(x) &= -\int_x^\infty D((1 + t^2)^m(t^{-1}D)^k t^{2\beta-1}\varphi(t))dt \\ &= -\int_x^\infty 2mt(1 + t^2)^{m-1}(t^{-1}D)^k t^{2\beta-1}\varphi(t)dt \\ &\quad - \int_x^\infty t(1 + t^2)^m(t^{-1}D)^{k+1}t^{2\beta-1}\varphi(t)dt, x \in I.\end{aligned}$$

As $2t \leq 1 + t^2, (t \in I)$, it follows that

$$|(1 + x^2)^m(x^{-1}D)^k x^{2\beta-1}\varphi(x)| \leq \int_0^\infty |(1 + t^2)^{m+1}(t^{-1}D)^{k+1}t^{2\beta-1}\varphi(t)|dt, x \in I.$$

Thus proof is completed. □

Theorem 4.2. *The space $H_{\alpha,\beta}$ is nuclear.*

Proof. Let $\varphi \in H_{\alpha,\beta}, m, k \in \mathbb{N}$. Now for $t \in I$ and $0 \leq k \leq m + 2$ we define $u_{t,k} \in H'_{\alpha,\beta}$ by the formula

$$\langle u_{t,k}, \varphi\rangle = (1 + t^2)^{m+2}(t^{-1}D)^k t^{2\beta-1}\varphi(t),$$

Let

$$V = \{\varphi \in H_{\alpha,\beta} : \sum_{k=0}^{m+2} \sup_{t\in I} |(1+t^2)^{m+2}(t^{-1}D)^k t^{2\beta-1}\varphi(t)| < 1\}.$$

One can easily note that V is a neighbourhood of the origin in $H_{\alpha,\beta}$, and that each $u_{t,k}(t \in I, 0 \le k \le m+2)$ belongs to V^0, the polar set of V. Thus a positive Radon measure σ may be defined on V^0 by the relation

$$\langle \sigma, \psi\rangle = \int_{V^0} \psi d\sigma = (m+1)\sum_{k=0}^{m+2} \int_0^\infty \psi(u_{t,k})(1+t^2)^{-1}dt, \psi \in C(V^0).$$

Now by using Lemma 4.1, we can infer that

$$\begin{aligned}\sum_{k=0}^{m} \sup_{x\in I} |(1+x^2)^m (x^{-1}D)^k x^{2\beta-1}\varphi(x)| &\le (m+1)\sum_{k=0}^{m+2}\int_0^\infty |(1+t^2)^{m+1}(t^{-1}D)^k t^{2\beta-1}\varphi(t)|dt \\ &= (m+1)\sum_{k=0}^{m+2} |\langle u_{t,k},\varphi\rangle|(1+t^2)^{-1}dt \\ &= \int_{V^0} |\langle u,\varphi\rangle|\, d\sigma(u),\ \varphi \in H_{\alpha,\beta}.\end{aligned}$$

Because the sets

$$V(m,\epsilon) = \{\varphi \in H_{\alpha,\beta} : \sum_{k=0}^{m} \sup_{x\in I} |(1+x^2)^m (x^{-1}D)^k x^{2\beta-1}\varphi(x)| < \epsilon\}, m \in \mathbb{N}, \epsilon > 0.$$

form a basis of neighbourhoods of the origin in $H'_{\alpha,\beta}$, the nuclearity of this space follows from Pietsch [4, Proposition 4.1.5].
This finishes the proof of the theorem. □

Now as applications of the Theorem 4.2, we have following corollaries.

Corollary 4.3. *The space $H'_{\alpha,\beta}$ is nuclear with respect to its strong topology.*

Proof. Proof is clear from Treves[5, proposition III. 50.6]. □

Corollary 4.4. *$H_{\alpha,\beta}$ (with its usual topology) and $H'_{\alpha,\beta}$ (with the strong topology) are Schwartz spaces.*

Proof. Following the technique used in Wong [6, Proposition 3.2.5], we can complete the proof. □

Corollary 4.5. *The space $H_{\alpha,\beta}$ is Montel and hence reflexive.*

Proof. By Horvath [3, corollary to Proposition 3.15.4], Frechet Schwartz spaces are Montel and by Horvath[3, corollary to proposition 3.9.1], Motel spaces are reflexive. Thus proof is completed. □

