Palestine Journal of Mathematics

Vol. 12(1)(2023) , 352–360

Quantum Calculus and Nörlund integrability

A. Sudhakar, Banitamani Mallik, *Birupakhya Prasad Padhy and Umakanta Misra

Communicated by V. Lokesha

MSC 2020 Classifications: 40E05, 05A30, 26A03

Keywords and phrases: q-calculus, Cesàro integrable, Nörlund integrability, Tauberian theorems, Jackson integral.

The authors would like to thank the Editor and the anonymous referees for their valuable comments and suggestions, which helped us to improve the quality of this manuscript.

Abstract. Quantum calculus is the modern technique vastly used for the investigations in calculus. Summability theory is the theory of assigning sums, that is fundamental in analysis, has wide range of applications concerning with infinite series and Fourier series. In the present article, we have established some properties on Nörlund integrability and Tauberian theorems by defining the Nörlund integrability method in quantum calculus.

1 Introduction

q-calculus often known as quantum calculus is a methodology that is similar to classical calculus but focuses on obtaining q-analogous findings without employing the concept of limits. It is a modern technique which is applied in different field of mathematics such as theory of numbers, combinatorics, orthogonal polynomials etc. since last decades of twentieth century. It has also wide applications in different field of science particularly in the areas like quantum theory, theory of relativity and statistical mechanics of physics. Hence q-calculus is a bridge by which mathematicians and physicist share their views in the same ground.
The theory of summability has a wide range of usage in analysis and practical mathematics. Engineers and physicists who work with the Fourier series or analytic continuation will find the concept useful in their work. The idea of summability has been used to a sequence of fuzzy numbers as well as ergodic theory theorems. A summability technique is essentially a function from the set. A summability method is a function that returns a value from a set of partial sums of a series. Summability is the theory of assigning sums, which is fundamental in analysis, function theory, topolgy, and functional analysis in its broadest sense. Summability approaches for continuous functions are becoming increasingly prominent in recent years. Currently, a number of significant classical works are being released.
Throughout the paper, the functions that are dealt with are real valued functions and the numbers are real. For any function $f(t)$ defined on $[0,\infty)$ let

$$s(t) \;=\; \int_0^t f(x)\,dx. \tag{1.1}$$

The above function $s(t)$ is Cesàro integrable, if there exists a finite number s such that

$$\frac{1}{t}\int_0^t s(x)dx \;=\; s,\; as\, t\to\infty. \tag{1.2}$$

For any non-decreasing function $p(t)$ on $[0,\infty)$, let

$$P(t) = \int_0^t p(x)\,dx,\; p(x)\neq 0,\; p(0)=0 \tag{1.3}$$

The function $s(t)$ is Nörlund integrable, if there is a finite number s for which

$$\frac{1}{P(x)}\int_0^x p(t)\ s(t)\ dt \ = \ s, \ as\ x \to \infty. \tag{1.4}$$

It is clear that, Nörlund integrable method is regular. That is, (1.4) is true whenever (1.2) holds. But (1.4) does not imply (1.2) always. However, by a suitable additional condition to $s(x)$, (1.4) implies (1.2). Such type of conditions are usually termed as Tauberian conditions, named after the famous mathematician A.Tauber, and theorem associated with those conditions is called Tauberian theorem.

2 q-calculus

In order to justify the article, some definitions and properties on q-calculus that are related to this work have been presented in this section.

2.1 Definitions

Definition 2.1. For a real valued function $\phi(t)$ and a number q, the q-differential of $\phi(t)$, usually denoted as $d_q\phi(t)$ is defined as [7]

$$d_q\phi(t) \ = \ \phi(qt) \ - \ \phi(t) \tag{2.1}$$

Clearly $d_qt \ = \ (q-1)t$. For any two functions $\phi(t)$ and $\psi(t)$, we have [7]

$$d_q(\phi(t)\psi(t)) \ = \ \phi(qt)d_q\psi(t) \ + \ \psi(t)d_q\phi(t) \tag{2.2}$$

Definition 2.2. $D_q\phi(t)$, the q-derivative of the $\phi(t)$, is defined by [7]

$$D_q\phi(t) \ = \ \frac{d_\phi(t)}{d_qt} \ = \ \frac{\phi(qt) \ - \ \phi(t)}{(q \ - \ 1)t} \tag{2.3}$$

In particular, if $\phi(t)$ is differentiable, then

$$lim_{t\to 1}D_q\phi(t) \ = \ \frac{d\phi(t)}{dt} \tag{2.4}$$

Clearly, Leibnitz notation $\frac{\phi(qx)}{dx}$ is a ratio of two infinitesimals, on the other hand q-derivative is simply a ratio.