Definition 4.6. For $\theta \in O$ and $T \in H'_{\alpha,\beta}, \theta T$ is defined by transposition

$$\langle \theta T, \varphi\rangle = \langle T, \theta\varphi\rangle, \varphi \in H_{\alpha,\beta}.$$

Theorem 3.4 guarantees that $\theta T \in H'_{\alpha,\beta}$ and that each map $T \to \theta T$ is continuous from $H'_{\alpha,\beta}$ to $H'_{\alpha,\beta}$. By applying the universal property of initial topologies, we also find that the map $\theta \to \theta T$ is continuous from O in to $H'_{\alpha,\beta}$ if the latter is equipped with its weak topology. Thus we have the following

Theorem 4.7. *The bilinear map*

$$\begin{aligned} O \times H'_{\alpha,\beta} \to H'_{\alpha,\beta} \quad & defined\ by \\ (\theta, T) \to \theta T & \end{aligned}$$

is separately continuous when $H'_{\alpha,\beta}$ is endowed with its weak topology.*

Given $a > 0$ and $(\alpha - \beta) \in \mathbb{R}, B_{\alpha,\beta,a}$ (see [6]) is the subspace of $H_{\alpha,\beta}$ formed by all those functions $\psi = \psi(x)$ infinitely differentiable on I such that $\psi(x) = 0(x \geq a)$, for which the quantities

$$\rho_k^{\alpha,\beta}(\psi) = \sup_{x \in I} |(x^{-1}D)^k x^{2\beta-1}\psi(x)| < \infty, k \in \mathbb{N}.$$

When equipped with the topology generated by the family of seminorms $\{\rho_k^{\alpha,\beta}\}_{k\in\mathbb{N}}, B_{\alpha,\beta,a}$ becomes a Frechet space. It is easy to see that $B_{\alpha,\beta,a} \subset B_{\alpha,\beta,b}$ if $0 < a < b$ and that $B_{\alpha,\beta,a}$ inherits from $B_{\alpha,\beta,a}$ its own topology. These facts allow us to define $B_{\alpha,\beta} = \cup_{a>0} B_{\alpha,\beta,a}$ as the inductive limit of the family $\{B_{\alpha,\beta,a}\}_{a>0}$. The space $B_{\alpha,\beta}$ turns out to be dense in $H_{\alpha,\beta}$.

Definition 4.8. Let $\theta \in C^\infty(I)$ be such that $(x^{-1}D)^k\theta(x)$ is bounded in a neighbourhood of zero for every $k \in \mathbb{N}$.If $T \in H'_{\alpha,\beta}$ then T lies in $B'_{\alpha,\beta}$, the dual space of $B_{\alpha,\beta}$, and $\theta T \in B'_{\alpha,\beta}$ may be consistently defined by the formula

$$\langle \theta T, \psi \rangle = \langle T, \theta\psi \rangle, \psi \in B_{\alpha,\beta}.$$

Now at this stage we are ready to prove that the space of multipliers of $H'_{\alpha,\beta}$ is precisely O.

Theorem 4.9. *Assume that $\theta \in C^\infty(I)$ is such that $(x^{-1}D)^k\theta(x), k \in \mathbb{N}$ is bounded in a neighbourhood of zero. If for every $T \in H'_{\alpha,\beta}$, the functional $\theta T \in B'_{\alpha,\beta}$ extended up to $H_{\alpha,\beta}$ as a member of $H'_{\alpha,\beta}$ in such a way that the map $\theta \to \theta T$ is continuous from $H'_{\alpha,\beta}$ into itself, then $\theta \in O$.*

Proof. If $\varphi \in H_{\alpha,\beta}$ then our hypothesis imply that the linear functional $T \to \langle \theta T, \varphi \rangle$ is continuous on $H'_{\alpha,\beta}$. By the reflexivity of $H_{\alpha,\beta}$, there exists $\psi \in H_{\alpha,\beta}$ satisfying

$$\langle \theta T, \varphi \rangle = \langle T, \psi \rangle, T \in H'_{\alpha,\beta}.$$