In particular, for $\phi(t) = \ t^m$, where m is a positive integer

$$D_qt^m \ = \ \frac{(qt)^m \ - \ t^m}{(q \ - \ 1)t} \ = \ \frac{q^m \ - \ 1}{q \ - \ 1}\ t^{m-1} \ = \ [m]\ t^{m-1}, \tag{2.5}$$

where

$$[m] \ = \ \frac{q^m \ - 1}{q \ - 1} \ = \ q^{m-1} \ + \ ... + 1\ .$$

For any two functions $\phi(t)$ and $\psi(t)$ and two constants a and b

$$(i)\ D_q(a\ \phi(t) \ + \ b\ \psi(t)) \ = \ a\ D_q\phi(t) \ + \ b\ D_q\psi(t) \tag{2.6}$$

$$(ii)\ D_q(\phi(t)\ \psi(t)) \ = \ \psi(t)\ D_q\phi(t) \ + \ \phi(qt)\ D_q\psi(t) \tag{2.7}$$

and

$$(iii)\ D_q\Big(\frac{\phi(t)}{\psi(t)}\Big) \ = \ \frac{\psi(t)\ D_q\phi(t) \ - \ \phi(t)\ D_q\psi(t)}{\psi(t)\ \psi(qt)} \tag{2.8}$$

Thus, similar to differentiation in classical calculus, we too derive product rule and quotient rule in q-differentiation. However, the chain rule for q-derivative, in general, does not hold. As an exceptional case for the functions of the form $\phi(v(t))$, where, for the constants α and β, $v(t) = \alpha t^{\beta}$

$$\begin{aligned} D_q[\phi(v(t))] = D_q[\phi(\alpha t^{\beta})] &= \frac{\phi(\alpha q^{\beta} t^{\beta}) - \phi(\alpha t^{\beta})}{qt - t} \\ &= \frac{\phi(\alpha q^{\beta} t^{\beta}) - \phi(\alpha t^{\beta})}{\alpha q^{\beta} t - \alpha t^{\beta}} \frac{\alpha q^{\beta} t^{\beta} - \alpha t^{\beta})}{qt - t} \\ &= \frac{\phi(q^{\beta} v) - \phi(v)}{q^{\beta} v - v} \frac{v(qt) - u(t)}{qt - t} \end{aligned}$$

and therefore,

$$D_q\phi(v(t)) = (D_{q^{\beta}}\phi(v(t)) \, . \, D_q v(t). \tag{2.9}$$

similar to classical calculus. On the other hand, for $v(t) = t + t^2$ and $v(t) = sin\, t$, the quantity $v(qt)$ can not be presented in some form of v easily. Hence, it is impossible to have chain rule, in general.

Definition 2.3. Suppose $D_q\phi(t) = \Phi(t)$. Then the function $\phi(t)$ is called a q-antiderivative of $\Phi(t)$. It is denoted by $\int \Phi(t)\, d_q(t)$.

It is to be noted that in quantum calculus, it is not necessarily true that $D_q\phi(t) = 0$ if and only if ϕ is constant. Further, if $\phi(t)$ is expressed as a power series $\phi(t) = \sum_{n=0}^{\infty} a_n\, t^n$, then $\phi(t)$ has a unique q-antiderivative up to a constant term, which is

$$\int \phi(t)\, d_q t = \sum_{n=0}^{\infty} \frac{a_n t^{n+1}}{[n+1]} + C \tag{2.10}$$

Definition 2.4. The Jackson integrals of $\phi(t)$ over the intervals $[0, k]$ and $[0, \infty)$ are defined respectively by

$$\int_0^k \phi(t)\, d_q t = (1-q)k \sum_{n=0}^{\infty} \phi(kq^n)\, q^n \tag{2.11}$$

$$\int_0^{\infty} \phi(t)\, d_q t = (1-q) \sum_{n=-\infty}^{\infty} \phi(q^n)\, q^n \tag{2.12}$$

if the series are absolute-convergent.

Thus, the integral in (2.11) is the sum of the area of an infinite number of rectangles, as shown in figure-1 [7].

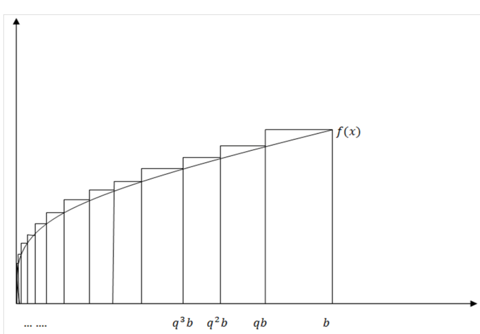

Figure 1. Jackson Integral

.

If $\Phi(t)$ is a q-antiderivative of $\phi(t)$ at $t = 0$ and $\Phi(t)$ is continuous, then we have

$$\int_a^b \phi(t)\, d_q t \;=\; \Phi(b) \;-\; \Phi(a) \tag{2.13}$$

Further, for any function ϕ, we have

$$D_q\Big(\int_0^t \phi(x)\, d_q x\Big) \;=\; \phi(t)\,. \tag{2.14}$$

In q-calculus, the change in order of integration is given by

$$\int_0^x \int_0^t \phi(s)\, d_q s\, d_q t \;=\; \int_0^x \int_{qs}^x \phi(s)\, d_q t\, d_q s \;. \tag{2.15}$$

Further, it is noted that, for $n \in \mathbb{R}$,

$$\int_0^x \phi(q^n t)\, d_q t \;=\; \frac{1}{q^n}\int_0^{xq^n} \phi(t)\, d_q t \;. \tag{2.16}$$

Definition 2.5. For $0 < q < 1$ and $z \in \mathcal{C}$ with $|z| < \frac{2}{1-q}$, the q-exponential function usually denoted by ε_q^z, is defined by[7]

$$\varepsilon_q^z \;=\; \sum_{n=0}^{\infty} \frac{z^n}{(n)!} \tag{2.17}$$

where $(n) \;=\; \frac{1+q+...+q^{n-1}}{\frac{1}{2}(1+q^{n-1})}$ and $(n)! \;=\; (1)(2)...(n)$.

Definition 2.6. For all $x \in \mathbb{R}$, q-trigonometric functions such as $q - sine$, $q - cosine$ are defined respectively as [7]

$$Sin_q x \;=\; \frac{\varepsilon_q^{ix} \;-\; \varepsilon_x^{-ix}}{2i} \quad \text{and} \quad Cos_q x \;=\; \frac{\varepsilon_q^{ix} \;+\; \varepsilon_x^{-ix}}{2} \tag{2.18}$$

such that $-1 \leq\; Cos_q x \;\leq 1$ and $-1 \leq\; Sin_q x \;\leq 1$

It is established by $Cie\acute{s}li\acute{n}ski$ [4] that

$$D_q Sin_q x \;=\; \frac{Cos_q x + Cos_q(qx)}{2} \quad \text{and} \quad D_q Cos_q x \;=\; -\,\frac{Sin_q x + Sin_q(qx)}{2} \tag{2.19}$$