In particular,

$$\langle \theta\varphi, v \rangle = \langle \theta v, \varphi \rangle = \langle v, \psi \rangle = \langle \psi, v \rangle, v \in B_{\alpha,\beta}.$$

Hence, $\theta\varphi = \psi \in H_{\alpha,\beta}$. As space of multipliers of $H_{\alpha,\beta}$ is O, we conclude tnat $\theta \in O$. Thus proof is completed □

5 Another topology on O.

Let $(\alpha - \beta)$ be any real number, and let $\mathfrak{B}_{\alpha,\beta}$ denote the family of all bounded subsets of $H_{\alpha,\beta}$. Throughout this section we shall assume that O is endowed with the topology generated by the family of seminorms

$$\eta_{B,k}^{\alpha,\beta} = \sup\{\eta_{\varphi,k}^{\alpha,\beta} : \varphi \in B\}, B \in \mathfrak{B}_{\alpha,\beta}, k \in \mathbb{N}. \tag{5.1}$$

Theorem 5.1. *The topological vector space O is locally convex, Hausdorff, nonmetrizable and complete.*

Proof. It is enough to check the completeness property . For, let $\{\theta_l\}_{l\in J}$ be a Cauchy net in O. As $\{\theta_l\}_{l\in J}$ is also Cauchy with respect to the topology considered on O in section 3 above, there exists $\theta \in O$ such that $\{\theta_l\}_{l\in J}$ converges to θ in that topology.
Let $B \in \mathfrak{B}_{\alpha,\beta}, k \in \mathbb{N}, \epsilon > 0$. By hypothesis there exists $l_0 = l_0(B, k, \epsilon) \in J$ such that

$$\eta_{B,k}^{\alpha,\beta}(\theta_l - \theta_{l'}) < \epsilon/2 \;\; l, l' \geq l_0.$$

Thus to every $\varphi \in B$ there corresponds $l' = l'(\varphi, k, \epsilon) \geq l_0$ satisfying

$$\eta_{\varphi,k}^{\alpha,\beta}(\theta_{l'} - \theta) < \epsilon/2.$$

If we combine the last two inequalities, it shows that

$$\eta_{B,k}^{\alpha,\beta}(\theta_l - \theta) < \epsilon, l \geq l_0.$$

Hence, $\{\theta_l\}_{l\in J}$ converges to θ in O.
This completes the proof. □

Theorem 5.2. *The bilinear map*

$$O \times H_{\alpha,\beta} \to H_{\alpha,\beta} \text{ defined by}$$
$$(\theta,\varphi) \to \theta\varphi \tag{5.2}$$

is hypocontinuous.

Proof. First we note that the topology defined on O is finer than that introduced in section 3. Any two spaces $H_{\alpha,\beta}$ and $H_{a,b}$ being isomorphic, this topology does not depend on the parameter $(\alpha - \beta)$. Now by Theorem 3.4 the bilinear map defined by (5.2) above is continuous. As $H_{\alpha,\beta}$ is a Frechet space, the uniform boundedness principle grantees the hypocontinuity with respect to the bounded subsets of O. On the otherside, we take $m, k \in \mathbb{N}$ and for every $\varphi \in H_{\alpha,\beta}$ and every $p \in \mathbb{N}, 0 \le p \le k$, define $\varphi_p \in H_{\alpha,\beta}$ by

$$\varphi_p(x) = (1+x^2)^m x^{2\alpha}(x^{-1}D)^{k-p}x^{2\beta-1}\varphi(x), x \in I.$$

The map $\varphi \to \varphi_p$ is continuous from $H_{\alpha,\beta}$ into $H_{\alpha,\beta}$ by Leibnitz's rule. Denoting by $B_p \in \mathfrak{B}_{\alpha,\beta}$ the image of $B \in \mathfrak{B}_{\alpha,\beta}$ through this map, it can be proved as in the part (i) of the remark preceding Theorem 3.1 that

$$\tau^{\alpha,\beta}_{m,k}(\theta\varphi) \le \sum_{p=0}^{k} \begin{pmatrix} k \\ p \end{pmatrix} \eta^{\alpha,\beta}_{B_p,p}(\theta), \theta \in O, \varphi \in B. \tag{5.3}$$