3 q-summability of integrals

For $0 < q < 1$, we denote $\mathbb{R}_{q,+} \;=\; (q^n \;: n \in Z)$. For a function $f(t)$ continuous on $[0, \infty)$, let

$$s(t) \;=\; \int_0^t f(x)\, d_q x. \tag{3.1}$$

Then $s(x)$ is q-Cesàro integrable, if there exists a finite number A such that $\lim_{x\to\infty}\; \sigma(s(x)) = \;A$, where

$$\sigma(s(x)) \;=\; \frac{1}{x}\int_0^x s(t)\, d_q t. \tag{3.2}$$

Further, for a non-decreasing function $p(t)$ delineated on $[0, \infty)$, let

$$P(t) \;=\; \int_0^t p(t)\, d_q t,\; p(t) \neq\; 0,\; p(0) = \;0\,. \tag{3.3}$$

Then the function $s(t)$, as defined in (3.1), is q-Nörlund integrable, if there is a finite number A such that $lim_{x\to\infty}\; \nu(s(t)) \;=\; A$, where

$$\nu(s(t)) \;=\; \frac{1}{P(t)}\int_0^t p(x)s(x)\, d_q x. \tag{3.4}$$

From the following theorem it is clear that q-Nörlund integrability is regular. That is

Theorem 3.1. *The existence of the limit* $\lim_{x\to\infty} s(t) = l$, *implies* $\lim_{x\to\infty} \nu(s(t)) = l$.

Proof. Since $\lim_{t\to\infty} s(t) = l$, for a given $\delta > 0$ there is a $k > 0$ such that $|s(t) - l| < \frac{\delta}{2}$, for all $t > k$ and a finite number K such that $|s(t) - l| < K$, for all t. Now

$$\begin{aligned}
|\nu(s(t)) - l| &= \Big|\frac{1}{P(t)}\int_0^t p(x)s(x)\,d_qx - \frac{1}{P(t)}\int_0^t l\,p(x)\,d_qx\Big| \\
&= \Big|\frac{1}{P(t)}\int_0^t p(x)(s(x) - l)\,d_qx\Big| \\
&\le \frac{1}{P(t)}\int_0^t |p(x)(s(x) - l)|\,d_qx \\
&\le \frac{1}{P(t)}\int_0^k |p(x)(s(x) - l)|\,d_qx + \frac{1}{P(t)}\int_k^t |p(x)(s(x) - l)|\,d_qx \\
&\le \frac{P(k)K}{P(t)} + \frac{\delta}{2}\frac{(P(t) - P(k))}{P(t)} \\
&\le \frac{P(k)K}{P(t)} + \frac{\delta}{2}.
\end{aligned}$$

Clearly $\lim_{t\to\infty} \frac{P(k)K}{P(t)} = 0$. Hence for every $\delta > 0$, there is a k_1 such that $|\frac{P(k)K}{P(t)}| \le \frac{\delta}{2}$, for $t > k_1$. Let $k_0 = max\,(k, k_1)$. Then, for $t > k_0$

$$|\nu(s(t)) - l| < \delta. \tag{3.5}$$

Hence, $\lim_{t\to\infty} \nu(s(t)) = l$.
Thus the proof is completed. □

But, in general the counter part is not always true. For $p(t) = 1$, the integral $\int_0^x \frac{\cos_q t + \cos_q(qt)}{2}\,d_qt$ is q-Nörlund integrable to zero, though $\int_0^x \frac{\cos_q t + \cos_q(qt)}{2}\,d_qt$ does not exist. As in the standard calculus, addition of some extra condition on $s(t)$, $\lim_{t\to\infty} \nu(s(t)) = s$ may imply $\lim_{t\to\infty} (s(t)) = s$. The main purpose of this article is to establish certain Tauberian theorems on q-Nörlund integrability of improper integrals. Before that certain results have been presented as follows.

Theorem 3.2. *If* $\nu(s(t))$ *is the q-Nörlund mean of the integral* $s(t)$, *then*

$$s(t) - \nu(s(t)) = q\,w(t), \tag{3.6}$$

where $w(t) = \frac{1}{P(t)}\int_0^t P(qx)\,f(x)\,d_qx$

Proof. We have

$$\begin{aligned}
\nu(s(t)) &= \frac{1}{P(t)}\int_0^t p(x)\,s(x)\,d_qx \\
&= \frac{1}{P(t)}\int_0^t p(x)\Big(\int_0^x f(y)\,d_qy\Big)\,d_qx \\
&= \frac{1}{P(t)}\int_0^t f(y)\Big(\int_{qy}^t p(x)d_qt\Big)\,d_qy \\
&= \frac{1}{P(t)}\int_0^t \Big(P(t) - P(qy)\Big)\,f(y)\,d_qy \\
&= \int_0^t f(y)\,d_qy - \frac{1}{P(t)}\int_0^t P(qy))\,f(y)\,d_qy
\end{aligned}$$