Therefore, (5.2) is $\mathfrak{B}_{\alpha,\beta}$- hypocontinuous.
Thus proof is completed. □

It must be observed that the topology generated on O by the seminorms(5.1) is compatible with the family

$$\eta^{\alpha,\beta}_{m,k;B}(\theta) = \sup\{\tau^{\alpha,\beta}_{m,k}(\theta\varphi) : \varphi \in B\}, m, k \in \mathbb{N}, B \in \mathfrak{B}_{\alpha,\beta}.$$

Indeed, let $k \in \mathbb{N}$. For each $p \in \mathbb{N}$ with $0 \le p \le k$, the map $\varphi \to \varphi_p$ defined from $H_{\alpha,\beta}$ by the formula

$$\varphi_p = x^{2\alpha}(x^{-1}D)^p x^{2\beta-1}\varphi(x), x \in I, (\alpha-\beta) \ge -\frac{1}{2}.$$

is continuous . As before we denote by $B_p \in \mathfrak{B}_{\alpha,\beta}$ the image of $B \in \mathfrak{B}_{\alpha,\beta}$ through this map. Now, the argument in the part (ii) of the remark preceding Theorem 3.1 shows that

$$\eta^{\alpha,\beta}_{B,k}(\theta) \le \sum_{p=0}^{k} \begin{pmatrix} k \\ p \end{pmatrix} \eta^{\alpha,\beta}_{0,k-p;B_p}(\theta), B \in \mathfrak{B}_{\alpha,\beta}, k \in \mathbb{N}, \theta \in O.$$

Along with (5.3) this estimate proves our assertion.

Theorem 5.3. *The bilinear map*

$$O \times H'_{\alpha,\beta} \to H'_{\alpha,\beta} \text{ defined by}$$
$$(\theta,\varphi) \to \theta\varphi$$

is separately continuous when $H'_{\alpha,\beta}$ is endowed either with its weak or with its strong topology.*

Proof. By following [5, Propositions II. 19.5 and II. 35.8], continuity in the second variable can be proved.
Let $T \in H'_{\alpha,\beta}, \theta \in O, B \in \mathfrak{B}_{\alpha,\beta}$. There exist $r \in \mathbb{N}$ and a constant $C > 0$ such that

$$|\langle T,\varphi\rangle| \le C \max_{0 \le m,k \le r} \tau^{\alpha,\beta}_{m,k}(\psi), \psi \in H_{\alpha,\beta}.$$

Thus

$$|\langle \theta T,\varphi\rangle| = |\langle T,\theta\varphi\rangle| \le C \max_{0 \le m,k \le r} \tau^{\alpha,\beta}_{m,k}(\theta\varphi), \varphi \in B, .$$

which leads to the inequality

$$\sup\{|\langle \theta T,\varphi\rangle| : \varphi \in B\} \le C \max_{0 \le m,k \le r} \eta^{\alpha,\beta}_{m,k;B}(\theta).$$

Thus proof is completed. □

Theorem 5.4. *The bilinear map*

$$O \times O \to O \text{ defined by}$$

$$(\theta, v) \to \theta v$$

is hypocontinuous.

Proof. Let $\mathfrak{B}$ denote the family of all bounded subsets of O. If $A \in B$ and $B \in \mathfrak{B}_{\alpha,\beta}$, a fortiori $AB \in \mathfrak{B}_{\alpha,\beta}$ (Theorem 5.2 and [3, Proposition 4.7.2]).Fix $m, k \in \mathbb{N}, \theta \in A, v \in O, \varphi \in B$, then

$$\eta^{\alpha,\beta}_{m,k;B}(\theta v) \leq \eta^{\alpha,\beta}_{m,k;AB}(v).$$

This completes the proof. □

Special cases:

(i) If we take $\alpha = \frac{1}{4} + \frac{\mu}{2}, \beta = \frac{1}{4} - \frac{\mu}{2}$ throughout this paper, then all the results studied in this paper reduce to the results studied in [2].

(ii) Author claims that, our results are stronger than that of [2].

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