Hence

$$s(t) - \nu(s(t)) = \frac{1}{P(t)}\int_0^t P(qy))\,f(y)\,d_qy = r(t) \tag{3.7}$$

□

4 Tauberian theorems for q-Nörlund integrability

An additional condition on a sequence of integrals is said to be a Tauberian condition for a given limitation method, which with the limitability of the sequence implies the convergence of the sequence. The theorem associated with the Tauberian condition that establishes the validity of the condition is usually called Tauberian theorem. The first theorem of this type was presented by the Hungerian-born Austrian mathematician Alfred Tauber [10]. So far classical summability method for series is concerned, $a_n = O(1/n)$ and $a_n = o(1/n)$ are the Tauberian conditions for Cesàro and Abel's method of the series $\sum a_n$ respectively. Such theorems are found in the book of Petersen [8]. We have

Theorem 4.1. *$b_n = O(\frac{1}{n})$ is a Tauberian-condition for $(C, 1)$ limitable of $\sum b_n$.*

Theorem 4.2. *$b_n = o(\frac{1}{n})$ is a Tauberian-condition for Abel's limitable of $\sum b_n$.*

Varheny[11], have established a Tauberian theorem on Nörlund summability method. Analogue to Tauber's first theorem, following theorem for Cesàro integrability for continuous function was found in the book of Hardy[6]:

Theorem 4.3. $\lim_{x\to\infty}$ *$x \frac{d}{dx}s(x) = 0$ is a Tauberian-condition for Cesàro integrable of an integral $s(x)$.*

Further, Hardy[6] also delineated slow oscillation concept for functions of real variables as defined by Schimidt[9]. Dealing with Cesàro summability of integrales, Canak and Totur ([1],[2]) established certain results of Schmidt type theorem. One of the result is

Theorem 4.4. *If A is Cesàro integrable of $s(x)$ and $s(x)$ is slowly oscillating, then $s(x)$ converges to A as $x \to \infty$.*

Very recently, dealing with q-Cesàro integrability, Canak et al[3] generalizes the result of Fitouchi and Brahim[5] establishing the following theorem:

Theorem 4.5. *If the function $s(t)$ is q-Cesàro integrable to s and its q-Cesàro mean be such that for all $\delta > 0$, there is a $k > 0$ with*

$$|\sigma(s(t)) - \sigma(s(qt))| < \delta, \tag{4.1}$$

for all $t > k$, then $\lim_{t\to\infty} s(t) = s$.

However, in the present paper we establish certain theorems on q-Nörlund integrability. We prove :

Theorem 4.6. *If $s(t)$ is q-Nörlund integrable to s and its q-Nörlund mean be such that for all $\delta > 0$, there is $t_o > 0$ such that*

$$|\nu(s(t)) - \nu(s(qt))| < \epsilon, \tag{4.2}$$

for all $t > t_o$, then $\lim_{t\to\infty} s(t) = s$.

Proof. The theorem is established in two steps. In the first step it is considered for $s = 0$ and in the later step it is established for $s \neq 0$. First of all we start with $s = 0$. From the definition of

$\nu(x)$, we have

$$\begin{aligned}
\nu(s(t)) \;-\; \nu(s(qt)) &= \frac{1}{P(t)}\int_0^t p(x)\, s(x)\, d_qx \;-\; \frac{1}{P(qt)}\int_0^{qt} p(x)\, s(t)\, d_qx \\
&= \frac{1}{P(t)}\int_0^{qt} p(x)\, s(x)\, d_qx \;+\; \frac{1}{P(t)}\int_{qt}^t p(x)\, s(x)\, d_qx - \frac{1}{P(qt)}\int_0^{qt} p(x)\, s(x)\, d_qx \\
&= \Big(\frac{1}{P(x)} - \frac{1}{P(qx)}\Big)\int_0^{qx} p(t)\, s(t)\, d_qt \;+\; \frac{1}{P(x)}\int_{qx}^x p(t)\, s(t)\, d_qt \\
&= \frac{P(qt)-P(t)}{P(t)P(qt)}\int_0^{qt} p(x)\, s(x)\, d_qx \;+\; \frac{1}{P(t)}\int_{qt}^t p(x)\, s(x)\, d_qx \\
&= \frac{P(qt)-P(t)}{P(t)}\nu(s(qt)) \;+\; \frac{s(t)}{P(t)}(P(t)-P(qt)) \\
&= \Big(1-\frac{P(qt)}{P(t)}\Big)(s(t) \;-\; \nu(s(t)) \;\leq (s(t) \;-\; \nu(s(t))
\end{aligned}$$

Hence

$$\nu(s(t)) \;-\; s(t) \leq \nu(s(qt)) \;-\; \nu(s(t)). \tag{4.3}$$

Therefore, by the condition, for every $\delta > 0$ there exists $t_1 > 0$ such that

$$|\nu(s(t)) \;-\; s(t)| \leq |\nu(s(qt)) \;-\; \nu(s(t))| \;<\; \frac{\delta}{2}. \tag{4.4}$$

Further, since q-Nörlund integrability of $s(t)$ tends to zero, for every $\delta > 0$ there is $t_2 > 0$ with

$$|\nu(s(t))| \;<\; \frac{\delta}{2}, \tag{4.5}$$

for all $t > t_2$.
Let $t_0 \;=\; Max(t_1, t_2)$. Then , for $t > t_0$

$$|s(t)| \;=\; |s(t) - \nu(s(t)) + \nu(s(t))| \;\leq\; |s(t) - \nu(s(t)| + |\nu(s(t))| < \;\frac{\delta}{2} \;+\; \frac{\delta}{2} = \delta. \tag{4.6}$$

which implies that $\lim_{t\to\infty}\; s(t) \;=\; 0.$
If $s \neq 0$, let $v(t) = s(t) - s$. Proceeding as above we can show that $\lim_{t\to\infty}\; v(t) \;=\; 0$ which implies that $\lim_{t\to\infty}\; s(t) \;=\; 0.$ □

From Theorem 4.6, we derive Theorem 4.7 as a corollary.

Theorem 4.7. *Let $s(t)$ be q-Nörlund integrable to s and be such that for all $\delta > 0$, there is $t_0 > 0$ such that*

$$|p(t)s(t) \;-\; p(qt)s(qt)| \;<\; \delta\;, \tag{4.7}$$

for all $t > t_0$, then $\lim_{t\to\infty}\; s(t) \;=\; s$.

Proof. For establishing the theorem, we need to show that $\nu(s(t))$, the q-Nörlund mean of the integral $s(t)$, satisfies the condition that satisfies $s(t)$. By the hypothesis, for every $\delta > 0$, there exists $t_1 > 0$ such that for all $t > k_1$

$$|s(t) \;-\; s(qt)| \;<\; \frac{\delta}{2} \tag{4.8}$$

For a positive finite number A, we have $|s(t) - s(qt)| < A$ for all t. Then

$$\begin{aligned}
|\nu(s(t)) \ - \ \nu(s(qt))| &= \Big|\frac{1}{P(t)}\int_0^t \ p(x)\ s(x)\ d_qx \ - \ \frac{1}{P(qt)}\int_0^{qt} \ p(x)\ s(x)\ d_qx\Big| \\
&= \Big|\frac{1}{P(t)}\int_0^t \ p(x)s(x)\ d_qx \ - \ \frac{q}{P(qt)}\int_0^t \ p(qx)\ s(qx)\ d_qt\Big| \\
&= \Big|\frac{1}{P(t)}\int_0^t \ p(t)\ s(x)\ d_qx \ - \frac{1}{P(t)}\int_0^t \ p(qx)\ s(qx)\ d_qx\ \Big| \\
&= \Big|\frac{1}{P(t)}\int_0^t \ [p(x)\ s(x) \ - p(qx)\ s(qx)]\ d_qx\ \Big| \\
&\le \frac{1}{P(t)}\int_0^t \ |p(x)\ s(x) \ - p(qx)\ s(qx)|\ d_qx \\
&\le \frac{1}{P(t)}\int_0^{t_1} |p(x)s(x) - p(qx)s(qt)|\ d_qx + \frac{1}{P(t)}\int_{t_1}^t |p(x)s(x) - p(qx)s(qx)|\ d_qx \\
&\le \frac{t_1 A}{P(t)} \ + \ \frac{\delta(t-t_1)}{2P(t)} \\
&\le \frac{t_1 A}{P(t)} \ + \ \frac{\delta}{2}
\end{aligned}$$

Since $\lim_{t\to\infty} \frac{t_1 A}{P(t)} \ = \ 0$, for every $\delta > 0$, there is t_2 such that $|\frac{t_1 A}{P(t)}| \le \frac{\delta}{2}$, for $t > t_2$. Let $t_0 \ = \ max\ (t_1, t_2)$. Then, for $t > t_0$

$$|\nu(s(t)) \ - \ \nu(s(qt))| \ < \ \delta. \tag{4.9}$$

Hence, using Theorem 4.6, the proof is completed. □

References

[1] $\dot{I}$.Canak and $\ddot{U}$.Totur , *A Tauberian theorem for Cesàro summability of integrals*, Appl. Math. Lett. , **24**(2011), no. 3, 391-395.

[2] $\dot{I}$.Canak and $\ddot{U}$.Totur, *Alternative proofs of some classical type of Tauberian theorems for the Cesàro summability of integrals*, Math. Comput. Modlling., **55**(2012), n0. 3-4 , 1558-1561.

[3] $\dot{I}$.Canak , $\ddot{U}$.Totur and S.A.Sezer., *Cesàro integrability and tauberian theorems in quantum calculus*, An. Ştiinţ. Univ. Al.I. Cuza Iaşi.Mat. (N.S.), **Tomul LXIV**(2018), 9-19.

[4] J.L.Cieśliński., *B Improved q-exponential and q-trigonometric functions*, Appl. Math. Lett. , **24** No. 12,(2011), 2110-2114.

[5] A.Fitouchi and K. Brahim., *Tauberian theorems in quantum calculus*, Jour. Nonlinear Math. Phys., **14**, No.3, (2007), 316-332.

[6] G.H.Hardy., *Divergent series, Clarendon Press, Oxford*, , (1949), xvi+396 pp.

[7] V.Kac and P.Cheung., *Quantum calculus*, Universitext, Springer-Verlag, New York, (2002), x+112 pp.

[8] G.M.Petersen, *Regular Matrix Transformations*,McGraw-Hill publishing company Limited, London., (1966).

[9] R.Schmidt, *Über divergenteÜ Folgen und lineare Mittelbildungen*, (German), Math. Z **22**(1925), no.1, 89-152.

[10] A.Tauber, *Ein satz Theorie der unendlichen Reihen*, (German), Monatsh. Math. Phys., **8**(1897), no. 1, 273-277.

[11] O.P.Varshney, *On Iyengar's Tauberian Theorem for Nölund summability*, Tohoku Math. Jour.(2)**16(1)** (1964), 105-120.

Author information

A. Sudhakar, Research Scholar, Department of Mathematics, Centurion University of Technology and Management, Paralakhemundi, Odisha, India.
E-mail: asesn99@gmail.com

Banitamani Mallik, Department of Mathematics, Centurion University of Technology and Management, Paralakhemundi, Odisha, India.
E-mail: banita.mallik@cutm.ac.in

*Birupakhya Prasad Padhy, Department of Mathematics, KIIT Deemed to be University, Bhubaneswar, Odisha, India.
E-mail: birupakhya.padhyfma@kiit.ac.in

Umakanta Misra, Department of Mathematics, National Institute of Science and Technology, Brahmapur, Odisha, India.
E-mail: umakanta_misra@yahoo.com

Received: January 8th, 2021
Accepted: March 7th, 2022